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Limit Analysis of FGM Circular Plates Subjected to Arbitrary Rotational Symmetric Loads

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Abstract— The limit load carrying capacity of functionally graded materials (FGM) circular plates subjected to an arbitrary rotationally symmetric loading has been computed. It is provided that the plate material behaves rigid perfectly plastic and obeys either the Square or the Tresca yield criterion. To this end the upper and lower bound principles of limit analysis are employed to determine the exact value for the limiting load. The correctness of the result are verified and finally limiting loads for two examples namely; through radius and through thickness FGM circular plates with simply supported edges are calculated, respectively and moreover, the values of critical loading factor are determined.

Keywords— Circular plate, FGM circular plate, Limit analysis, Lower and Upper bound theorems.

I. Introduction

STUDYING the behavior of structures in the plastic range is of great practical importance. Limit analysis theorems, as powerful devices, have been mostly applied to the case of rigid-plastic materials and led to the design of more efficient structures. Traditional elastic design cannot, in general, present realistic estimates for the true load carrying capacity. As a result, structures that have been manufactured according to elastic design principles are relatively heavy and expensive. Application of the limit analysis approach results in the design of lighter structures than those obtained through elastic design. Such structures are frequently utilized, especially in aerospace transportation devices. Limit analysis is based on lower and upper bound theorems, which results in corresponding estimates for the critical value of the load (the limit load of the structure). If these two bounds coincide, the exact solution of the problem and/or true limit load of the structure is obtained. However, it has to be emphasized that the exact solution of the described problem should satisfy all of the followings: (a) kinematic admissibility of the deformation field, (b) static admissibility of the stress field, and (c) satisfaction of the yield criterion, all over the plate.

Limit analysis theorems, and some of their applications in the structural analysis, were first introduced in the 1950's. Among the most fundamental works on this subject, the contributions of Prager [1], Horne [2], Mansfield [3], and Hodge [4] are remarkable. Some of the most notable works on the subject have been reviewed extensively in Save and Massonnet [5] and Sobotka [6]. In an attempt to analyse the plastic behaviour of plates subjected to some general kind of loading, Ghorashi [7] considered constant thickness circular plates obeying either the Square or Tresca yield criterion and obtained exact solutions for the case of rotationally symmetric load. Also Ghorashi et al. [8] considered the variable thickness plates which are used extensively. Furthermore, in another attempt, Guowei [9] has derived the limit load of orthotropic circular plates. But so far there is no reported work on the limit analysis of other types of material such as functionally graded materials or FGMs.

In the present paper, the load carrying capacity of FGM circular plates under arbitrary rotational symmetric loadings has been considered. In this way, the simple supported circular plate is considered to be through radius FGM as well as through thickness FGM. The upper and lower bound theorems of limit analysis are utilized and the critical loading factor is computed. Ultimately the outcome of the paper is illustrated through several case studies.

II. Description of FGMs properties

Advances in material synthesis technologies have spurred the development of a new class of materials, called functionally graded materials (FGMs), with promising applications in aerospace, transportation, energy, electronics and biomedical engineering [10-12]. In applications involving severe thermal gradients (e.g. thermal protection systems), FGMs exploit the heat, oxidation and corrosion resistance typical of ceramics, and the strength, ductility and toughness typical of metals. In a typical FGM, the volume fractions of the constituents are varied gradually over a macro scale geometrical dimension such as coating thickness. Within an FGM, the different phases have different functions to suit the needs of the material. The material gradients induced by the

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spatial variations of the material properties make FGMs behave differently from common homogeneous, isotropic materials and traditional composites. Plastic limit analysis of these kinds of FGM plates should be updated based on the yield properties of the plates.

