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(λ, μ)-Intuitionistic Fuzzy Subgroups of Groups with Operators

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***Abstract*—**The aim of this paper is to introduce the concepts of the (λ, μ)-intuitionistic fuzzy subgroups and (λ, μ)-intuitionistic fuzzy normal subgroups of groups with operators, and to investigate their properties and characterizations based on M-group homomorphism.

***Keywords*—**Intuitionistic fuzzy group, (λ, μ)-intuitionistic fuzzy subgroup of groups with operators, (λ, μ)-intuitionistic fuzzy normal subgroup of groups with operators, M-group homomorphism.

I. Introduction

THE fuzzy set is an effective generalization of the classical set. In 1965, Zadeh [1] first raised the fuzzy set. In 1986, the Bulgarian Scholar K. Atannassov [2] introduced the intuitionistic fuzzy sets (IFS). After that, the two theories were extensively applied to many mathematical fields. Based on the two theories, W. X. Gu [3] raised the definition of fuzzy group with operators; [4]-[6] researched intuitionistic fuzzy relations, martingale theory and topological spaces; [7]-[10] studied intuitionistic fuzzy subgroups and some properties are discussed; [11] gave the definition of $(\lambda,\mu)-$ intuitionistic fuzzy subgroups; [12] defined the $(\lambda,\mu)-$ intuitionistic fuzzy implicative ideals of BCI-algebras.

At first, this paper gives the concepts of the (λ, μ)-intuitionistic fuzzy subgroups and (λ, μ)-intuitionistic fuzzy normal subgroups of groups with operators. Secondly, it is proven that A is a (λ, μ)-intuitionistic fuzzy subgroup or (λ, μ)-intuitionistic fuzzy normal subgroup of a group G with operators if and only if cut sets of A are subgroup or normal subgroup of G. Thirdly, some properties are discussed. Finally, in the sense of M-group homomorphism between two classical groups, the image and the preimage of the (λ, μ)-intuitionistic fuzzy subgroups and (λ, μ)-intuitionistic fuzzy normal subgroups of groups with operators are studied, which enriches and expands the theory of the IFS and group.

II. Preliminaries

In this paper, we always assume $0 \le \lambda < \mu \le 1$.

Let $IFG[G]$ and $IFNG[G]$ be the intuitionistic fuzzy subgroups and intuitionistic fuzzy normal subgroups of G.

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Definition 1. [13] Let $A: X \to [0,1]$ be a mapping. If there exist $a \in (0,1]$ and $x \in A$ such that

$$A(y)=\begin{cases} a, & y=x; \\ 0, & y\ne x. \end{cases}$$

Then A is called a fuzzy point, and denoted by x_a.

Definition 2. [2] Let X be any nonempty set. An intuitionistic fuzzy subset A of X is an object of the following form

$$A=\left\{\left\langle x,\mu_A(x),\nu_A(x)\right\rangle \middle| x\in X\right\}.$$

where $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ define the degree of membership and the degree of non-membership of the element $x \in X$ respectively and for every $x \in X$,

$$0 \le \mu_A(x)+\nu_A(x) \le 1.$$

Let $IFS[X]$ be the IFS of X.

Definition 3. [2] Let X be any nonempty set, $A, B \in IFS[X]$ and

$$A=\left\{\left\langle x,\mu_A(x),\nu_A(x)\right\rangle \middle| x\in X\right\},$$
$$B=\left\{\left\langle x,\mu_B(x),\nu_B(x)\right\rangle \middle| x\in X\right\}.$$

The rules and operations are as follows:

