

# Linear Instability of Wake-Shear Layers in Two-Phase Shallow Flows

Inta Volodko and Valentina Koliskina

**Abstract**—Linear stability analysis of wake-shear layers in two-phase shallow flows is performed in the present paper. Two-dimensional shallow water equations are used in the analysis. It is assumed that the fluid contains uniformly distributed solid particles. No dynamic interaction between the carrier fluid and particles is expected in the initial moment. The stability calculations are performed for different values of the particle loading parameter and two other parameters which characterize the velocity ratio and the velocity deficit. The results show that the particle loading parameter has a stabilizing effect on the flow while the increase in the velocity ratio or in the velocity deficit destabilizes the flow.

**Keywords**—Linear stability, Shallow flows, Wake-shear flows.

## I. INTRODUCTION

A flow is considered to be shallow if the transverse length scale of the flow is much larger than water depth. Shallow flows often occur in nature and engineering. One example is wake flow behind obstacles (such as islands).

Methods of linear stability theory are often used to analyze shallow flows [1] – [4]. Three different flow regimes in shallow wake flows, namely, steady bubble, unsteady bubble and vortex street are identified experimentally in [5]. In addition, theoretical studies in [2] – [4] showed that the three regimes are related to convective/absolute instability of the flow.

Linear stability of two-phase flows in deep water is analyzed in [6], [7]. Such flows often occur in applications. Examples include liquid-gas bubble or particle-laden flows. The analysis in [6] and [7] is based on some simplifying assumptions. First, it is assumed that small perturbations imposed on the flow have no effect on the particles during the initial moment. Second, the particle distribution is assumed to be uniform. The dynamic interaction of particles with fluid is considered in [8]. The case of differential particle loading is analyzed in [9]. In general, it is found in [6]-[9] that if the particle loading parameter increases then the flow becomes more stable. Linear stability of two-phase shallow wake flows is analyzed in [10].

Experimental analysis of the structure of shallow wake

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flows [5] shows that there are also shear layers across the wake. The effect of shear on the stability of wake flows is analyzed in [11], [12] for the case of deep water flows. Linear instability of wake-shear layers in shallow water is studied in [13] where it is shown that the increase in the velocity deficit destabilizes the flow.

In the present paper we perform a linear stability analysis of wake-shear layers in two-phase shallow flows under the simplifying assumptions used in [6], [7].

## II. LINEAR STABILITY ANALYSIS

Consider the two-dimensional shallow water equations of the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} + \frac{c_f}{2h} u \sqrt{u^2 + v^2} = B(u^p - u), \quad (2)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} + \frac{c_f}{2h} v \sqrt{u^2 + v^2} = B(v^p - v), \quad (3)$$

where  $u$  and  $v$  are the depth-averaged velocity components in the  $x$  and  $y$  directions, respectively,  $u^p$  and  $v^p$  are the velocity components of the particles,  $h$  is water depth,  $c_f$  is the friction coefficient,  $p$  is the pressure and  $B$  is the particle loading parameter (see [6], [7]). The system (1) – (3) is derived under some simplifying assumptions that are discussed in [6] and [7] in detail. In particular, the dynamic interaction of particles and the fluid is not considered in the model.

Introducing the stream function by the relations

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (4)$$

and eliminating the pressure we transform the system (1) – (3) to the form

$$(\Delta \psi)_t + \psi_y (\Delta \psi)_x - \psi_x (\Delta \psi)_y + \frac{c_f}{2h} \Delta \psi \sqrt{\psi_x^2 + \psi_y^2} \quad (5)$$

$$+ \frac{c_f}{2h \sqrt{\psi_x^2 + \psi_y^2}} (\psi_y^2 \psi_{yy} + 2\psi_x \psi_y \psi_{xy} + \psi_x^2 \psi_{xx}) + B \Delta \psi = 0$$

$$\text{where } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Assuming that  $\psi_{0y}(y) = u_0(y)$  is the base flow solution and imposing small perturbations on the base flow, the stream function can be written in the form

$$\psi = \psi_0(y) + \varepsilon\psi_1(x, y, t) + \varepsilon^2\psi_2(x, y, t) + \dots \quad (6)$$

Substituting (6) into (5) and keeping only linear terms with respect to  $\varepsilon$ , we obtain

$$L\psi_1 = 0, \quad (7)$$

where

$$L\psi_1 = \psi_{1xx} + \psi_{1yy} + \psi_{0y}(\psi_{1xx} + \psi_{1yy}) - \psi_{0yy}\psi_{1x} + \frac{c_f}{2h}[(\psi_{1xx} + 2\psi_{1yy})\psi_{0y} + 2\psi_{1y}\psi_{0yy}] + B(\psi_{1xx} + \psi_{1yy})$$