In our analysis, for a simply supported FGM plate it is assumed that the compositional variation of ceramic and metal phases can be approximated by an idealized power-law equation as Eq. (1) for the through radius FGM

$$V_1 = \left(\frac{r}{R}\right)^n \tag{1}$$

and as Eq. (2) for through thickness FGM plates:

$$V_1 = \left(\frac{2z}{t}\right)^n \tag{2}$$

Here, V_1 represents the volume fraction of the ceramic phase, z is the location measured from the mid surface, t is the total thickness of the FGM layer, r is the radial coordinate, R is the radius of the plate and n is the power exponent (see Fig. (1)). Note that, at any location, $V_1 + V_2 = 1$, where V_2 is the volume fraction of the metal phase. In our model, the composition is pure ceramic at the support edge (for through radius FGM) and at the surfaces (for through thickness FGM) while it is pure metal at the centre (for through radius FGM) and at the mid surface (for through thickness FGM). When n = 1, the composition changes linearly through the thickness or radius, while $n = 1/2$ or 2 corresponds to the quadratic or parabolic distribution. Since we need to specify a likely range of the unknown variable in the inverse analysis, it is assumed that n ranges from 1/3 to 3. Any value outside this range is not usually desired since such an FGM would contain too much of one phase. (For example when n = 1/3 or 3, one phase has 75% of the total volume.)

In the case of through thickness, when the lower surface is metal and upper surface is ceramic and the Eq. (2) should be modified and can be expressible as $V_1 = \left(\frac{z}{t}\right)^n$, but the similar analysis should be done.

In the current analysis, the so-called "modified rule of mixtures" described in [13-16] is adopted. If the composite is treated as isotropic, its uni-axial stress and strain can be decomposed into:

$$\sigma_f = \sigma_1 V_1 + \sigma_2 V_2 \quad and \quad \varepsilon_f = \varepsilon_1 V_1 + \varepsilon_2 V_2 \tag{3}$$

Where σ_1, σ_2 and ε_1 , ε_2 are the stresses and strains of the ceramic and metal under uni-axial stress and strain conditions, respectively. The normalized ratio of the stress to strain transfer is then defined by a parameter q to describe the ratio of stress-to-strain transfer.

$$q = \frac{\sigma_1 - \sigma_2}{\left|\varepsilon_1 - \varepsilon_2\right|} \quad , \ 0 \le q < \infty \tag{4}$$

Combining Eqs. (3) and (4) and setting $E_f = \frac{\sigma_f}{\varepsilon_f}$, one may obtain the following expression for the effective modulus [17]:

$$E_f = \left[V_2 E_2 \frac{q + E_1}{q + E_2} + (1 - V_2) E_1\right] / \left[V_2 \frac{q + E_1}{q + E_2} + (1 - V_2)\right] \tag{5}$$

For applications involving plastic deformation of ceramic/metal (brittle/ductile) composites, it is assumed that the composite yields once the metal constituent yields. With these assumptions, the yield stress, σ_Y , of the composite may be obtained as follows [17]:

$$\sigma_Y(V_2) = \sigma_0 \left[V_2 + \frac{q + E_2}{q + E_1} \frac{E_1}{E_2} (1 - V_2)\right] \tag{6}$$

where E_1 , E_2 are Young's module of the constituent phases, and σ_0 denotes the yield stress of the metal (phase 2). The above equation indicates that the yield stress of the composite depends on the yield stress of the metal, the volume fraction of the metal, Young's module of the constituent phases, and the parameter q.

III. Problem Statement

Fig. (1) shows a FGM circular plate of radius R and thickness t which is subjected to an arbitrary rotationally symmetric loading $f(r)$ per unit area. Primarily, it is assumed that the FGM properties of the plate in its undeformed shape are a function of radius only, then another problem is considered where the FGM properties of the plate is a function of thickness only. Rewriting the loading function as $\mu f(r)$, where μ is called the load factor, the main purpose of this paper is to obtain the critical value of μ , i.e. μ_{cr} , which generates the collapse mechanism in the plate.

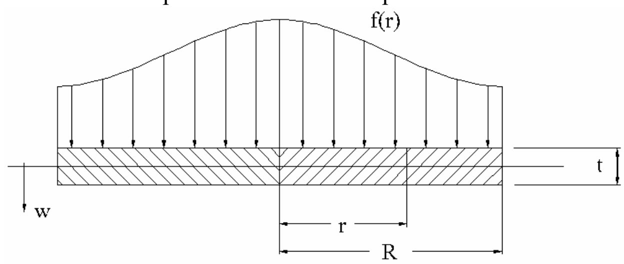

Fig (1). FGM plate with rotationally symmetric loading

In general, the collapse mechanism consists of a few sets of yield lines along which the plate becomes fully plastic through the thickness. The formation of the collapse mechanism is usually considered as the last stage of load carrying function.

