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Lagrangian geometrical model of the rheonomic mechanical systems

Camelia Frigioiu, Katica (Stevanovic) Hedrih, and Iulian Gabriel Bîrsan

Abstract—In this paper we study the rheonomic mechanical systems from the point of view of Lagrange geometry, by means of its canonical semispray. We present an example of the constraint motion of a material point, in the rheonomic case.

Keywords—Lagrange's equations, mechanical system, non-linear connection, rheonomic Lagrange space.

I. Introduction

THE main purpose of the present paper is to study the rheonomic Lagrangian mechanical systems. The geometric study of the sclerhonomic mechanical systems given by Lagrange equations with the external forces a priori given has been investigated in many papers, as [4], [6], [8], [10]. The works [2], [3], [9], [11], [13] extend the geometric investigation of nonconservative sclerhonomic mechanical systems, using the associated evolution non-linear connection.

In this paper, we study the dynamical system of the rheonomic Lagrangian mechanical systems, whose evolution curves are given, on the phase space $TM \times R$, by Lagrange equations. Then one can associate to the considered mechanical system a vector field S on the phase space, which is named the canonical semispray. The integral curves of the canonical semispray are the evolution curves of the rheonomic mechanical system.

The article is organized as follows. In the next section we briefly recall some basic notions on rheonomic Lagrange geometry. In the third section we employ a method similar to that used in the geometrization of sclerhonomic Lagrange mechanical systems, [11], and we obtain a non-linear connection for the rheonomic Lagrangian system with external forces. The geometry of the semispray will determine the geometry of the associated dynamical system on the phase space. We obtain the canonical non-linear connection and the metrical connection which depends by the external forces of the mechanical system.

In the last section, we apply these results to a concrete rheonomic mechanical system: the constrained motion of a material particle on a time varying surface. The kinetic potential as a difference of the system's kinetic energy and its potential energy express the Lagrange function introduced in the geometrical approach of the dynamical system. It is visible that for the considered rheonomic mechanical system is easy to apply the previous theory for obtaining geometrical descriptions of this mechanical system.

Camelia Frigioiu is with the Department of Mathematics, Faculty of Sciences, "Dunărea de Jos" University of Galaţi, 111 Domneasca, 800201 Galaţi, România, (corresponding author; e-mail: cfrigioiu@ugal.ro)

Katica (Stevanovic) Hedrih is with Faculty of Mechanical Engineering, University of Nis, Serbia, (e-mail:katicah@mi.sanu.ac.yu)

Iulian Gabriel Bîrsan is with the Faculty of Mechanical Engineering, "Dunărea de Jos" University of Galaţi, 111 Domneasca, 800201 Galaţi, România, (e-mail:igbirsan@ugal.ro)

II. Preliminaries

We start with a short review of the basic used notions and concepts of the Lagrange geometry and their terminology. For more, see [12].

Let M be a smooth C^∞ manifold of finite dimension n, and (TM, π, M) be its tangent bundle. We consider the manifold $TM \times R$ and we shall use the differentiable structure on $TM \times R$ as the product of the manifold TM and R.

In this paper the indices $i, j, k, \dots$ run over the set $\{1, 2, \dots, n\}$.

The manifold

$$E = TM \times R$$

is a $(2n+1)$dimensional, real manifold and the local coordinates in a chart will be denoted by (x^i, y^i, t).

The natural basis of tangent space T_uE at the point $u \in U \times (a, b)$ is given by

$$\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right).$$

$\chi(E)$ is the $C^\infty(E)$-module of (smooth) vector fields defined on E.

On the manifold E a vertical distribution V is introduced, generated by $n+1$ local vector fields $\left(\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \dots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial t}\right)$,

$$V : u \in E \to V_u \subset T_uE \tag{1}$$

as well as the tangent structure, [1],

$$J : \chi(E) \to \chi(E),$$

given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}; J\left(\frac{\partial}{\partial y^i}\right) = 0; J\left(\frac{\partial}{\partial t}\right) = 0, \tag{2}$$

for $i, j, k = 1, 2, ..., n$.

The tangent structure J is globally defined on E and it is an integrable structure.

A semispray on E, [12], is a vector field $S \in \chi(E)$ which has the property

$$JS = C, \tag{3}$$

where $C = y^i \frac{\partial}{\partial y^i}$ is the Liouville vector field.

Locally, a semispray S has the form

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y,t)\frac{\partial}{\partial y^i} - G^0(x,y,t)\frac{\partial}{\partial t}, \quad (4)$$

where $G^i(x,y,t)$ and $G^0(x,y,t)$ are the coefficients of S.

The integrals curves of the semispray S are the solutions of the following system of differential equations

$$\frac{dx^i}{d\tau} = y^i(\tau); \frac{dy^i}{d\tau} + 2G^i(x(\tau), y(\tau), t(\tau)) = 0;$$
$$\frac{dt}{d\tau} + G^0(x(\tau), y(\tau), t(\tau)) = 0. \quad (5)$$

A *non-linear connection* on E is a smooth distribution:

$$N : u \in E \to N_u \subset T_u E, \quad (6)$$

which is supplementary to the vertical distribution V:

$$T_u E = N_u \oplus V_u, \quad \forall u = (x,y,t) \in E. \quad (7)$$

In the following we set $t = y^0$ and we introduce the Greek indices $\alpha, \beta, \dots$ ranging on the set $\{0,1,2,...,n\}$.

