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L^1-Convergence of Modified Trigonometric Sums

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Abstract—The existence of sine and cosine series as a Fourier series, their L^1-convergence seems to be one of the difficult question in theory of convergence of trigonometric series in L^1-metric norm. In the literature so far available, various authors have studied the L^1-convergence of cosine and sine trigonometric series with special coefficients. In this paper, we present a modified cosine and sine sums and criterion for L^1-convergence of these modified sums is obtained. Also, a necessary and sufficient condition for the L^1-convergence of the cosine and sine series is deduced as corollaries.

Keywords—Conjugate Dirichlet kernel, Dirichlet kernel, L^1-convergence, modified sums.

I. Introduction

LET

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \tag{1}$$

and

$$g(x) = \sum_{k=1}^{\infty} b_k \sin kx \tag{2}$$

be the trigonometric cosine and sine series.

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx$$

and

$$\tilde{S}_n(x) = \sum_{k=1}^{n} b_k \sin kx$$

be the partial sums of the series (1) and (2) respectively.

Convex sequence. ([1], Vol. I, p. 4) A sequence $\{a_k\}$ is said to be convex if $\Delta^2 a_k \geq 0$.
where $\Delta^2 a_k = \Delta(\Delta a_k)$ and $\Delta a_k = a_k - a_{k+1}$.

Quasi-convex sequence. ([1], Vol. II, p. 202) A sequence $\{a_k\}$ is said to be quasi-convex if

$$\sum_{k=1}^{\infty} n \left|\Delta^2 a_k\right| < \infty$$

The work on L^1-convergence of trigonometric series with special coefficients was introduced by Young [2] and Kolmogorov [3] by taking classes of convex sequences and quasi-convex sequences.

Theorem 1. [2], [3] *If $\{a_k\} \downarrow 0$ and $\{a_k\}$ is convex or even quasi-convex, then for the convergence of the series (1) in the metric space L^1, it is necessary and sufficient that* $a_k \log k = o(1), \quad k \to \infty$.

Class S. [4], [5] A sequence $\{a_k\}$ belongs to class S, if $a_k \to \infty$ as $k \to \infty$ and there exists a sequence of numbers $\{A_k\}$ such that
(i) $A_k \downarrow 0$ as $k \to \infty$.
(ii) $\sum\limits_{k=0}^{\infty} A_k < \infty$.
(iii) $|\Delta a_k| \leq A_k$, for all k.
The class S is usually called as Sidon-Teljakovskii class.
A quasi-convex null sequence satisfies conditions of the class S by choosing:

$$A_k = \sum_{m=k}^{\infty} |\Delta^2 a_m|.$$

Teljakovskii [5] generalized Theorem 1 for the cosine series (1) with coefficients satisfying the conditions of the class S in the following form:

Theorem 2. [5] *Let $\{a_k\}$ be the sequence of the cosine series (1) belongs to the class S, then a necessary and sufficient condition for L^1-convergence of (1) is* $a_k \log k = o(1), \quad k \to \infty$.

It is well known that if a trigonometric series converges in L^1-metric to a function $f \in L^1(T)$, then it is the Fourier series of the function f. Riesz ([1], Vol. II, Ch. VIII, 22) gave a counter example to show that in L^1-metric, the converse of the above said result does not hold good. L^1-convergence of trigonometric series with special coefficients have been studied by various authors. During the literature survey, It can be observed that many authors have introduced modified trigonometric sums "*as these sums approximate their limits better than the classical trigonometric series in the sense that these sums converge in L^1- metric to the sum of trigonometric series whereas the classical series itself may not*". Rees and Stanojevic [6], Kumari and Ram [7], [8], Hooda, Ram and Bhatia [9], Kaur, Bhatia and Ram [10], Kaur and Bhatia [11], [12], Braha and Xhevat [13], Krasniqi [14], [15] have introduced new modified trigonometric sums and studied their L^1-convergence under various classes of coefficient sequences.

Rees and Stanojevic [6] have introduced modified cosine sum as

$$g_n(x) = \frac{1}{2}\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} (\Delta a_j) \cos kx \tag{3}$$

Garrett and Stanojevic [16], Ram [17], Singh and Sharma [18] studied the L^1-convergence of cosine sum (3) under different set of conditions on the coefficients.