Therefore, having evaluated μ_{cr} , the load carrying capacity of the plate can be expressed as $\mu_{cr} f(r)$. As is customary in limit analysis, it is assumed that all defections are negligibly small until the collapse mechanism is fully formed. To be compatible, the material is supposed to be a rigid-perfectly plastic one. This assumption implies that the sequence of formation of the yield lines prior to collapse is not significant for the results. Furthermore, it is assumed that the material obeys either the Square or the Tresca yield criterion, as illustrated in Fig. 2. In this figure, m_r m_θ and m_u stand for the radial, tangential, and ultimate (fully plastic) bending moments per unit length, respectively. It is well known that the ultimate bending moment per unit length for a plate with constant thickness t and yield stress σ_y is:

$$m_u = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y z dz = \frac{1}{4}\sigma_y t^2 \tag{7}$$

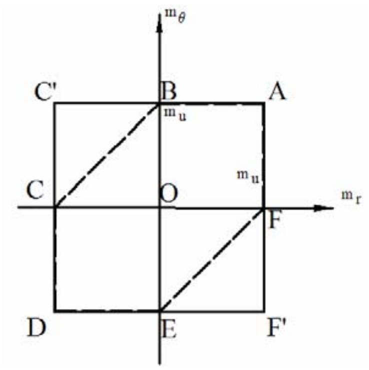

Fig (2). Square (continuous) and Tresca (dashed) yield diagrams

For the case of a through radius FGM, according to Eq. (6) it has been shown that, the yield bending stress can be defined as [19],

$$\sigma_y = \sigma_a + \sigma_b \left(\frac{r}{R}\right)^n \tag{8}$$

Hence, the ultimate bending moment per unit length can be defined as:

$$m_u = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y(r) z dz = \frac{1}{4}\sigma_y(r) t^2 \tag{9}$$

Clearly, for a variable thickness plate, $m_u(r)$ would no longer yield a constant value. As a result, if the yield criterion is expressed in terms of the bending moments per unit length, the yield diagram clearly would change for differing points of the plate. This representation is depicted in Fig. (2). According to Eq. (9) the formulation is as the same as the formulation of the isotropic homogeneous circular plates with variable thickness [8].

On the other hand for the case of through thickness FGM, it has been shown that [19]:

$$\sigma_y = \sigma_a + \sigma_b \left(\frac{2z}{t}\right)^m \tag{10}$$

So one can find the ultimate bending moment per unit length as:

$$m_u = \int_{-\frac{t}{2}}^{\frac{t}{2}} \sigma_y(z) z dz = m^* \tag{11}$$

Where m^* is constant and is independent of *z*. Therefore, the same formulation as for the case of isotropic homogeneous material can be used.

IV. Critical load factor for through radius simply supported FGM plates

For a simply supported plate under transverse loading, it may be observed that stress states lie on the line AB in Fig. (2), for which the two yield criteria are identical. Hence, the analysis would result in the same outcome if either criterion applied. According to Koiter's rule [1], the deformation field and hence the collapse mechanism should be compatible with the assumed yield criterion. It may be easily verified that the frustum collapse mechanism is compatible with the segment AB of the two yield criteria shown in Fig. (2). Hence, it can be assumed that the frustum collapse mechanism would be generated in the plate. Fig. (3) illustrates the geometric parameters of the frustum shaped collapse mechanism. It should be noted that this mechanism can be regarded as a generalization of the simple conical one (with $r_f = 0$).