Using the method of normal modes we seek the solution to (7) in the form

$$\psi_1(x, y, t) = \phi_1(y) \exp[ik(x - ct)], \quad (8)$$

where  $\phi_1(y)$  is the amplitude of the normal perturbation. Substituting (8) into (7) we obtain the modified Rayleigh equation of the form

$$\phi_1''(-ikc + iku_0 + 2Su_0 + B) + Su_{0y}\phi_1' + \phi_1(ik^3c - ik^3u_0 - iku_{0yy} - Sk^2u_0 - Bk^2) = 0 \quad (9)$$

where  $S = \frac{c_f b}{2h}$  is the bed-friction number (see [1]) and  $b$  is the half-width of the wake. The boundary conditions are

$$\phi_1(\pm\infty) = 0 \quad (10)$$

Problem (9) – (10) describes the linear stability of the base flow  $u = u_0(y)$ . In fact, (9) – (10) is an eigenvalue problem where the eigenvalues are  $c = c_r + ic_i$ . The flow  $u = u_0(y)$  is said to linearly stable if all  $c_i$  are negative and unstable if at least one  $c_i$  is positive.

The base flow velocity profile in the present study is chosen in the form

$$u_0(y) = 1 - \frac{f}{\cosh^2 y} + r \tanh y, \quad (11)$$

where  $r$  is the velocity ratio (i.e., the velocity difference across the layer divided by the mean velocity) and  $f$  is the wake deficit parameter (i.e., the velocity deficit divided by the mean ambient velocity). The cases of a wake flow or a mixing layer are obtained from (11) if  $r = 0$  or  $f = 0$ , respectively.

The presence of the two parameters  $f$  and  $r$  allows one to investigate the effect of shear on the stability of wake flows. The profile (11) was suggested in [12] for the stability analysis of wake-shear layers in deep water and is adopted in the present study.

Problem (9) – (10) is solved numerically by means of the pseudospectral collocation method. The Chebyshev polynomials are chosen as the base functions. Since problem

(9) – (10) is defined on the infinite interval  $-\infty < y < +\infty$ ,

we use the transformation  $z = \frac{2}{\pi} \arctan y$  to map the

interval  $(-\infty, +\infty)$  onto  $(-1, 1)$ . The solution to (9) – (10) (in terms of the variable  $z$ ) is sought in the form

$$\phi_1(z) = \sum_{k=0}^N a_k (1 - z^2) T_k(z), \quad (12)$$

where  $T_k(z)$  is the Chebyshev polynomial of degree  $k$ . The factor  $1 - z^2$  is added to (12) in order to simplify the numerical solution of (9), (10) since the boundary conditions (10) (in terms of the variable  $z$ ) are satisfied automatically. The collocation points are

$$z_j = \cos \frac{\pi j}{N}, \quad j = 0, 1, \dots, N. \quad (13)$$

Evaluating the function  $\phi_1(z)$  and its derivatives at the collocation points (13) we obtain the generalized eigenvalue problem of the form

$$(A - \lambda B)a = 0 \quad (14)$$

where  $A$  and  $B$  are complex-valued matrices and

$$a = (a_1 a_2 \dots a_N)^T,$$

where the subscript  $T$  denotes the transpose.

Problem (14) is solved numerically by means of IMSL routine DGVCCG. Marginal stability curves for different values of  $f$  and  $B$  are shown in Figs. 1 – 3. The velocity ratio (the value of  $r$ ) is fixed at  $r = 0.4$ . The increase in the value of the parameter  $f$  corresponds to the increase in the velocity deficit in the wake.

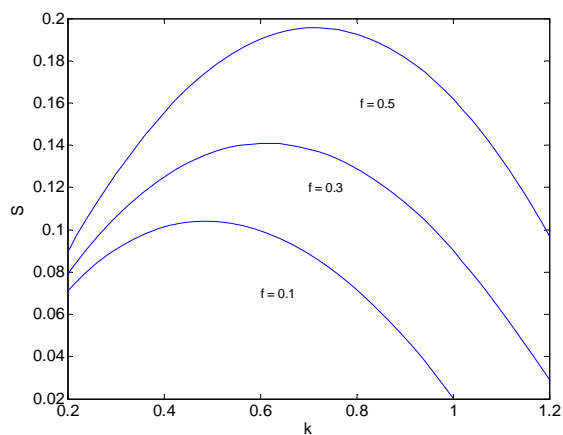


Fig. 1 Marginal stability curves for different values of  $f$  at  $B = 0$  and  $r = 0.4$ .

