

Laplace Adomian Decomposition Method Applied to a Two-Dimensional Viscous Flow with Shrinking Sheet

M. A. Koroma, S. Widatalla, A. F. Kamara, and C. Zhang

Abstract—Our aim in this piece of work is to demonstrate the power of the Laplace Adomian decomposition method (LADM) in approximating the solutions of nonlinear differential equations governing the two-dimensional viscous flow induced by a shrinking sheet.

Keywords—Adomian polynomials, Laplace Adomian decomposition method, Padé Approximant, Shrinking sheet.

I. INTRODUCTION

THE flow induced by a shrinking sheet is such that the velocity at the boundary is towards a fixed point. This flow, about which very little is known, has been the subject of investigation by researchers in recent times. C. Y. Wang discussed the Stagnation flow towards a shrinking sheet [1]. M. Miklavcic and C. Y. Wang investigated the properties of the flow due to a shrinking sheet with suction [2]. Noor and Ishak investigated MHD flow and heat transfer adjacent to permeable shrinking sheet embedded in a porous medium [3]. Naeem Faraz, Yasir Khan and Ahmet Yildirim applied the variational iteration algorithm in finding an analytic solution to a two-dimensional viscous flow with shrinking sheet [4]. In this work, we shall apply the LADM to the shrinking sheet problem investigated by Naeem Faraz, Yasir Khan and Ahmet Yildirim and compare results to demonstrate its effectiveness [4].

The LADM has been successfully applied by researchers to find reliable approximate solutions to nonlinear partial differential equations, see [5]-[6]-[7]-[8].

II. EQUATIONS GOVERNING THE MOTION

We consider the continuity equation and the Navier- Stokes equation for a compressible steady state fluid flow

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$$\begin{aligned} \nabla \cdot \mathbf{q} &= 0 \\ \mathbf{q} \cdot \nabla \mathbf{q} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{q} \end{aligned} \quad (1)$$

with Cartesian forms given by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (3)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \quad (4)$$

$$u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \quad (5)$$

u , v and w are the components of velocity in the x , y and z directions respectively, p is the pressure, ρ is the density and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.

The boundary conditions for the flow are:

$$\begin{aligned} u &= -ax, v = -a(m-1)y, w = -W, y = 0 \\ u &\rightarrow \infty \text{ as } y \rightarrow \infty \end{aligned} \quad (6)$$

a is a shrinking constant and W is the velocity of suction.

Consider the similarity transformations

$$u = axf'(\eta), v = -a(m-1)yf'(\eta), \eta = \sqrt{\frac{a}{\nu}}z \quad (7)$$

Substituting these expressions in (2) and integrating gives

$$w = -am\sqrt{va}f(\eta) \tag{8}$$

Substituting (7) and (8) in (5) and integrating gives

$$\frac{p}{\rho} = -\frac{w^2}{2} + v\frac{\partial w}{\partial z} + C, \tag{9}$$

where C is a constant of integration.

From (9) it is clear that p is a function of z only and so

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \tag{10}$$

Now (3) yields

$$axf'(\eta)(af'(\eta)) - m\sqrt{av}f(\eta)\left(ax\sqrt{\frac{a}{v}}f''(\eta)\right) = v\left(axf'''(\eta)\times\left(\frac{a}{v}\right)\right).$$

So

$$f'''(\eta) = (f'(\eta))^2 - mf(\eta)f''(\eta) \tag{11}$$

Substituting in (4) gives the same result as in (11).

By considering our similarity transformations in (7) and the boundary conditions in (6) we obtain

$$\begin{aligned} f'(0) &= -1, f(0) = W/(m\sqrt{av}) = k \\ f'(\infty) &= 0 \end{aligned} \tag{12}$$

Thus, our problem is to solve (11) together with the boundary conditions in (12).

III. SOLUTION PROCEDURE

In this work, we shall apply the LADM to the axisymmetrically shrinking sheet problem with $a > 0, m = 2, k = 2$ and compare our results with those obtained by Naeem Faraz and others.

We illustrate the LADM by considering the nonlinear differential equation

$$f^n(\eta) = P(\eta) - mQ(\eta) \tag{13}$$

where $f^n(\eta)$ is the n th derivative of f with respect to η and $P(\eta)$ and $Q(\eta)$ are nonlinear terms.