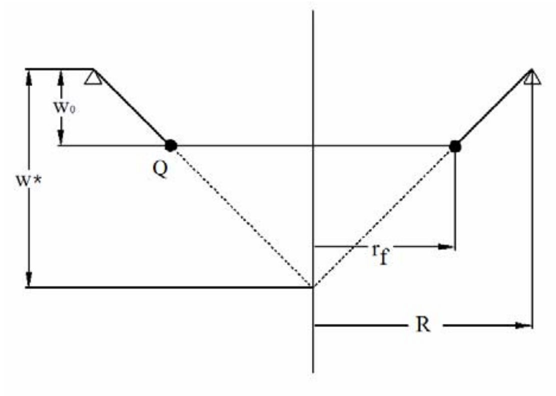

Fig (3). The frustum collapse mechanics

A. Upper Bound Solution

Under presented configuration for the plate collapse in Fig. (3), if the central part of the plate with radius r_f , has a constant deflection of w_0 , then when r_f changes from zero to R, then we have;

$$w_0 = w^*(1-\frac{r_f}{R}) \quad (12)$$

in which w^* is the deflection of the plate center when we have a central hinge and R is the radius of the plate. The total internal (dissipative) work may now be written as follows:

$$W_i = 2\pi r_f m_u(r_f)\frac{w^*}{R} + 2\pi\frac{w^*}{R}\int_{r_f}^{R} m_u(r)dr \quad (13)$$

where, the first term on the right-hand side is due to the work at the internal circular hinge Q. It has been calculated by the multiplication of the ultimate bending moment per unit length, length of the circular hinge at Q, and the angle of rotation at this point, $\frac{w^*}{R}$. The second term represents the internal work of the conical part of the mechanism and has been calculated in a similar way.

For the external work done by the applied loading, one obtains,

$$W_e = \int_0^{2\pi}\left[\int_0^R \mu f(r)w(r)rdr\right]d\theta \quad (14)$$

where $w(r)$ represents the defection function of the plate. Eq. (14), for a frustum collapse mechanism shown in Fig. (3), can be rewritten as:

$$W_e = 2\pi\left[\int_0^{r_f} \mu f(r)w_0 rdr + \int_{r_f}^{R} \mu f(r)w^*(1-\frac{r}{R})rdr\right] \quad (15)$$

or by using Eq. (12)

$$W_e = 2\pi\mu w^*\left[\int_0^{r_f} rf(r)(1-\frac{r_f}{R})dr + \int_{r_f}^{R} f(r)(1-\frac{r}{R})rdr\right] \quad (16)$$

As a result of the implementation of the virtual work principle and the upper bound theorem of limit analysis, upper bound estimation for the critical load factor, i.e. μ_u, may be obtained by equating the internal work,

Eq. (13) to the external work, Eq. (16). The final result is:

$$\mu_u = \frac{\frac{1}{R}\left[r_f m_u(r_f) + \int_{r_f}^{R} m_u(r)dr\right]}{\left[\int_0^{r_f} rf(r)(1-\frac{r_f}{R})dr + \int_{r_f}^{R} f(r)(1-\frac{r}{R})rdr\right]} \quad (17)$$

B. lower bound solution

In order to apply the lower bound theorem, the equation of equilibrium of the plate should be considered. Furthermore, a compatible stress field which does not violate the assumed yield criterion should be introduced. The equation of equilibrium for a circular plate with rotationally symmetric geometry, loading, and boundary conditions is as follows [1]:

$$(rm_r)' = m_\theta - \int_0^r \xi[\mu f(\xi)]d\xi \quad (18)$$

where ξ is a dummy variable of integration. Integration of Eq. (18) gives:

$$m_r(r) = \frac{1}{r}\int_0^r m_\theta(\xi)d\xi - \frac{\mu}{r}\int_0^r\left[\int_0^{r_2}\xi[\mu f(\xi)]d\xi\right]dr_2 + \frac{C_1}{r} \quad (19)$$

where, r_2 is another dummy variable of integration, and C_1 is an integration constant. The finiteness of the results at the plate center states that C_1 should vanish. For simply supported boundary conditions, the frustum shaped collapse mechanism and either of the two yield criteria shown in Fig. (2). A discontinuous distribution for m_θ will result in as:

$$m_\theta = \begin{cases} m_u(r) & r_f \le r \le R \\ unknown & 0 \le r \le r_f \end{cases} \quad (20)$$