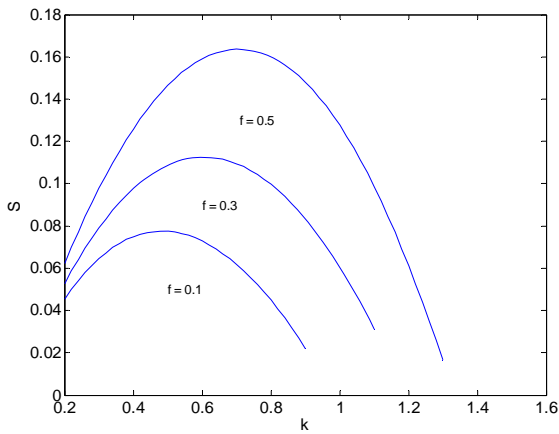


Fig. 2 Marginal stability curves for different values of  $f$  at  $B = 0.02$  and  $r = 0.4$ .

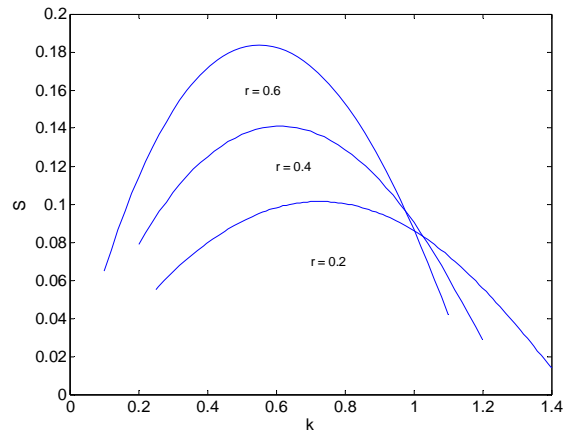


Fig. 4 Marginal stability curves for different values of  $r$  at  $B = 0$  and  $f = 0.3$ .

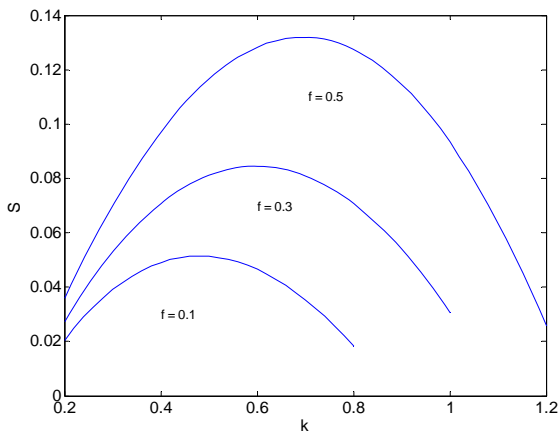


Fig. 3 Marginal stability curves for different values of  $f$  at  $B = 0.04$  and  $r = 0.4$ .

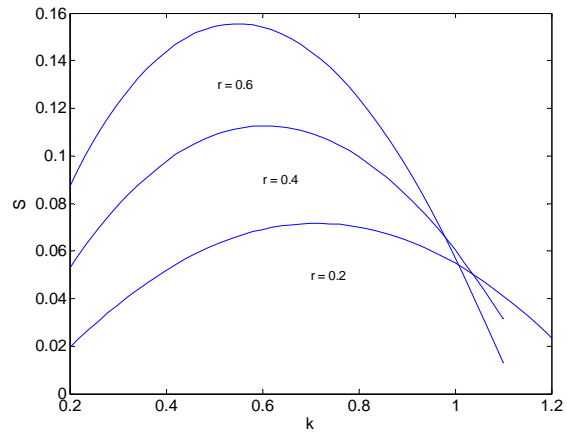


Fig. 5 Marginal stability curves for different values of  $r$  at  $B = 0.02$  and  $f = 0.3$ .

As can be seen from Figs. 1 – 3, the increase of the velocity deficit leads to less stable flow (the values of the parameter  $S$  on the marginal stability curves also increase). In addition, the flow becomes more stable when the particle loading parameter  $B$  increases.

Marginal stability curves for different values of  $r$  and  $B$  are shown in Figs. 4 – 6. The velocity deficit (the value of  $f$ ) is fixed at  $f = 0.3$ .