The local basis adapted to the (7) decomposition is

$$\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^\alpha}\right), \quad (8)$$

where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^\alpha(x,y,t)\frac{\partial}{\partial y^\alpha} \quad (9)$$

and $(N_i^\alpha(x,y,t))$ are the local coefficients of the non-linear connection N on E.

The dual basis of (8) is $(\delta x^i, \delta y^\alpha)$, with

$$\delta x^i = dx^i; \delta y^i = dy^i + N_j^i dx^j; \quad (10)$$
$$\delta y^0 = \delta t = dt + N_i^0 dx^i.$$

A differentiable rheonomic Lagrangian is a scalar function

$$L : TM \times R \to R$$

of the class C^∞ on the manifold $\tilde{E} = E \setminus \{(x,0,0), x \in M\}$ and continuous for all the points $(x,0,0) \in TM \times R$.

The $d-$tensor field with the components

$$g_{ij}(x,y,t) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} \quad (11)$$

is of type $(0,2)$ and symmetric.

It is called *the fundamental* or *the metric tensor field* of the rheonomic Lagrangian $L(x,y,t)$.

The rheonomic Lagrangian $L(x,y,t)$ is called *regular* if

$$rank(g_{ij}) = n, \text{on } \tilde{E}.$$

A rheonomic Lagrange space is a pair $RL^n = (M, L(x,y,t))$, where L is a regular rheonomic Lagrangian and its fundamental tensor g_{ij} has constant signature on $\tilde{E}$.

For a rheonomic Lagrange space $RL^n = (M,L)$ exists a non-linear connection N defined on $\tilde{E}$, whose coefficients (N_j^α) are completely determined by L, called *the canonical non-linear connection*, [12]. Its coefficients are as follows

$$N_j^i = \frac{1}{4}\frac{\partial}{\partial y^j}\left[g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^k}y^k - \frac{\partial L}{\partial x^h}\right)\right]; \quad (12)$$

$$N_j^0 = \frac{1}{2}\frac{\partial^2 L}{\partial t \partial y^j}.$$

The almost complex structure is a $\mathcal{F}(\tilde{E})-$linear mapping

$$\mathbb{F} : \chi(E) \to \chi(E),$$

given by

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}; \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}; \mathbb{F}\left(\frac{\partial}{\partial t}\right) = 0. \quad (13)$$

A $d-$connection is a linear connection $D_X, X \in \chi(E)$ (in Koszul's sense) on $E = TM \times R$ which maps horizontal vector fields onto horizontals ones and vertical vector fields onto verticals ones.

An $N-$linear connection on $\tilde{E}$ is a $d-$connection D on $\tilde{E}$, such that

$$(D_X \mathbb{F})(Y) := D_X \mathbb{F} Y - \mathbb{F}(D_X Y) = 0, \forall X, Y \in \chi(\tilde{E}). \quad (14)$$

With respect to the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^\alpha}\right)$, the $N-$linear connection has the coefficients

$$D\Gamma = \left(L^i_{jh}(x,y,t), C^i_{j\alpha}(x,y,t)\right),$$

where the functions $\left(L^i_{jh}(x,y,t)\right)$ under a coordinate transformation on E behave as the local coefficients of a linear connection on the manifold M and the functions $\left(C^i_{j\alpha}(x,y,t)\right)$ define $d-$tensor fields.

The $h-$ and $v-$covariant derivatives of a $d-$vector field $\left(X^i\right)$, with respect to the $N-$linear connection $D\Gamma$ are given by:

$$X^i_{|k} = \frac{\delta X^i}{\delta x^k} + X^h L^i_{hk},$$

respectively

$$X^i \mid_\alpha = \frac{\partial X^i}{\partial y^\alpha} + X^h C^i_{h\alpha}, \quad (y^0 = t).$$

An $N-$linear connection D is called *the metrical $N-$ linear connection* for a rheonomic Lagrange space if

$$g_{ij|k} = 0; \quad g_{ij} \mid_\alpha = 0, \quad (15)$$

where ${}_{|k}$ and $|_\alpha$ are $h-$ and v_α-derivations, respectively.

The metrical N-linear connection $C\Gamma = \left(L^i_{jk}, C^i_{j\alpha}\right)$, with the coefficients:

$$L^i_{jk} = \frac{1}{2}g^{ih}\left(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{jh}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^h}\right);$$
$$C^i_{jk} = \frac{1}{2}g^{ih}\left(\frac{\partial g_{hk}}{\partial y^j} + \frac{\partial g_{jh}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^h}\right); \quad (16)$$
$$C^i_{j0} = \frac{1}{2}g^{ih}\frac{\partial g_{jh}}{\partial t}$$

is called *the canonical metrical $N-$ linear connection*.

This connection depends only on the fundamental function L of the Lagrange space.

III. Rheonomic Lagrangian Mechanical Systems

A rheonomic Lagrangian mechanical system is the triplet

$$\Sigma = (M, L(x,y,t), F(x,y,t)), \tag{17}$$

where $RL^n = (M, L(x,y,t))$ is a rheonomic Lagrange space and $F(x,y,t)$ is a vertical vector field:

$$F(x,y,t) = F^i(x,y,t)\frac{\partial}{\partial y^i}. \tag{18}$$

The tensor $g_{ij}(x,y,t)$ of the rheonomic Lagrange space RL^n is the fundamental tensor of the mechanical system Σ.

