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Laplace Technique to Find General Solution of Differential Equations without Initial Conditions

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***Abstract*—**Laplace transformations have wide applications in engineering and sciences. All previous studies of modified Laplace transformations depend on differential equation with initial conditions. The purpose of our paper is to solve the linear differential equations (not initial value problem) and then find the general solution (not particular) via the Laplace transformations without needed any initial condition. The study involves both types of differential equations, ordinary and partial.

***Keywords*—**Differential Equations, Laplace Transformations.

I. Introduction

DIFFERENTIAL equation has wide usage and applications, for example it used for studying electrochemical cells [1], Harmonic Resonant [2], Control [3], Fractional Differential Equations [4], Transient Heat Conduction [5], Relaxation Model [6], electrical engineering problems [7], and others. Let $f(t,y)$ be continuous at all points (t,y) in some rectangle

$$R : |t - t_0| < a, |y - y_0| < b \tag{1}$$

and bounded in R, say,

$$|f(t, y)| \le k \tag{2}$$

for all (t,y) in R, then the initial value problem

$$y' = f(t, y), y(t_0) = y_0 \tag{3}$$

has at least one solution $y(t)$ [8]-[11].

This solution defined for all t in the interval

$$|t - t_0| < \alpha$$

If $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous for all (t,y) in that rectangular R and bounded say,

$$|f| \le k, \left|\frac{\partial f}{\partial y}\right| < M \tag{4}$$

for all (t,y) in R initial value problem (3) has one solution $y(t)$.

This solution is defined at least for all t in that $|t - t_0| < \alpha$. It can be obtained by Picard's iteration method that is the sequence

$$y_0, y_1, y_2, \ldots, y_n, \ldots \tag{5}$$

where

$$y_n(t) = y_0 + \int_{t_0}^{t} f(t, y_{n-1}(t))dt \tag{6}$$

Equation (6) converges to that solution $y(t)$ [12].

If $f(t)$ is piecewise regular and of exponent order

$$|f(t)| < Me^{\alpha t} \tag{7}$$

with abscissa which converges to α_0, then for any number $s_0 > \alpha_0$,

$$L(f(t)) = \int_0^{\infty} f(t)e^{-st}dt \tag{8}$$

converges uniformly for all values of s such that $s \ge s_0$ where $L(f(t))$ is called the Laplace transformation of $f(t)$ and is denoted by $F(s)$. Equivalently $f(t)$ is called the inverse Laplace transformation of $F(s)$ and is denoted by [13], [14].

$$f(t) = L^{-1}(F(s)) \tag{9}$$

where

$$f(t) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} F(s)e^{st}ds \tag{10}$$

II. New Main Result

All previous studies used Laplace transformations for finding the particular (not general) solution of following n-the order linear non homogenous differential equation with constant coefficients with initial conditions (initial value problem):

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = f(t) \tag{11}$$

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$$y_0^{(m-1)}=b_{m-1}, m=1,2,\cdots,n-1 \tag{12}$$

In this section an expanding of the previous usage is introduced to involve the studying of finding the general solution (not particular) of (11) without initial conditions (12) using the method of Laplace transformations. The homogenous form of (11) is represented as

$$a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=0 \tag{13}$$

then the characteristic equation of (13) is

$$g_n(s)=a_0s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n \tag{14}$$

By taking the Laplace transformations of left hand side of (11), one can has

$$\begin{aligned}
&L(a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny)=\\
&a_0L(y^{(n)})+a_1L(y^{(n-1)})+\cdots+a_{n-1}L(y')+a_nL(y)=\\
&a_0s^nL(y)-a_0s^{n-1}y_0-a_0s^{n-2}y_0'-\cdots-a_0y_0^{(n-1)}+\\
&a_1s^{n-1}L(y)-a_1s^{n-2}y_0-a_1s^{n-3}y_0'-\cdots-a_1y_0^{(n-2)}+\\
&\vdots\\
&a_{n-3}s^3L(y)-a_{n-3}s^2y_0-a_{n-3}sy_0'-a_{n-3}y_0^{(2)}+\\
&a_{n-2}s^2L(y)-a_{n-2}sy_0-a_{n-2}y_0'+\\
&a_{n-1}sL(y)-a_{n-1}y_0'+a_nL(y)
\end{aligned}$$

It implies that:

$$L(a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny)=g_n(s)Y(s)+r_{n-1}(s) \tag{15}$$

where $r_{n-1}(s)$ is a polynomial of degree n-1.