Since, in the region $r_f \le r \le R$ radial yield lines are generated, m_θ is equal to $m_u(r)$ but nothing can be said about the values of m_θ inside the circle $r = r_f$, except that they should satisfy the yield criterion under consideration. Using Eqs. (19) and (20) for $r \ge r_f$, one obtains:

$$m_r(r) = \frac{1}{r}\left[\int_0^{r_f} m_\theta(\xi)d\xi + \int_{r_f}^{r} m_\theta(\xi)d\xi\right] - \frac{\mu}{r}\int_0^r\left[\int_0^{r_2}\xi f(\xi)d\xi\right]dr_2 \quad (21)$$

In the first integral, m_θ is unknown. In order to evaluate this integral, one may evaluate Eq. (21) at $r = r_f$:

$$m_u(r_f) = \frac{1}{r_f}\left[\int_0^{r_f} m_\theta(\xi)d\xi + \int_{r_f}^{r} m_\theta(\xi)d\xi\right] - \frac{\mu}{r_f}\int_0^{r_f}\left[\int_0^{r_2}\xi f(\xi)d\xi\right]dr_2 \quad (22)$$

Therefore,

$$\int_0^{r_f} m_\theta(\xi)d\xi = r_f m_u(r_f) + \mu\int_0^{r_f}\left[\int_0^{r_2}\xi f(\xi)d\xi\right]dr_2 \quad (23)$$

Substation of Eq. (23) into Eq. (21) results in:

$$m_r(r) = \frac{1}{r}\left\{r_f m_u(r_f) + \mu\int_0^{r_f}\left[\int_0^{r_2}\xi f(\xi)d\xi\right]dr_2 + \int_{r_f}^{r} m_\theta(\xi)d\xi\right\} - \frac{\mu}{r}\int_0^r\left[\int_0^{r_2}\xi f(\xi)d\xi\right]dr_2 \quad (24)$$

Finally, since the plate is simply supported, the radial bending moment at the plate edge is zero, i.e.

$$m_r(r = R) = 0 \quad (25)$$

Utilizing Eqs. (20), (24) and (25), one obtains:

$$0 = \frac{1}{R}\left\{r_f m_u(r_f) + \mu\int_0^{r_f}\left[\int_0^{r_2}\xi f(\xi)d\xi\right]dr_2 + \int_{r_f}^{R} m_\theta(\xi)d\xi\right\} - \frac{\mu}{R}\int_0^R\left[\int_0^{r_2}\xi f(\xi)d\xi\right]dr_2 \quad (26)$$

Therefore, the following lower bound answer for the critical load factor, i.e. μ_l, may be obtained:

$$\mu_l = \frac{r_f m_u(r_f) + \int_{r_f}^{R} m_u(r)dr}{\int_{r_f}^{R}\left[\int_0^r \xi f(\xi)d\xi\right]dr} \quad (27)$$

Considering the special case of the conical collapse mechanism with $r_f = 0$, Eq. (27) reduces to the one reported in Ref. [7].

It should be noted that Eq. (27) has been obtained through the application of the plate equation of equilibrium, Eq. (18). However, it can be considered as a complete lower bound solution only if the yield criterion is satisfied and the collapse mechanism with a rigid inner disk is valid. A necessary condition for satisfaction of the yield criterion, which ensures that Eq. (27) is a lower bound solution, is that the integral of m_θ over the inner circular disk of the plate should not exceed the plate load carrying capacity in this region. Hence, using Eq. (23),

$$r_f m_u\left(r_f\right)+\mu\int_0^{r_f}\left[\int_0^{r_2}\xi f\left(\xi\right)d\xi\right]dr_2\leq\int_0^{r_f}m_u\left(\xi\right)d\xi \tag{28}$$

If the inner circular disk is rigid (corresponding to a frustum collapse mechanism) then condition given in Eq. (28) with strict inequality sign would be satisfied. It thus presents a necessary condition for the rigidity of the inner circular disk.