Using the variational problem of the integral action of $L(x,y,t)$, we introduce the evolution equations of Σ by:

The evolution equations of the rheonomic Lagrangian mechanical system Σ are the following Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial y^i}\right) - \frac{\partial L}{\partial x^i} = F_i(x,y,t); \quad y^i = \frac{dx^i}{dt}, \tag{19}$$

where $F_i(x,y,t) = g_{ij}(x,y,t)F^j(x,y,t)$.

Proposition 3.1: The Lagrange equations (19) are equivalent to the equations

$$\frac{d^2x^i}{dt^2} + 2\Gamma^i(x,y,t) = \frac{1}{2}F^i(x,y,t), \tag{20}$$

where

$$2\Gamma^i = 2G^i(x,y,t) + N_0^i(x,y,t),$$

$$2G^i = \frac{1}{2}g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^s}y^s - \frac{\partial L}{\partial x^h}\right), \tag{21}$$

and $N_0^i(x,y,t) = \dfrac{1}{2}g^{ih}\dfrac{\partial^2 L}{\partial t \partial y^h}$.

The equations (20) are called *the evolution equations* of the mechanical system Σ. The solutions of these equations are called *evolution curves* of the mechanical system Σ.

Theorem 3.1: a)The vector field $\breve{S}$ given by:

$$\breve{S} = y^i\frac{\partial}{\partial x^i} - 2\breve{\Gamma}^i(x,y,t)\frac{\partial}{\partial y^i} + a\frac{\partial}{\partial t} \tag{22}$$

is a semispray on $T\tilde{M} \times R$.

b) The semispray $\breve{S}$ is a dynamical system on $T\tilde{M} \times R$ depending only on the rheonomic Lagrangian mechanical system Σ.

c) The integral curves of $\breve{S}$ are the evolution curves of Σ given by (20).

Proof:

a) As G^i and N_0^i are the local coefficients of the canonical semispray of the rheonomic Lagrange space RL^n and F^i are the components of a $d-$vector field it follows that $G^i - \dfrac{1}{2}F^i$ and N_0^i are also a local coefficients of a local semispray.

The vector field $\breve{S}$ is globally defined on $\tilde{E}$ and $J\breve{S} = C$.

So $\breve{S}$ is a semispray on $\tilde{E}$.

c) The integral curves of $\breve{S}$ satisfy the system (20), so they are the evolution curves of the rheonomic mechanical system Σ . ∎

We call this semispray *the canonical evolution semispray* of the mechanical system Σ.

We can say:

The geometry of the rheonomic Lagrangian mechanical system Σ is the geometry of the pair $(RL^n, \breve{S})$, where RL^n is a rheonomic Lagrange space and $\breve{S}$ is the evolution semispray.

Afterwards we investigate the variation of the energy

$$E_L = y^i\frac{\partial L}{\partial y^i} - L$$

along the evolution curves of the rheonomic mechanical system Σ.

Straightforward calculations lead to the following results:

Theorem 3.2: On the evolution curves of the rheonomic mechanical system Σ, the variation of energy E_L is given by:

$$\frac{dE_L}{dt} = y^iF_i(x,y,t) - \frac{\partial L}{\partial t}.$$

Theorem 3.3: The canonical non-linear connection $\breve{N}$ of the mechanical system Σ has the coefficients $(\breve{N}_j^i, \breve{N}_j^0)$:

$$\breve{N}_j^i = \frac{1}{4}\frac{\partial}{\partial y^j}\left[g^{ih}\left(\frac{\partial^2 L}{\partial y^h \partial x^k}y^k - \frac{\partial L}{\partial x^h}\right)\right] - \frac{1}{4}\frac{\partial F^i}{\partial y^j} = \frac{\partial \breve{G}^i}{\partial y^j}; \tag{23}$$

$$\breve{N}_j^0 = \frac{1}{2}\frac{\partial^2 L}{\partial t \partial y^j},$$

with $\breve{G}^i = 2G^i(x,y,t) - \dfrac{1}{2}F^i(x,y,t)$.

Let us consider the adapted basis to the distributions $\breve{N}$ and V:

$$\left\{\frac{\breve{\delta}_i}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t}\right\}, \tag{24}$$

where

$$\frac{\breve{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \breve{N}_j^i(x,y,t)\frac{\partial}{\partial y^j} - \breve{N}_j^0(x,y,t)\frac{\partial}{\partial t} + \frac{1}{4}\frac{\partial F^j}{\partial y^i}\frac{\partial}{\partial y^j}. \tag{25}$$