One can write the Laplace transformation of right hand side of (15) as

$$L(f(t))=\frac{H(s)}{p_m(s)} \tag{16}$$

where $H(s)$ is a function of s and, $p_m(s)$ is a polynomial of degree m, then from (15) and (16), one can have

$$Y(s)=\frac{H(s)-p_m(s)r_{n-1}(s)}{g_n(s)p_m(s)} \tag{17}$$

$$Y(s)=\frac{H(s)}{g_n(s)p_m(s)}-\frac{r_{n-1}(s)}{g_n(s)} \tag{18}$$

then by using inverse Laplace transformation, y(t) becomes

$$y(t)=L^{-1}(-\frac{r_{n-1}(s)}{g_n(s)})+L^{-1}(\frac{H(s)}{g_n(s)p_m(s)}) \tag{19}$$

where the first part represents the homogenous solution $y_h(t)$ and the second part represents the particular solution R (t), and so it agreement with (18).

Remark 1: In [15], $f(t)$ is solved as exponent and trigonometric functions. In this work $f(t)$ expanded to general functions.

Case 1: If $K(s)=H(s)-p_m(s)r_{n-1}(s)$ and $H(s)$ is a polynomial equation, then $K(s)$ is a polynomial equation such that

$$Y(s)=\frac{K_v(s)}{g_n(s)p_m(s)}, v<n+m \tag{20}$$

In this case one can use the method of partial frictions in order to solve inverse Laplace of (20) and then find $y(t)$.

Case 2: If $H(s)$ not a polynomial, then one can write

$$Y(s)=\frac{H(s)}{g_n(s)p_m(s)}-\frac{r_{n-1}(s)}{g_n(s)} \tag{21}$$

For solving (21), one can use the convolution method of first part. For solving second part, the partial frictions method can use.

We are in a position to introduce the following proposition:

Proposition 1: The general solution $y(t)$ of linear equation represent the inverse Laplace transforms where

$$y(t)=L^{-1}(\frac{H(s)-p_m(s)r_{n-1}(s)}{g_n(s)p_m(s)}) \tag{22}$$

The studying of general solution of homogenous linear equation (22) was introduced as follows:

By the same manner of proof of Proposition (1), one can have

$$Y(s)=\frac{H(s)-p_m(s)r_{n-1}(s)}{g_n(s)p_m(s)} \tag{23}$$

Since $f(t)=0$ then $h(s)=0$. So (23) becomes

$$Y(s)=\frac{Q_{n-1}(s)}{g_n(s)} \tag{24}$$

Therefore one can use the method of partial frictions in order to solve inverse Laplace of (24) and then find $y(t)$.

Case 3: If all the roots of characteristic equation $g_n(s)$ are distinct, then

$$Y(s)=\frac{A_1}{s-t_1}-\frac{A_2}{s-t_2}-\cdots-\frac{A_n}{s-t_n} \tag{25}$$

Therefore by using the inverse of (25), one can have

$$y(t) = A_1 e^{t_1 t} + A_2 e^{t_2 t} + \cdots + A_n e^{t_n t} \tag{26}$$

which represents the general solution.

Case 4: If $t_1 = t_2$, then

$$Y(s) = \frac{Q_{n-1}(s)}{(s-t_1)^2 (s-t_3)\cdots(s-t_n)} \tag{27}$$

$$Y(s) = \frac{A_1}{s-t_1} + \frac{A_2}{(s-t_1)^2} + \frac{A_3}{s-t_3} + \cdots + \frac{A_n}{s-t_n} \tag{28}$$

Therefore by using the inverse of (28), one can have

$$y(t) = A_1 e^{t_1 t} + A_2 t e^{t_2 t} + \cdots + A_n e^{t_n t} \tag{29}$$

which represents the general solution.

Case 5 If $t_1 = t_2 = \cdots = t_r \neq t_{r+1} \neq \cdots \neq t_n$ then

$$Y(s) = \frac{Q_{n-1}(s)}{(s-t_1)^r (s-t_{r+1})\cdots(s-t_n)} \tag{30}$$

therefore by using the inverse of (30), one can have

$$y(t) = A_1 e^{t_1 t} + A_2 t e^{t_1 t} + \cdots A_r t^r e^{t_1 t} + A_{r+1} e^{t_{r+1} t} + \cdots + A_n e^{t_n t} \tag{31}$$

which represents the general solution.