V. Critical Load Factor for Through Thickness Simply Supported FGM Plates

Since for a through thickness FGM plate, m_u is constant at each point, it behaves similar to a homogeneous isotropic plate. Thus, for a simply supported plate, a simple conical collapse mechanism, corresponding to the following kinematically admissible deformation field, can be assumed:

$$W=W_0(1-\frac{r}{R}) \tag{29}$$

Where $W\left(r\right)$ is the plate deflection at radius r, and W_0 is the deflection at the centre of the plate. It then follows that m_r and the tangential curvature k_θ are both positive throughout the plate. Therefore according to Koiter's rule, the compatible stress states, lie on AB in Fig. 2. Since the two criteria are coincident in this region, the same results for the critical load factor would be obtained and corresponding results of Ref. [18] are applicable to the present problem, hence:

$$\mu_{cr}=\frac{Rm_u}{\int_0^R\left[\int_0^{r_1}\xi f\left(\xi\right)d\xi\right]dr_1} \tag{30}$$

A. Solution scheme and case studies

In this section, critical loading factor is investigated for the following loading function:

$$f(r)=a+b\left(\frac{r}{R}\right)+c\left(\frac{r}{R}\right)^2 \tag{31}$$

This loading function is expressed in a general form and it incorporates different type of loadings i.e., uniformly distributed with *b=c=0*, linearly varying loadings *(c=0)* as well as parabolic loading.

The material property for each phase is considered as shown in Table 1 and $q=4.5\,GPa$ [19].

Table 1
Material properties of Ti and TiB [19]

Materials	Young's modulus (GPa)	Yield stress (MPa)
Ti	107	450
TiB	375	____

Based on the values listed in Table and yielding model represented by Eq. (6), we can conclude that:

$$\sigma_Y=450+13.365V_1 \tag{32}$$

B. simply supported through radius FGM circular plate

It is evident from Eq. (8) that m_u is a function of radius and is in the form of:

$$m_u=A+B\left(\frac{r}{R}\right)^n \tag{33}$$

Considering Eqs. (17), (27), (31) and (33) we obtain the exact critical loading factor for a simply supported through radius FGM circular plate obeying square or Tresca yielding function as below:

$$\mu_l=\mu_u=\frac{60R^2\left(\left(A+B+An\right)R+Bnr_f\left(\frac{r_f}{R}\right)^n\right)}{\left(1+n\right)\left(\left(10a+5b+3c\right)R^5-r_f^3\left(10aR^2+5bRr_f+3cr_f^2\right)\right)} \tag{34}$$

$$\mu_l=\mu_u=\frac{20R^3\left(3A+2B+B\left(\frac{r_f}{R}\right)^{\frac{3}{2}}\right)}{\left(10a+5b+3c\right)R^5-r_f^3\left(10aR^2+5bRr_f+3cr_f^2\right)} \tag{35}$$

The result given in Eq. (34) is derived for *n=1, 2* whereas in Eq. (35) *n=1/2*. Since we have obtained the analytical form of the solution, as it is seen there is no need to use material properties shown in Table 1 to compute numerical value.

C. simply supported through thickness FGM circular plate

Considering Eqs. (11) and (31) we are able to obtain the exact value for the critical loading factor of a simply supported through thickness FGM circular plate obeying square or Tresca yield function as:

$$\mu_{cr}=\frac{60m_u}{\left(10a+5b+3c\right)R^2} \tag{36}$$

D. Verification

For a homogeneous isotropic material, it is obvious that yield stress is constant, therefore, according to Eq. (33) we have:

$$A = \frac{1}{4}\sigma_Y t^2, \quad B = 0 \tag{37}$$

Considering Eqs. (17), (27) and (37) and assuming that a constant uniform pressure p is applied on the plate, we conclude that:

$$\mu_{cr} = \frac{3\sigma_Y t^2}{2pR^2(1-\lambda^3)}; \quad \lambda = \frac{r_f}{R} \tag{38}$$

The above result is the same as shown in Eq. (25) in [8].

Since the formula derived for a through thickness simply supported FGM circular plate is the same for a homogeneous plate, it is not necessary to verify Eq. (30).

VI. Conclusions

Limit analysis of simply supported circular FGM plates with constant thickness subjected to an arbitrary rotational symmetric loading has been discussed using upper and lower bound principles. In this way, general equations for the critical load factor considering the Square and Tresca yield criteria have been obtained. For the simply supported through radius or through thickness FGM circular plates, it was shown that the results obtained for either of yield criterion are the same. For a parabolic type of loading the exact solution is derived for simply supported FGM plates.

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