The Lie brackets of the local vector fields from the adapted basis (24) are as the following:

$$\left[\frac{\breve{\delta}}{\delta x^j}, \frac{\breve{\delta}}{\delta x^h}\right] = \breve{R}_{jh}^i\frac{\partial}{\partial y^i} + \breve{R}_{jh}^0\frac{\partial}{\partial t};$$

$$\left[\frac{\breve{\delta}}{\delta x^j}, \frac{\partial}{\partial t}\right] = \frac{\partial \breve{N}_j^i}{\partial t}\frac{\partial}{\partial y^i} + \frac{\partial \breve{N}_j^0}{\partial t}\frac{\partial}{\partial t}; \tag{26}$$

$$\left[\frac{\breve{\delta}}{\delta x^j}, \frac{\partial}{\partial y^h}\right] = \frac{\partial \breve{N}_j^i}{\partial y^h}\frac{\partial}{\partial y^i} + \frac{\partial \breve{N}_j^0}{\partial y^h}\frac{\partial}{\partial t};$$

$$\left[\frac{\breve{\delta}}{\delta y^j}, \frac{\partial}{\partial y^h}\right] = \left[\frac{\partial}{\partial y^j}, \frac{\partial}{\partial t}\right] = \left[\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right] = 0,$$

where

$$\breve{R}_{jh}^i = \frac{\breve{\delta}\breve{N}_j^i}{\delta x^h} - \frac{\breve{\delta}\breve{N}_h^i}{\delta x^j}; \quad \breve{R}_{jh}^0 = \frac{\breve{\delta}\breve{N}_j^0}{\delta x^h} - \frac{\breve{\delta}\breve{N}_h^0}{\delta x^j}. \tag{27}$$

The dual basis $\{dx^i, \breve{\delta}y^i, \breve{\delta}t\}$ is given by

$$\breve{\delta}y^i = dy^i + \breve{N}_j^i dx^j - \frac{1}{4}\frac{\partial F^i}{\partial y^j}dx^j; \quad \breve{\delta}t = dt + \breve{N}_i^0 dx^i. \tag{28}$$

One can demonstrate the following theorem

Theorem 3.4: The canonical non-linear connection $\breve{N}$ is integrable if and only if $\breve{R}^i_{jh} = 0$ and $\breve{R}^0_{jh} = 0$.

Proof:

As $\dfrac{\breve{\delta}}{\delta x^j}$ are generators for the horizontal distribution $\chi^h(TM \times R)$ it results that $\breve{N}$ is integrable if and only if their Lie brackets are horizontal, which means that $\left[\dfrac{\breve{\delta}}{\delta x^j}, \dfrac{\breve{\delta}}{\delta x^h}\right] \in \chi^h(TM \times R)$. Using (26) we have the already mentioned conclusion. ■

On the manifold $TM \times R$ an important geometric structure, whose existence is equivalent to the existence of the canonical non-linear connection $\breve{N}$, is given by the $\mathcal{F}(TM \times R)$-linear mapping $\mathbb{F} : \chi(TM \times R) \to \chi(TM \times R)$,

$$\mathbb{F}\left(\frac{\breve{\delta}}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}; \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{\breve{\delta}}{\delta x^i}; \mathbb{F}\left(\frac{\partial}{\partial t}\right) = 0. \tag{29}$$

The structure $\mathbb{F}$ has the following properties:

1)$\mathbb{F}$ depends on the rheonomic Lagrangian mechanical system Σ, only;

2) $\mathbb{F}$ is a tensor field of $(1,1)$-type on the manifold $\tilde{TM} \times R$ and $F \circ F = -Id + \dfrac{\partial}{\partial t} \otimes dt$;

3)It is an almost complex structure if and only if the curvature tensor $\breve{R}^\alpha_{jk}$ of the evolution non-linear connection vanishes.

Theorem 3.5: The canonical metrical $\breve{N}-$connection of the rheonomic mechanical system Σ, $C\Gamma(\breve{N})$, has the coefficients given by the generalized Christoffel symbols:

$$\breve{L}^i_{jk} = \frac{1}{2} g^{ih}\left(\frac{\breve{\delta} g_{hk}}{\delta x^j} + \frac{\breve{\delta} g_{jh}}{\delta x^k} - \frac{\breve{\delta} g_{jk}}{\delta x^h}\right),$$
$$\breve{C}^i_{jk} = \frac{1}{2} g^{ih}\left(\frac{\partial g_{hk}}{\partial y^j} + \frac{\partial g_{jh}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^h}\right), \tag{30}$$
$$\breve{C}^i_{j0} = \frac{1}{2} g^{ih}\frac{\partial g_{jh}}{\partial t}.$$

The relation between the coefficients (30) and the coefficients of the canonical metrical $N-$connection of the rheonomic Lagrange space RL^n is given by the following theorem.

Theorem 3.6: The local coefficients of the canonical metrical $\breve{N}-$linear connection D of the mechanical system Σ have the following form

$$\breve{L}^i_{jk} = L^i_{jk} + \frac{1}{2} g^{is}\left(C_{skh}\frac{\partial F^h}{\partial y^j} + C_{jsh}\frac{\partial F^h}{\partial y^k} - C_{jkh}\frac{\partial F^h}{\partial y^s}\right); \tag{31}$$
$$\breve{C}^i_{jk} = C^i_{jk}; \;\; \breve{C}^i_{j0} = C^i_{j0},$$

where $C\Gamma(N) = (L^i_{jk}, C^i_{j\alpha})$ is the $N-$canonical metrical connection of Lagrange space RL^n and

$$C_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial y^j} + \frac{\partial g_{ij}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^i}\right).$$

Proof:

If we use the Lagrange equations and the coefficients L^i_{jk} of the $N-$ canonical metrical connection of rheonomic Lagrange space we obtain first equality (31).