1. $A\cap B=\left\{\left\langle x,\mu_A(x)\wedge\mu_B(x),\nu_A(x)\vee\nu_B(x)\right\rangle \middle| x\in X\right\}$;
2. $A\cup B=\left\{\left\langle x,\mu_A(x)\vee\mu_B(x),\nu_A(x)\wedge\nu_B(x)\right\rangle \middle| x\in X\right\}$;
3. $\underset{j\in J}{\cap} A_j=\left\{\left\langle x,\underset{j\in J}{\wedge}\mu_{A_j}(x),\underset{j\in J}{\vee}\nu_{A_j}(x)\right\rangle \middle| x\in X\right\}$;
4. $\underset{j\in J}{\cup} A_j=\left\{\left\langle x,\underset{j\in J}{\vee}\mu_{A_j}(x),\underset{j\in J}{\wedge}\nu_{A_j}(x)\right\rangle \middle| x\in X\right\}$.

where $A_j=\left\{\left\langle x,\mu_{A_j}(x),\nu_{A_j}(x)\right\rangle \middle| x\in X\right\}\in IFS[X]$, $j=1,2,\cdots$, J is the index sets.

Definition 4. [8] Let X, Y be any two nonempty sets and $f: X \to Y$ be a mapping. Let $A \in IFS[X]$ and

$$A=\left\{\left\langle x,\mu_A(x),\nu_A(x)\right\rangle \middle| x\in X\right\}.$$

Then $F_f : IFS[X] \to IFS[Y]$ and $F_f(A)$ are also the IFS of Y, and

$$F_f(A) = \left\{\left\langle y, F_f(\mu_A)(y), F_f(\nu_A)(y)\right\rangle y \in Y\right\}.$$

where

$$F_f(\mu_A)(y) = \begin{cases} \sup\left\{\mu_A(x) \middle| f(x) = y, x \in X\right\}, & f^{-1}(y) \neq \varnothing; \\ 0, & f^{-1}(y) = \varnothing. \end{cases}$$

$$F_f(\nu_A)(y) = \begin{cases} \inf\left\{\nu_A(x) \middle| f(x) = y, x \in X\right\}, & f^{-1}(y) \neq \varnothing; \\ 1, & f^{-1}(y) = \varnothing. \end{cases}$$

Definition 5. [8] Let X, Y be any two nonempty sets and $f : X \to Y$ be a mapping. Let $B \in IFS[Y]$ and

$$B = \left\{\left\langle y, \mu_B(y), \nu_B(y)\right\rangle \middle| y \in Y\right\}.$$

Then $F_f^{-1} : IFS[Y] \to IFS[X]$ and $F_f^{-1}(B)$ are also the IFS of X, and

$$F_f^{-1}(B) = \left\{\left\langle x, F_f^{-1}(\mu_B)(x), F_f^{-1}(\nu_B)(x)\right\rangle \middle| x \in X\right\}.$$

Definitions 4 and 5 are called the extension principle of IFS. Denote $\langle I\rangle = \{\langle a,b\rangle : a,b \in [0,1]\}$.

Definition 6. [12] Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in S\}$ be an IFS in a set S. For $\langle\alpha,\beta\rangle \in \langle I\rangle$, the set $A_{\langle\alpha,\beta\rangle} = \{x \in S : \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta\}$ is called a cut set of A.

Definition 7. [8] Let G be a group and

$$A = \left\{\left\langle x, \mu_A(x), \nu_A(x)\right\rangle \middle| x \in X\right\} \in IFS[G].$$

If for $\forall x, y \in G$,

1. $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$, $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$,
2. $\mu_A(x^{-1}) \geq \mu_A(x)$, $\nu_A(x^{-1}) \leq \nu_A(x)$.

Then A is called the intuitionistic fuzzy subgroup of G.

Definition 8. [11] Let $A = \left\{\left\langle x, \mu_A(x), \nu_A(x)\right\rangle \middle| x \in G\right\}$ be the intuitionistic fuzzy subgroup of G. If for $\forall x, y \in G$,

1. $\mu_A(xy) \vee \lambda \geq (\mu_A(x) \wedge \mu_A(y)) \wedge \mu$,
   $\nu_A(xy) \wedge \mu \leq (\nu_A(x) \vee \nu_A(y)) \vee \lambda$,
2. $\mu_A(x^{-1}) \vee \lambda \geq \mu_A(x) \wedge \mu$, $\nu_A(x^{-1}) \wedge \mu \leq \nu_A(x) \vee \lambda$.