Taking the Laplace transform of both sides of (13) gives

$$s^n L[f(\eta)] - \sum_{i=0}^{n-1} s^{n-1-i} f^i(0) = L[P(\eta) - mQ(\eta)] \tag{14}$$

where L denotes the Laplace transform, $f^i(0)$ is the i th derivative of f at $\eta = 0$ and $f^0(0) = f(0)$.

Equation (14) simplifies to:

$$L[f(\eta)] = \sum_{i=0}^{n-1} s^{-1-i} f^i(0) + s^{-n} L[P(\eta) - mQ(\eta)].$$

If we put $H(s) = \sum_{i=0}^{n-1} s^{-1-i} f^i(0)$ and take the inverse

Laplace transform of both sides of the above expression we obtain

$$f(\eta) = L^{-1}H(s) + L^{-1}s^{-n}L[P(\eta) - mQ(\eta)]. \tag{15}$$

The LADM assumes the expansion of $f(\eta)$ as a series of

$$\text{the form } f(\eta) = \sum_{n=0}^{\infty} f_n(\eta). \tag{16}$$

The nonlinear terms are decomposed as

$$P(\eta) = \sum_{n=0}^{\infty} A_n \text{ and } Q(\eta) = \sum_{n=0}^{\infty} B_n \text{ [8].} \tag{17}$$

Substituting (17) in (15) gives

$$\sum_{n=0}^{\infty} f_n(\eta) = H_0(\eta) + L^{-1}s^{-n}L\left[\sum_{n=0}^{\infty} (A_n - mB_n)\right], \tag{18}$$

where $H_0(\eta) = L^{-1}s^{-n}H(s)$.

Consider a first approximation to $f(\eta)$ in (18) of the form

$$f_0(\eta) = H_0(\eta). \tag{19}$$

Higher iterates of the series solution can be obtained from the recurrence relation

$$f_{n+1}(\eta) = L^{-1}s^{-n}L(A_n - mB_n), n \geq 0 \tag{20}$$

where A_n and B_n are the Adomian polynomials.

Now putting $m = k = 2$ in (11) and (12) gives

$$f'''(\eta) = (f'(\eta))^2 - 2f(\eta)f''(\eta), \tag{21}$$

together with the boundary conditions

$$f'(0) = -1, f(0) = 2 \tag{22}$$

$$f'(\infty) = 0$$

$$\text{Let } \alpha = f''(0) \tag{23}$$

By using the result in (14) we have

$$s^3 L[f(\eta)] - (2s^2 - s + \alpha) = L[(f'(\eta))^2 - 2f(\eta)f''(\eta)]$$

which by (18) gives

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta) = 2 - \eta + \frac{\alpha}{2} \eta^2 + L^{-1} s^{-3} L[A_n - 2B_n] \tag{24}$$

where the nonlinear terms are given by

$$(f'(\eta))^2 = \sum_{n=0}^{\infty} A_n \text{ and } f(\eta)f''(\eta) = \sum_{n=0}^{\infty} B_n \tag{25}$$

Thus our first approximation is given by

$$H_0 = f_0 = 2 - \eta + \frac{\alpha}{2} \eta^2 \tag{26}$$

and from (20) we have the recurrence relation

$$f_{n+1} = L^{-1} s^{-3} L[A_n - 2B_n], n \geq 0 \tag{27}$$

The Adomian polynomials are given by

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\left(\sum_{i=0}^n \lambda^i f'_i \right)^2 \right]_{\lambda=0},$$

$$B_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[\left(\sum_{i=0}^n \lambda^i f_i \right) \left(\sum_{i=0}^n \lambda^i f''_i \right) \right]_{\lambda=0} \tag{28}$$

Thus

$$A_0 = (f'(0))^2$$

$$A_1 = 2f'_0 f'_1$$

$$A_2 = 2f'_0 f'_2 + (f'_1)^2$$

$$A_3 = 2f'_0 f'_3 + 2f'_1 f'_2$$

.....

$$\tag{29}$$

and

$$B_0 = f_0 f''_0$$

$$B_1 = f_0 f''_1 + f_1 f''_0$$

$$B_2 = f_0 f''_2 + f_1 f''_1 + f_2 f''_0$$

$$B_3 = f_0 f''_3 + f_1 f''_2 + f_2 f''_1 + f_3 f''_0$$

.....