The equalities $\breve{C}^i_{jk} = C^i_{jk}$; $\breve{C}^i_{j0} = C^i_{j0}$ are straightforward. ■

The metric tensor $g_{ij}(x, y, t)$ of the rheonomic Lagrange space RL^n and the canonical non-linear connection $\breve{N}$ allow us to introduce a pseudo-Riemannian structure $\mathbb{G}$ on the manifold $\tilde{TM} \times R$. This is given by the $\breve{N}-$lift $\mathbb{G}$ of the fundamental tensor g_{ij}:

$$\mathbb{G} = g_{ij} dx^i \otimes dx^j + g_{ij}\breve{\delta} y^i \otimes \breve{\delta} y^j + \breve{\delta} t \otimes \breve{\delta} t \tag{32}$$

The metric $\breve{N}-$lift $\mathbb{G}$ has the following properties:

1) $\mathbb{G}$ depends on the rheonomic Lagrangian mechanical system Σ only;

2) $\mathbb{G}$ is a pseudo-Riemannian structure on the manifold $\tilde{TM} \times R$.

The model $\left(\tilde{TM} \times R, \mathbb{G}, \mathbb{F}\right)$ gives a fair geometric description of the rheonomic Lagrangian mechanical system.

IV. Example of rheonomic mechanical system

Let us consider the concrete rheonomic mechanical system of a constrained motion of a material particle of mass m and position vector $\overline{r} = x^1\overline{i} + x^2\overline{j} + x^3\overline{k}$, [14], with the rheonomic constraint given by $\tilde{f}(x^1, x^2, x^3, t) = 0$. The material particle is under the action of the linear, or non-linear spring force with potential $\Pi = -U(\overline{r})$. The material particle cannot move subject to constraints without considering certain constraint reaction due to the constraint expressed by $\overline{F}_{wN} = \lambda\, grad\, \tilde{f}\,(x^1, x^2, x^3, t)$, for ideal constraint, where λ is Lagrange's multiplier.

For non ideal constraint, with friction coefficient μ, the constraint reaction due to the constraint can be expressed by

$$\overline{F}_w = \overline{F}_{wN} + \overline{F}_{wT} = \lambda\, grad\, \tilde{f}(x^1, x^2, x^3, t) -$$
$$-\mu\overline{v}|\lambda\, grad\, \tilde{f}\,(x^1, x^2, x^3, t)|.$$

The material particle is under the external dumping force, linear proportional with the material particle velocity, expressed by $\overline{F}_{w,\overline{v}} = -b\overline{v}$. Let $\overline{F}(t) = X\overline{i} + Y\overline{j} + Z\overline{k}$ be the active force.

In the considered case, the material particle is limited by rheonomic constraint as the moving surface, and their position is defined by three coordinates x^1, x^2 and x^3, but the material particle have two degrees of freedom and we need to chose two coordinates as the generalized coordinates of their motion. Let the generalized coordinates be:

$$q^1 = x^1; \;\; q^2 = x^2$$

and because the constraint is of rheonomic nature, we consider a rheonomic coordinate $q^0 = \phi(t)$. Therefore we can express the third position coordinate $x^3 = f(q^1, q^2, q^0)$, as [7].

The kinetic energy of the material particle motion can be expressed by the two generalized coordinates and the third rheonomic coordinate in the following form:

$$E_k = \frac{1}{2} m\overline{v}^2 = \tag{33}$$
$$= \frac{1}{2} m\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \left(\dot{q}^1\frac{\partial f}{\partial q^1} + \dot{q}^2\frac{\partial f}{\partial q^2} + \dot{q}^0\frac{\partial f}{\partial q^0}\right)^2\right].$$

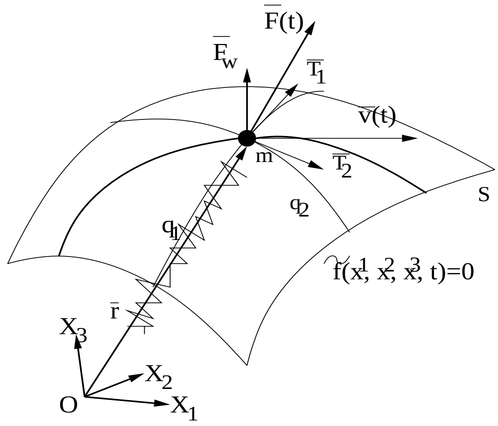

Fig. 1.

When the constraint is defined by $x^3 = \frac{1}{\ell}\left(q^1 - \ell\Omega t\right)^2$, we can introduce the rheonomic coordinate $q^0 = \ell\Omega t$ and the previous constrain will become as follows $x^3 = \frac{1}{\ell}\left(q^1 - q^0\right)^2$.