Kumari and Ram ([7], [8]) introduced new modified cosine and sine sums as

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{a_j}{j}\right) k \cos kx$$

and

$$g_n(x) = \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left(\frac{b_j}{j}\right) k \sin kx$$

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and have studied their L^1-convergence under the condition that the coefficients $\{a_n\}$ and $\{b_n\}$ belong to different classes of sequences.

Hooda, Ram and Bhatia [9] introduced a new modified cosine sums as

$$f_n(x) = \frac{1}{2}\left(a_1 + \sum_{k=0}^{n} \Delta^2 a_k\right) + \sum_{k=1}^{n}\left(a_{k+1} + \sum_{j=k}^{n} \Delta^2 a_j\right) \cos kx$$

and have studied its L^1-convergence to a cosine trigonometric series and also deduced a result of Teljakovskii [19] as a corollary.

Kaur, Bhatia and Ram [10] introduced new modified sine sums as

$$K_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\Delta b_{j-1} - \Delta b_{j+1}) \sin kx$$

and have studied the L^1-convergence of modified sine sums under a different class.

Kaur and Bhatia [11] have introduced new modified cosine and sine sums as

$$\frac{a_0}{2} + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(a_j \cos jx)$$

and

$$\sum_{k=1}^{n}\sum_{j=k}^{n} \Delta(b_j \sin jx)$$

and studied their integrability and L^1-convergence.

Krasniqi [14] have introduced new modified cosine and sine sums as

$$H_n(x) = -\frac{1}{2\sin x}\sum_{k=0}^{n}\sum_{j=k}^{n} \Delta\left[(a_{j-1} - a_{j+1})\cos jx\right]$$

and

$$G_n(x) = \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n} \Delta\left[(b_{j-1} - b_{j+1})\sin jx\right]$$

and studied the L^1-convergence of these modified cosine and sine sums with semi convex coefficients.

In 2013, Krasniqi [15] have introduced new modified cosine sum as

$$G_n(x) = \frac{a_0}{2} + \sum_{k_1=1}^{n}\sum_{k_2=k_1}^{n}\sum_{k_3=k_2}^{n} \Delta^2(a_{k_3}\cos k_3 x),$$

where $\Delta^2 a_k = \Delta(\Delta a_k) = a_k - 2a_{k+1} + a_{k+2}$
and studied its L^1-convergence.

We introduce here a new modified cosine and sine sums as

$$f_n(x) = \sum_{k=1}^{n}\left[\sum_{j=k}^{n}\left(\Delta a_{j+1} + \sum_{i=j}^{n}\Delta^3 a_i\right)\right]\cos kx$$

where $\Delta^3 a_k = \Delta^2 a_k - \Delta^2 a_{k+1}$
and

$$g_n(x) = \sum_{k=1}^{n}\left[\sum_{j=k}^{n}\left(\Delta b_{j+1} + \sum_{i=j}^{n}\Delta^3 b_i\right)\right]\sin kx$$

where $\Delta^3 b_k = \Delta^2 b_k - \Delta^2 b_{k+1}$
and study the L^1-convergence of these modified cosine and sine sums under the class S.

As usual $D_n(x)$ and $\tilde{D}_n(x)$ will denote the Dirichlet and its conjugate kernels respectively, defined by:

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n}\cos kx$$

and

$$\tilde{D}_n(x) = \sum_{k=1}^{n}\sin kx$$

II. Lemmas

The following lemmas are used in the proof of main results:

Lemma 1. [20] *Let $n \geq 1$ and let r be a nonnegative integer, $x \in [\epsilon, \pi]$. Then $|\tilde{D}_n^r(x)| \leq C_\epsilon \frac{n^r}{x}$ where C_ϵ is a positive constant depending on ϵ, $0 < \epsilon < \pi$ and $\tilde{D}_n(x)$ is the conjugate Dirichlet kernel.*

Lemma 2. [20] $||D_n^r(x)||_{L^1} = O(n^r \log n), \quad r = 0, 1, 2, 3, ...$ *where $D_n^r(x)$ represents the r^{th} derivative of Dirichlet kernel.*

Lemma 3. [20] $||\tilde{D}_n^r(x)||_{L^1} = O(n^r \log n), \quad r = 0, 1, 2, 3, ...$ *where $\tilde{D}_n^r(x)$ represents the r^{th} derivative of conjugate Dirichlet kernel.*

III. Main Results

We establish the following results.