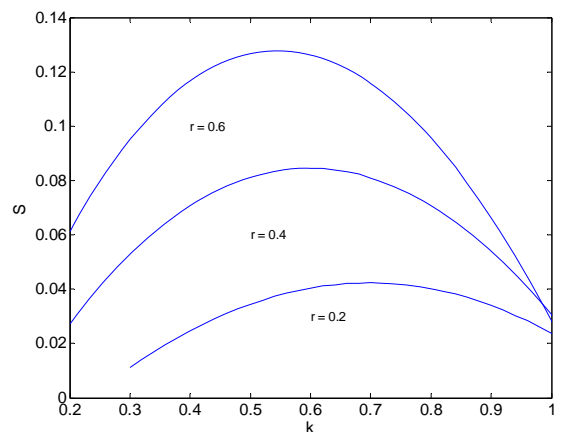


Fig. 6 Marginal stability curves for different values of  $r$  at  $B = 0.04$  and  $f = 0.3$ .

It follows from Figs. 4 – 6 that for a fixed velocity deficit ( $f$ ) the increase in the velocity ratio ( $r$ ) destabilizes the flow. The stabilizing effect of the particle loading parameter  $B$  is clearly seen in Figs. 4 – 6.

The critical values of  $S$  ( $S_c = \max_k S$ ) for different values of the particle loading parameter  $B$  and  $f = 0.3$  are shown in Fig. 7.

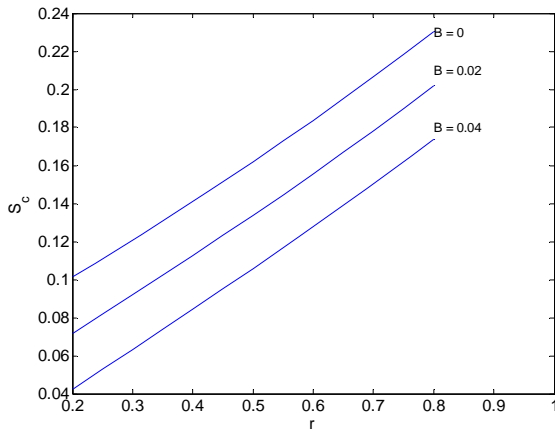


Fig. 7 Critical values of the parameter  $S$  versus  $r$  at  $f = 0.3$  and different values of  $B$ .

As can be seen from the figure, the critical values of  $S$  grow almost linearly as a function of  $r$ .

The critical values of  $S$  versus  $f$  are plotted in Fig. 8 for different values of  $B$ .

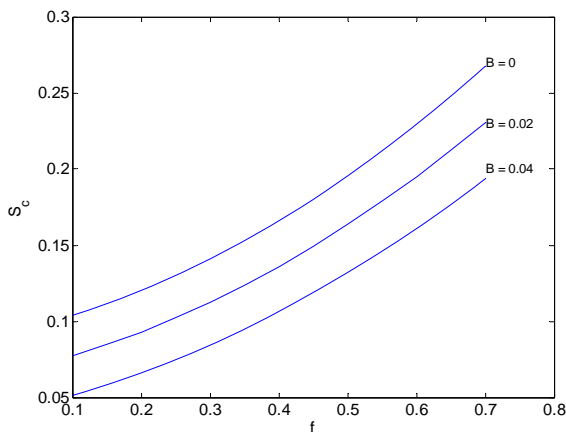


Fig. 8 Critical values of the parameter  $S$  versus  $f$  at  $r = 0.3$  and different values of  $B$ .

The increase of the velocity deficit results in more unstable flow (the parameter  $S$  increases). Note that the growth is not linear, the critical values of the stability parameter increase faster as the parameter  $f$  increases.

Several conclusions can be drawn from Figs. 1 – 8. First, the particle loading parameter  $B$  has a stabilizing influence on

the flow. Second, the increase of the wake velocity deficit for a fixed velocity ratio (this amount is related to the magnitude of shear) results in less stable flow. Third, if the velocity deficit is fixed at a constant level, the flow becomes more unstable when the velocity ratio increases.

### III. DISCUSSION

Linear stability analysis of wake-shear layers in two-phase shallow flows is presented in this paper. It is known that there are regions of absolute and convective instability in the parameter space for wake flows (if the velocity deficit is large enough) and for mixing layers (if the velocity ratio of two streams moving in the opposite direction is large enough), [2]-[4]. Since the base flow profile used in our paper simulates the presence of shear in wake flows, it is plausible to assume that wake-shear layers for two-phase shallow flows can be absolutely unstable in some regions of the parameter space. The authors are currently working on this problem.

In addition, it is shown in [3] that if the bed-friction number  $S$  is slightly smaller than the critical value then the development of the most unstable mode is described by the complex Ginzburg-Landau equation. Examples of calculation of the coefficients of the Ginzburg-Landau equation for two-phase shallow wake flows are given in [10]. It is shown in [10] that for shallow wake flows the instability is supercritical (e.g., finite amplitude equilibrium is possible). Application of the Ginzburg-Landau model to wake-shear layers in two-phase shallow flows is the topic for future work.

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