Now we are in a position to introduce the following proposition:

Proposition 2: The general solution $y(t)$ of homogenous linear equation

Equation (31) represents the inverse Laplace transforms where

$$y(t) = L^{-1}\left(\frac{Q_{n-1}(s)}{g_n(s)}\right) \tag{32}$$

III. Examples

For clear above propositions, one can present the following examples:

Example 1: (Linear Differential Equation): For solving the differential equation

$$y' + y = \cos(t)$$

then by using Laplace transformations of given equation, one can have

$$g_1(s) = s+1,\ p_2(s) = s^2+1,\ H(s) = s$$

So $Y(s)$ (according with case (1)) becomes

$$Y(s) = \frac{K_1(s)}{g_1(s)\, p_2(s)}$$

$$Y(s) = \frac{K_1(s)}{(s+1)(s^2+1)}$$

Then by using partial frictions

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2 s + A_3}{s^2+1}$$

$$Y(s) = A_1 \frac{1}{s+1} + A_2 \frac{s}{s^2+1} + A_3 \frac{1}{s^2+1}$$

finally by using inverse Laplace transformations, the general solution has the form

$$y(t) = A_1 e^{-t} + A_2 \cos(t) + A_3 \sin(t)$$

From theory of solving differential equation it is known that the number of constants in general solution must be equal to the order of given differential equation. Two constants are calculated. By derivative, one can have

$$y'(t) = -A_1 e^{-t} - A_2 \sin(t) + A_3 \cos(t)$$

So

$$(A_3 - A_2)\sin(t) + (A_3 + A_2)\cos(t) = \cos(t)$$

and

$$A_2 = A_3 = \frac{1}{2} \cdot$$

Then the general solution is

$$y(t) = A_1 e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)$$

Example 2: Nonlinear Differential Equations: For solving the differential equation

$$y' + y = u_a(t)$$

Then by using Laplace transformations, one can have

$$g_1(s) = s+1,\ p_1(s) = s,\ H(s) = e^{-as}$$

So $Y(s)$ (according with case (2)) becomes

$$Y(s) = \frac{h(s)}{g_1(s)\, p_1(s)} - \frac{r_0(s)}{g_1(s)}$$

$$Y(s) = \frac{e^{-as}}{(s+1)(s)} - \frac{r_0(s)}{(s+1)}$$

Then for using convolution theorem of first part and partial frictions of its second part, $Y(s)$ has the form:

$$Y(s) = (\frac{e^{-as}}{s})(\frac{1}{S+1}) - \frac{A}{(s+1)}$$

By using inverse Laplace transforms, the general solution is

$$y(t) = \int_0^t u_a e^{-\lambda} d\lambda - Ae^{-t}$$

$$y(t) = u_a(1 - e^{-t}) - Ae^{-t}$$

Example 3: Homogenous Differential Equation: For solving the homogenous differential equation

$$y'' + y = 0$$

Then by using Laplace transformations, one can have

$$g_2(s) = s^2 + 1$$

So $Y(s)$ (according with case (3)) becomes

$$Y(s) = \frac{Q_1(s)}{g_2(s)}$$

$$Y(s) = \frac{Q_1(s)}{(s^2-1)}$$

$$Y(s) = \frac{Q_1(s)}{(s-1)(s+1)} = \frac{A_1}{S-1} + \frac{A_2}{S+1}$$

By using inverse Laplace transforms, the general solution is

$$y(t) = A_1 e^t + A_2 e^{-t}$$

Example 4: Partial Differential Equation: For solving the linear partial differential equation

$$u_{tt} - u_{xx} - 2u_t = e^x \sin(t)$$

One can solve it with respect to variable t, then

$$g(s) = s^2 - 2s \quad , \quad p(s) = s^2 + 1 \quad , H(s,x) = e^x$$

Then the solution is

$$v(x,t) = L^{-1}(\frac{-r(s)}{s^2-2s}) + L^{-1}(\frac{e^x}{(s^2-2s)(s^2+1)})$$

$$v(x,t) = A + Be^{2t} + e^x L^{-1}(\frac{Cs+D}{s^2+1} + \frac{E}{s-2} + \frac{F}{s})$$

By calculating v_t, v_{tt}, v_{xx}, the resulting is

$$c = \frac{1}{4} \quad , D = -\frac{1}{4} \quad , E = 0 \quad , F = 0$$

So the general solution has the form

$$v(x,t) = A + Be^{2t} + \frac{1}{4}e^x \cos(t) - \frac{1}{4}e^x \sin(t)$$

IV. Conclusion

In summary, we have shown here that it is possible to solve linear differential equation via Laplace transformations without needed any initial condition. This solution involves the general and the equation is in famous or general design (not in initial value problem). Our proposed method shares many common features with recent theoretical studies of solutions. All our finding results are in good agreement with the recent solved methods.

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