Theorem 3. *Let $\{a_k\}$ belong to the class S and $n^2 a_n = o(1)$, then $||f - f_n||_{L^1} = o(1)$ as $n \to \infty$.*

Proof. we have

$$\begin{aligned} f_n(x) &= \sum_{k=1}^{n}\left[\sum_{j=k}^{n}\left(\Delta a_{j+1} + \sum_{i=j}^{n}\Delta^3 a_i\right)\right]\cos kx \\ &= \sum_{k=1}^{n}\left[\sum_{j=k}^{n}\left(\Delta a_{j+1} + \Delta^2 a_j - \Delta^2 a_{n+1}\right)\right] \\ &= \sum_{k=1}^{n}\left[a_k - a_{n+1} - \Delta^2 a_{n+1}(n-k+1)\right]\cos kx \end{aligned}$$

$$= \sum_{k=1}^{n} a_k \cos kx - a_{n+1}D_n(x) - (n+1)\Delta^2 a_{n+1}D_n(x) + \Delta^2 a_{n+1}\tilde{D}'_n(x)$$

Apply Abel's transformation, we get

$$= \sum_{k=1}^{n} \Delta a_k D_k(x) - (n+1)(a_{n+1} - 2a_{n+2} + a_{n+3})D_n(x) + (a_{n+1} - 2a_{n+2} + a_{n+3})\tilde{D}'_n(x)$$

Since $D_n(x)$ is bounded and $|\Delta a_k| \leq A_k \quad \forall \quad k = 1, 2, ...$

Hence by given hypothesis and Lemma 1:
$\lim_{n\to\infty} f_n(x) = f(x)$ exists in $(0, \pi)$

Next, we consider

$$f(x) - f_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + (n+1)\Delta^2 a_{n+1}D_n(x)$$

$$+ a_{n+1}D_n(x) - \Delta^2 a_{n+1}\tilde{D}'_n(x)$$

Apply Abel's transformation, we have

$$= \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) + (n+1)\Delta^2 a_{n+1}D_n(x) - \Delta^2 a_{n+1}\tilde{D}'_n(x)$$

$$\int_0^\pi |f(x) - f_n(x)|dx =$$

$$\int_0^\pi \Bigg| \sum_{k=n+1}^{\infty} \Delta a_k D_k(x) + (n+1)a_{n+1}D_n(x) -2(n+1)a_{n+2}D_n(x) + (n+1)a_{n+3}D_n(x) - a_{n+1}\tilde{D}'_n(x) +2a_{n+2}\tilde{D}'_n(x) - a_{n+3}\tilde{D}'_n(x)\Bigg| dx$$

$$\le \int_0^\pi \left| \sum_{k=n+1}^{\infty} A_k \frac{\Delta a_k}{A_k} D_k(x)\right| dx + (n+1)|a_{n+1}| \int_0^\pi |D_n(x)|dx + 2(n+1)|a_{n+2}| \int_0^\pi |D_n(x)|dx + (n+1)|a_{n+3}| \int_0^\pi |D_n(x)|dx + |a_{n+1}| \int_0^\pi |\tilde{D}'_n(x)|dx + 2|a_{n+2}| \int_0^\pi |\tilde{D}'_n(x)|dx + |a_{n+3}| \int_0^\pi |\tilde{D}'_n(x)|dx$$