$$\tag{30}$$

From (26), (27), (29) and (30) we obtain

$$f_1 = \frac{(1-4\alpha)}{6} \eta^3 \tag{31}$$

Similarly, we obtain:

$$f_2 = \frac{(-1+4\alpha)}{6} \eta^4 + \frac{(1-4\alpha)}{60} \eta^5 + \frac{(-\alpha+4\alpha^2)}{360} \eta^6; \tag{32}$$

$$f_3 = \frac{(2-8\alpha)}{15} \eta^5 + \frac{(-1+4\alpha)}{30} \eta^6 + \frac{(1-16\alpha^2)}{504} \eta^7$$

$$+ \frac{(-\alpha+4\alpha^2)}{1008} \eta^8 + \frac{(\alpha^2-\alpha^3)}{90720} \eta^9 \tag{33}$$

$$f_4 = \frac{4(-1+4\alpha)}{45} \eta^6 - \frac{4(-1+4\alpha)}{105} \eta^7 +$$

$$\frac{(-1+4\alpha)(3+8\alpha)}{630} \eta^8 - \frac{(-1+4\alpha)(7+152\alpha)}{45360} \eta^9$$

$$+ \frac{\alpha(-1+4\alpha)(87+19\alpha)}{453600} \eta^{10} - \frac{31\alpha^2(-1+4\alpha)}{712800} \eta^{11}$$

$$- \frac{\alpha^3(-1+4\alpha)}{2138400} \eta^{12} \tag{34}$$

Now from $f(\eta) = f_0 + f_1 + f_2 + f_3 + f_4 + \dots$ we obtain the series;

$$f(\eta) = 2 - \eta + \frac{\alpha}{2} \eta^2 + \frac{(1-4\alpha)}{6} \eta^3 + \frac{(-1+4\alpha)}{6} \eta^4 +$$

$$\frac{31(1-4\alpha)}{20} \eta^5 + \frac{(-1+4\alpha)(44+\alpha)}{360} \eta^6 +$$

$$\begin{aligned} & \frac{(-1+4\alpha)(102+20\alpha)}{2520}\eta^7 + \frac{(-1+4\alpha)(8+23\alpha)}{1680}\eta^8 \\ & + \frac{(-1+4\alpha)(7+123\alpha-604\alpha^2)}{90720}\eta^9 \\ & + \frac{(-1+4\alpha)(87+194\alpha)}{453600}\eta^{10} - \frac{31\alpha^2(-1+4\alpha)}{712800}\eta^{11} + \\ & - \frac{\alpha^3(-1+4\alpha)}{2138400}\eta^{12} + \dots \end{aligned} \tag{35}$$

IV. DETERMINATION OF THE FREE PARAMETER α

It is worth noting here that the infinity condition cannot be applied directly on $f'(\eta)$, the derivative of the series expansion of $f(\eta)$ given in (35), since it appears to be divergent. To overcome this difficulty and improve on the convergence of $f'(\eta)$, we determine the [M, M] Padé approximants of $f'(\eta)$ which are the most suitable for expressing series expansions as rational functions. Padé approximants have proven very useful in the manipulation of high order series expansions of functions. We start by differentiating f , then determine the [M, M] Padé approximants of the resulting series and finally apply the boundary condition $f'(\infty) = 0$ by equating the coefficient of the highest power of η in the numerator to zero. Solving the resulting polynomial for α in Maple 13 gives the average value of the free parameter $\alpha = f''(0)$.

TABLE I
COMPARISON OF VALUES OF $\alpha = f''(0)$

| Padé approximant | VIM | ADM | LADM |
|------------------|----------------|----------------|----------------|
| [1,1] | 0.224748 | 0.292893 | 0.292893 |
| [2,2] | Complex number | Complex number | Complex number |
| [3,3] | 0.294748 | Complex number | Complex number |
| [4,4] | 0.308086 | 0.247723 | 0.248126 |
| [5,5] | 0.249556 | 0.24893 | 0.249279 |

where VIM means variational iteration method, ADM means Adomian decomposition method and LADA means Laplace Adomian decomposition method.