The kinetic energy can be written

$$E_k = \frac{1}{2}m\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \frac{4}{\ell^2}\left(\dot{q}^1 - \dot{q}^0\right)^2\left(q^1 - q^0\right)^2\right].$$

The matrix of the mass inertia moment tensor is:

$$\mathbf{A} = m\begin{pmatrix} 1 + \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \\ 0 & 1 & 0 \\ -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \end{pmatrix}.$$

The virtual work $\delta\mathbf{W}$ of the active force on the virtual displacements $\delta\vec{r}$

$$\delta\vec{r} = \delta q^1\vec{i} + \delta q^2\vec{j} + \frac{2}{\ell}\left[\delta q^1\left(q^1 - q^0\right) - \delta q^0\left(q^1 - q^0\right)\right]\vec{k}$$

is given by:

$$\delta\mathbf{W} = \left(\vec{F}, \delta\vec{r}\right)q^0;$$

$$\delta\mathbf{W} = \left[X + \frac{2}{\ell}Z\left(q^1 - q^0\right)\right]\delta q^1 + Y\delta q^2 - \frac{2}{\ell}Z\left(q^1 - q^0\right)\delta q^0.$$

The generalized components of the active force, for the generalized coordinates q^1 and q^2 and the rheonomic coordinate q^0, are given by

$$Q_1 = X + \frac{2}{\ell}Z\left(q^1 - q^0\right); Q_2 = Y; Q_0 = -\frac{2}{\ell}Z\left(q^1 - q^0\right).$$

For the active force induced by the spring:

$$\vec{F} = -c\vec{r} = -cq^1\vec{i} - cq^2\vec{j} - \left[c\frac{1}{\ell}\left(q^1 - q^0\right)^2 + mg\right]\vec{k},$$

the generalized force components are:

$$Q_1 = -cq^1 - c\frac{2}{\ell^2}\left(q^1 - q^0\right)^3 - \frac{2}{\ell}mg\left(q^1 - q^0\right); Q_2 = -cq^2;$$

$$Q_0 = c\frac{2}{\ell^2}\left(q^1 - q^0\right)^3 - \frac{2}{\ell}mg\left(q^1 - q^0\right).$$

The potential energy can be expressed in the form:

$$\mathbf{E_p} = \Pi = -\int_0^{\vec{r}}\left(\vec{F}, d\vec{r}\right) =$$

$$= \frac{c}{2}\vec{r}^2 = \frac{c}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2 + \frac{1}{\ell^2}\left(q^1 - q^0\right)^4\right].$$

The kinetic potential is difference of the kinetic energy and the potential energy and we can express *the Lagrangian function* in the following form:

$$L = E_k - E_p = E_k - \Pi\left(q^1, q^2, f\left(q^1, q^2, q^0\right)\right) =$$

$$= \frac{1}{2}m\left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \frac{4}{\ell^2}\left[\left(\dot{q}^1 - \dot{q}^0\right)\left(q^1 - q^0\right)\right]^2\right] -$$

$$-\frac{c}{2}\left[\left(q^1\right)^2 + \left(q^2\right)^2 + \frac{1}{\ell^2}\left(q^1 - q^0\right)^4\right].$$

L is a rheonomic regular Lagrangian.

The matrix of the d-tensor field with the components

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}$$

is the fundamental (or metric) tensor field of the Lagrangian corresponding to the mechanical rheonomic system of one material particle moving along moving surface, $x^3 = \frac{1}{\ell}\left(q^1 - \ell\Omega t\right)^2$ as a rheonomic constraint:

$$\mathbf{G} = \left(g_{ij}\right)\Big|_{\downarrow i=1,2,0}^{\rightarrow j=1,2,0} = \left(\frac{1}{2}\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\right)_{\downarrow i=1,2,0}^{\rightarrow j=1,2,0}$$

$$\mathbf{G} = \frac{1}{2}\mathbf{A} =$$

$$= \frac{1}{2}m\begin{pmatrix} 1 + \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \\ 0 & 1 & 0 \\ -\frac{4}{\ell^2}\left(q^1 - q^0\right)^2 & 0 & \frac{4}{\ell^2}\left(q^1 - q^0\right)^2 \end{pmatrix}$$

where

$$g_{11} = \frac{1}{2}a_{11}(q) = \frac{1}{2}m\left[1 + \frac{4}{\ell^2}\left(q^1 - q^0\right)^2\right]$$

$$g_{22} = \frac{1}{2}a_{22}(q) = \frac{1}{2}m$$

$$g_{00} = \frac{1}{2}a_{00}(q) = \frac{1}{2}m\frac{4}{\ell^2}\left(q^1 - q^0\right)^2,$$

$$g_{12} = \frac{1}{2}a_{12}(q) = 0$$

$$g_{01} = \frac{1}{2}a_{01}(q) = -\frac{1}{2}m\frac{4}{\ell^2}\left(q^1 - q^0\right)^2,$$

$$g_{02} = \frac{1}{2}a_{02}(q) = 0,$$

where $(q) = \left(q^1, q^2, q^0\right)$ and q^0 is the rheonomic coordinate, depending of time t.

We can see that the fundamental tensor field of the considered Lagrangian is half *of the mass inertia moment tensor matrix*.

The extended Lagrange system of differential equations of the second order, has the following form:

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^1}-\frac{\partial E_k}{\partial q^1}-\frac{\partial E_p}{\partial q^1}=Q_1;$$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^2}-\frac{\partial E_k}{\partial q^2}-\frac{\partial E_p}{\partial q^2}=Q_2;$$