It is well known that $\int_0^\pi |D_n(x)|dx \sim \log n$ (see [1]) and using Lemma 3, we get

$$\le \sum_{k=n+1}^{\infty} \Delta A_k \int_0^\pi \left| \sum_{i=0}^{k} \frac{\Delta a_i}{A_i} D_i(x)\right| dx + (n+1)|a_{n+1}|\log n + 2(n+1)|a_{n+2}|\log n + (n+1)|a_{n+3}|\log n + |a_{n+1}|n\log n + 2|a_{n+2}|n\log n + |a_{n+3}|n\log n$$

$$\le C\sum_{k=n+1}^{\infty} (k+1)\Delta A_k + n(n+1)|a_{n+1}| + 2n(n+1)|a_{n+2}| + n(n+1)|a_{n+3}| + n^2|a_{n+1}| + 2n^2|a_{n+2}| + n^2|a_{n+3}|$$

Since, $\{a_k\} \in S$ then $\sum_{k=n+1}^{\infty} (k+1)\Delta A_k = o(1)$ as $n \to \infty$ and by given hypothesis

The conclusion of main result holds.

Corollary 1. *Let the coefficient $\{a_k\}$ belong to the class S and $n^2 a_n = o(1)$ as $n \to \infty$, then the necessary and sufficient condition for the L^1-convergence of cosine series (1) is $\lim_{n\to\infty} a_n \log n = 0$.*

Proof. we notice that

$$\begin{aligned} ||f - S_n||_{L^1} &= ||f - f_n + f_n - S_n||_{L^1} \\ &\le ||f - f_n||_{L^1} + ||f_n - S_n||_{L^1} \end{aligned}$$

$$\le \int_0^\pi |f(x) - f_n(x)|dx + \int_0^\pi |a_{n+1}D_n(x) - \Delta^2 a_{n+1}\tilde{D}'_n(x) + (n+1)\Delta^2 a_{n+1}D_n(x)|dx$$

$$\le \int_0^\pi |f(x) - f_n(x)|dx + \int_0^\pi |a_{n+1}D_n(x) + (n+1)(a_{n+1} - 2a_{n+2} + a_{n+3})D_n(x) - (a_{n+1} - 2a_{n+2} + a_{n+3})\tilde{D}'_n(x)|dx$$

$$\le \int_0^\pi |f(x) - f_n(x)|dx + |a_{n+1}| \int_0^\pi |D_n(x)|dx + (n+1)|a_{n+1}| \int_0^\pi |D_n(x)|dx + 2(n+1)|a_{n+2}| \int_0^\pi |D_n(x)|dx + (n+1)|a_{n+3}| \int_0^\pi |D_n(x)|dx + |a_{n+1}| \int_0^\pi |\tilde{D}'_n(x)|dx + 2|a_{n+2}| \int_0^\pi |\tilde{D}'_n(x)|dx + |a_{n+3}| \int_0^\pi |\tilde{D}'_n(x)|dx$$

We know $\int_0^\pi |D_n(x)|dx \sim \log n$ (see [1]) and using Lemma 3, we have:

$$\le \int_0^\pi |f(x) - f_n(x)|dx + |a_{n+1}|\log n + (n+1)|a_{n+1}|\log n + 2(n+1)|a_{n+2}|\log n + (n+1)|a_{n+3}|\log n + |a_{n+1}|n\log n + 2|a_{n+2}|n\log n + |a_{n+3}|n\log n$$

$$\le \int_0^\pi |f(x) - f_n(x)|dx + |a_{n+1}|\log n + n(n+1)|a_{n+1}| + 2n(n+1)|a_{n+2}| + n(n+1)|a_{n+3}| + n^2|a_{n+1}|\log n + 2n^2|a_{n+2}| + n^2|a_{n+3}|$$

Since, $\int_0^\pi |f(x) - f_n(x)|dx = 0$ as $n \to \infty$ and by given conditions.

The conclusion of the corollary follows.

Theorem 4. *Let the coefficient $\{b_k\}$ belong to the class S and $n^2 b_n = o(1)$, then $||g - g_n||_{L^1} = o(1)$ as $n \to \infty$.*

Corollary 2. *Let $\{b_k\}$ belong to the class S and $n^2 b_n = o(1)$ as $n \to \infty$, then the necessary and sufficient condition for the L^1-convergence of sine series (2) is $\lim_{n\to\infty} b_n \log n = 0$.*

The proofs of Theorem 4 and Corollary 2 are similar to the proofs of Theorem 3 and Corollary 1, therefore we omit.

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