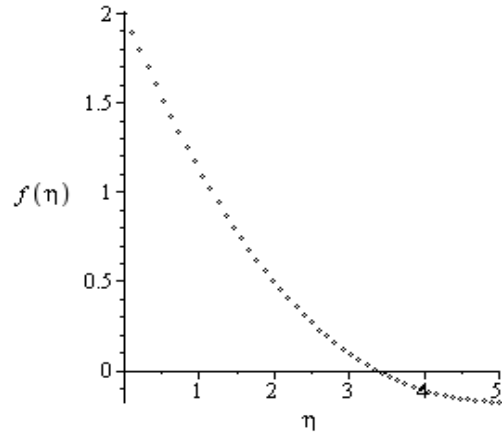


Fig. 1 Graphical solution of $f(\eta)$ by LADM
 $k=2, m=2, \alpha = 0.249279$

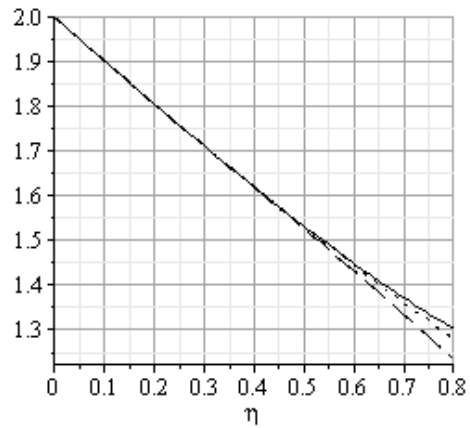


Fig. 2 Comparison of solutions of $f(\eta)$
ADA LDA VIM

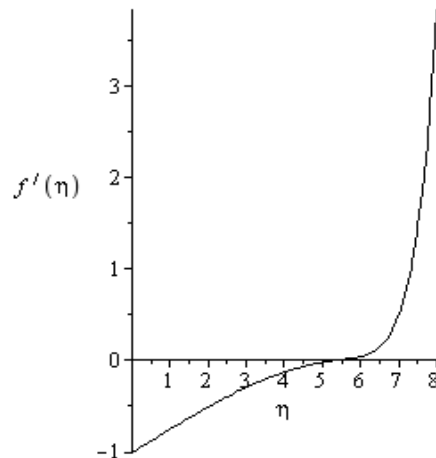


Fig. 3 Graphical solution of $f'(\eta)$ by LADM
 $k=2, m=2, \alpha = 0.249279$

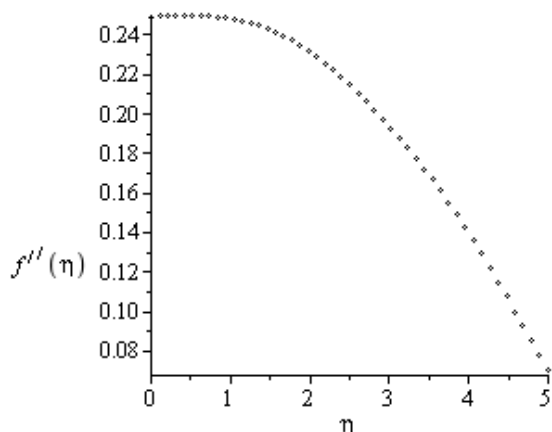


Fig. 4 Graphical solution of $f''(\eta)$ by LADM
 $k=2, m=2, \alpha = 0.249279$

V. DISCUSSION OF RESULTS AND CONCLUSION

In this work we determined the numerical value of the free parameter $f''(0) = \alpha$ by the use of Padé approximants in Maple 13.

Table I gives a comparison of values of α obtained by our method (LADM), Adomian decomposition method (ADM) and Variational iteration method (VIM) [4]. Our results, as shown in Table I, are in close agreement with those obtained by other methods. Fig. 1 presents a graphical solution of $f(\eta)$, by our method, for the case of an axisymmetrically shrinking plate ($k = 2, m = 2$), using our average value of α obtained by taking the [5, 5] padé approximant of $f'(\eta)$. In fig. 2 we compared graphical solutions of our method with other methods. The graph indicates that our solution is in close agreement with those of ADM and VIM. Fig. 3 and Fig. 4 present graphical solutions of $f'(\eta)$ and $f''(\eta)$.

From (7) it can be observed that an understanding of the behaviour of $f'(\eta)$ is key towards determining the velocity components u and v . Equations (8) and (9) also outline the importance of an understanding of the behaviour of $f(\eta)$ in determining the velocity component w and the flow pressure p . Thus, any method that leads to a solution of $f(\eta)$ and its derivatives is useful in finding a solution of the problem that is governed by (2) to (5).

The advantages of the LADM are:

- it does not require discretization, linearization, perturbation or restrictive assumptions that may affect the solution;
- the iterates are easily calculated and
- It approximates the function $f(\eta)$ after a few iterates (as demonstrated in (35)).

It is thus clear that the LADM is a very powerful tool in finding approximate solutions to nonlinear differential equations encountered in problems of Fluid Dynamics.

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