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}^0}-\frac{\partial E_k}{\partial q^0}-\frac{\partial E_p}{\partial q^0}=Q_0+Q_{00},$$

and for $q^0=\ell\Omega t$, $\dot{q}^0=\ell\Omega$, $\ddot{q}^0=0$, it becomes:

$$\ddot{q}^1+\frac{c}{m}\frac{q^1}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]}+\frac{2c}{m\ell^2}\frac{\left(q^1-q^0\right)^3}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]}=$$
$$=-\frac{4}{\ell^2}\frac{\left(\dot{q}^1-\dot{q}^0\right)^2\left(q^1-q^0\right)}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]}-\frac{2}{\ell}\frac{g\left(q^1-q^0\right)}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]};$$

$$\ddot{q}^2+\frac{c}{m}q^2=0,$$

and from the third equation of the Lagrange system we can find the rheonomic constraint force Q_{00}

$$\frac{d}{dt}\left[-\frac{4}{\ell^2}\left(\dot{q}^1-\dot{q}^0\right)\left(q^1-q^0\right)^2\right]+\frac{4}{\ell^2}\left(\dot{q}^1-\dot{q}^0\right)^2\left(q^1-q^0\right)=$$
$$=-\frac{2c}{m\ell^2}\left(q^1-q^0\right)^3-\frac{2}{\ell}mg\left(q^1-q^0\right)+Q_{00}. \tag{34}$$

The theoretical form of the Lagrange equations is:

$$\ddot{q}^i+2G^i(q,\dot{q},t)+N_0^i(q,\dot{q},t)=\frac{1}{2}Q^i(q,\dot{q},t);\quad \dot{q}^i=\frac{dq^i}{dt},$$

with the coefficients

$$2G^i(q,\dot{q},t)=$$
$$=mg^{i1}\left[\left(\dot{q}^1\right)^2\frac{\partial^2 f}{\partial\left(q^1\right)^2}+\left(\dot{q}^2\right)^2\frac{\partial^2 f}{\partial\left(q^2\right)^2}+2\frac{\partial^2 f}{\partial q^1\partial q^2}\dot{q}^1\dot{q}^2+\right.$$
$$\left.+\frac{\partial^2 f}{\partial q^2\partial t}\dot{q}^2\right]\frac{\partial f}{\partial q^1}-mg^{i1}\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial t}-mg^{i1}\dot{q}^2\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}+$$
$$+mg^{i2}\left[\left(\dot{q}^1\right)^2\frac{\partial^2 f}{\partial\left(q^1\right)^2}+\left(\dot{q}^2\right)^2\frac{\partial^2 f}{\partial\left(q^2\right)^2}+2\frac{\partial^2 f}{\partial q^1\partial q^2}\dot{q}^1\dot{q}^2+\right.$$
$$\left.+\frac{\partial^2 f}{\partial q^1\partial t}\dot{q}^1\right]\frac{\partial f}{\partial q^2}-mg^{i2}\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial t}-mg^{i2}\dot{q}^1\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial q^1}+$$
$$+g^{i1}\frac{\partial\Pi}{\partial q^1}+g^{i2}\frac{\partial\Pi}{\partial q^2}+\left(g^{i1}\frac{\partial f}{\partial q^1}+g^{i2}\frac{\partial f}{\partial q^2}\right)\frac{\partial\Pi}{\partial f};i=1,2.$$

Using the theoretical form of coefficients

$$N_0^i(q,\dot{q},t)=\frac{1}{2}g^{ih}\frac{\partial^2 L}{\partial q^h\partial t}$$

we obtain:

$$N_0^i(q,\dot{q},t)=mg^{i1}\left(2\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^1\partial t}\dot{q}^1+\dot{q}^2\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}+\right.$$
$$\left.+\dot{q}^2\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial q^1}+\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial q^1}\right)+$$
$$+mg^{i2}\left(2\frac{\partial f}{\partial q^2}\frac{\partial^2 f}{\partial q^2\partial t}\dot{q}^2+\dot{q}^1\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}+\dot{q}^1\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial q^1}\right.$$
$$\left.+\frac{\partial^2 f}{\partial q^2\partial t}\frac{\partial f}{\partial t}+\frac{\partial^2 f}{\partial t^2}\frac{\partial f}{\partial q^2}\right);\quad i=1,2. \tag{35}$$

One obtain

$$2G^1(q,\dot{q},t)+N_0^1(q,\dot{q},t)=\frac{4}{\ell^2}\frac{\left(\dot{q}^1-\dot{q}^0\right)^2\left(q^1-q^0\right)}{\left[1+\frac{4}{\ell^2}\left(q^1-q^0\right)^2\right]};$$

$$2G^2(q,\dot{q},t)+N_0^2(q,\dot{q},t)=0.$$

We define the following functions

$$2\overset{0}{\Gamma^i}=2G^i(q,\dot{q},t)+N_0^i(q,\dot{q},t)-\frac{1}{2}Q^i(q,\dot{q},t).$$

So, $\overset{0}{S}$ given by

$$\overset{0}{S}=y^i\frac{\partial}{\partial x^i}-2\overset{0}{\Gamma^i}(x,y,t)\frac{\partial}{\partial y^i}+\frac{\partial}{\partial t}$$

is *the evolution semispray of the mechanical system* $\overset{0}{\Sigma}$.

The integral curves of $\overset{0}{S}$ are *the evolution curves of the mechanical system* $\overset{0}{\Sigma}$.

The canonical non-linear connection $\overset{0}{N}$ of mechanical system, $\overset{0}{\Sigma}$, depending only on the rheonomic Lagrangian mechanical system, has the coefficients $(\overset{0}{N^i_j},\overset{0}{N^0_j})$ given by

$$\overset{0}{N^i_1}(q,\dot{q},t)=\frac{\partial G^i(q,\dot{q},t)}{\partial\dot{q}^1}-\frac{1}{4}\frac{\partial F^i(q,\dot{q},t)}{\partial\dot{q}^1}=$$
$$=mg^{i1}\left(2\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial(q^1)^2}\dot{q}^1+2\dot{q}^2\frac{\partial^2 f}{\partial q^1\partial q^2}\frac{\partial f}{\partial q^1}\right)+$$
$$+mg^{i2}\left(2\dot{q}^1\frac{\partial^2 f}{\partial(q^1)^2}\frac{\partial f}{\partial q^2}+2\dot{q}^2\frac{\partial^2 f}{\partial q^1\partial q^2}\frac{\partial f}{\partial q^2}+\right.$$
$$\left.+\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}-\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^2\partial t}\right)-\frac{1}{2}\frac{\partial Q^i}{\partial\dot{q}^1};$$

$$\overset{0}{N^i_2}(q,\dot{q},t)=\frac{\partial G^i(q,\dot{q},t)}{\partial\dot{q}^2}-\frac{1}{4}\frac{\partial F^i(q,\dot{q},t)}{\partial\dot{q}^2}=$$
$$=mg^{i1}\left(2\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^1\partial q^2}\dot{q}^1+2\dot{q}^2\frac{\partial^2 f}{\partial(q^2)^2}\frac{\partial f}{\partial q^1}+\right.$$
$$\left.+\frac{\partial f}{\partial q^1}\frac{\partial^2 f}{\partial q^2\partial t}-\frac{\partial^2 f}{\partial q^1\partial t}\frac{\partial f}{\partial q^2}\right)+$$
$$+mg^{i2}\left(2\dot{q}^1\frac{\partial^2 f}{\partial q^1\partial q^2}\frac{\partial f}{\partial q^2}+2\dot{q}^2\frac{\partial^2 f}{\partial(q^2)^2}\frac{\partial f}{\partial q^2}\right)-\frac{1}{2}\frac{\partial Q^i}{\partial\dot{q}^2},$$
$$i=1,2.$$

Clearly, the canonical non-linear connection $\overset{0}{N}$ determines the metrical $\overset{0}{N}$-linear connection $C\Gamma(\overset{0}{N})$.

V. Conclusion

We present the Lagrangian geometric model of the rheonomic mechanical systems and apply it to a concrete rheonomic mechanical system: the material particle motion along moving surface.

We can see that the fundamental (or metric) tensor field of the considered Lagrangian is half of the mass inertia moment tensor matrix.

The integral curves of the evolution semispray $\overset{0}{S}$ are the evolution curves of the mechanical system.

The canonical non-linear connection for the concrete rheonomic mechanical system have the coefficients expressed using the rheonomic constraints of the mechanical system.

It is visible that for considered rheonomic mechanical system is easy to obtain geometrical description and the dynamical properties.

References

[1] M. Anastasiei, *On the geometry of time-dependent Lagrangians*, Mathematical and Computing Modelling, 20, no4/5, Pergamon Press,1994, pp.67-81.

[2] I. Bucataru, R. Miron, *Finsler-Lagrange Geometry. Applications to dynamical systems*, Editura Academiei Romane, Bucureşti, 2007.

[3] S. S. Chern, Z. Shen, *Riemann-Finsler Geometry*, Nankai Tracts in Mathematics, 6, World Scientific, Publishing Co. Pte. Ltd., Hackensack, NJ.

[4] M. Crampin, F. A. E. Pirani, *Applicable Differential Geometry*, London Math. Society, Lectures Notes Series, 59, Cambridge University Press, 1986.

[5] C. Frigioiu, C. Gheorghieş, *Mechanical systems and their associated Lagrange geometries*, International Journal of Computer Mathematics, Volume 87 Issue 12, 2010, pp.2846-2856.

[6] C. Godbillon, *Geometrie Differentielle et Mecanique Analytique*, Herman, Paris, 1969.

[7] K. Hedrih (Stevanovic), *Rheonomic Coordinate method Applid to Nonlinear Vibration Systems with Hereditary Elements*, Facta Universitatis, Series "Mechanics, Automatic Control and Robotics", 10(2000), pp.1111-1135.

[8] J. Klein, *Espaces Variationnels et Mecanique*, Ann. Inst Fourier, Grenoble, 12(1962), pp.1-124.

[9] O. Krupkova, *The geometry of Ordinary Variationel Equations*, Springer-Verlag, 1997.

[10] M. de Leon, P.R. Rodriguez, *Methods of Differential Geometry in Analitical Mechanics*, North-Holland, 1989.

[11] R. Miron, *Dynamical Systems of Lagrangian and Hamiltonian Mechanical Systems*, Advanced Studies in Pure Math., 24(2006), pp.165-199.

[12] R. Miron, M. Anastasiei, I. Bucataru, *The Geometry of Lagrange Spaces*, in Handbook of Finsler Geometry, P.L.Antonelli, ed., Kluwer Acad. Publ.FTPH, 2003, pp. 969-1124.

[13] R. Miron, C. Frigioiu, *Finslerian Mechanical Systems*, Algebras, Groups and Geometries, 22(2005), pp.151-168.

[14] V. A. Vujicic, K. Hedrih(Stevanovic), *The rheonomic constraints change force*, Facta Universitatis, Series "Mechanics, Automatic Control and Robotics", 1(1993), pp. 313-322.