Then A is called the $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of G.

Definition 9. [11] Let $A = \left\{\left\langle x, \mu_A(x), \nu_A(x)\right\rangle \middle| x \in G\right\}$ be the $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of G. If for $\forall x, y \in G$,

$$\mu_A(xyx^{-1}) \vee \lambda \geq \mu_A(y) \wedge \mu,\ \nu_A(xyx^{-1}) \wedge \mu \leq \nu_A(y) \vee \lambda.$$

Then A is called the $(\lambda, \mu)-$ intuitionistic fuzzy normal subgroup of G.

Definition 10. [3] A group with operators in an algebraic system consisting of a group, a set M and a function defined in the product set $M \times G$ and having values in G such that, if ma denotes the element in G determined by the element a of G and the element m of M, then

$$m(ab) = (ma)(mb),$$

Holds for any a, b in G, m in M. We shall usually use the phrase "G is an $M-$ group" to a group with operators.

Definition 11. [14] A subgroup A of $M-$ group is said to be an $M-$ subgroup if mx in A for every m in M and x in A.

Definition 12. [14] Let G_1 and G_2 both be $M-$ groups, f be a homomorphism from G_1 onto G_2

$$f(mx) = mf(x), m \in M, x \in G,$$

Then f is called a $M-$ homomorphism.

Proposition 1. [10] Let G be a $M-$ group, e be the identity element of G, and $A \in IFG[G]$. Then for $\forall x \in G$,

$$\mu_A(x) \leq \mu_A(e),\ \nu_A(x) \geq \nu_A(e).$$

Proposition 2. [11] Let $A = \left\{\left\langle x, \mu_A(x), \nu_A(x)\right\rangle \middle| x \in G\right\}$ be the $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of G and e be the identity element. Then

$$\mu_A(e) \vee \lambda \geq \mu_A(x) \wedge \mu,\ \nu_A(e) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

Proposition 3. [11] Let A be the intuitionistic fuzzy subset. Then A is a $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of G iff for $\forall x, y \in G$,

$$\mu_A(x^{-1}y) \vee \lambda \geq (\mu_A(x) \wedge \mu_A(y)) \wedge \mu,$$
$$\nu_A(x^{-1}y) \wedge \mu \leq (\nu_A(x) \vee \nu_A(y)) \vee \lambda.$$

Proposition 4. [11] Let A be the intuitionistic fuzzy subset. Then A is a $(\lambda, \mu)-$ intuitionistic fuzzy subgroup of G iff for $\forall \alpha, \beta \in (\lambda, \mu)$, $A_{\langle\alpha,\beta\rangle}$ is the subgroup when $A_{\langle\alpha,\beta\rangle} \neq \varnothing$, where $\langle\alpha,\beta\rangle \in \langle I\rangle$.

Proposition 5. [11] Let $f: G_1 \to G_2$ be a surjective homomorphism of groups. If A be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of G_1, then $f(A)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of G_2.

Proposition 6. [11] Let $f: G_1 \to G_2$ be a homomorphism of groups. If B be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of G_2, then $f^{-1}(B)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of G_1.

Proposition 7. [11] Let A be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of G .Then A is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of G iff for $\forall x, y \in G$,

$$\mu_A(xy) \vee \lambda \geq \mu_A(yx) \wedge \mu,\ \nu_A(xy) \wedge \mu \leq \nu_A(yx) \vee \lambda.$$

Proposition 8. [11] Let $f: G_1 \to G_2$ be a surjective homomorphism of groups. If A be a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of G_1, then $f(A)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of G_2.

Proposition 9. [11] Let $f: G_1 \to G_2$ be a homomorphism of groups. If B be a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of G_2, then $f^{-1}(B)$ is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of G_1.

III. $(\lambda,\mu)-$ Intuitionistic Fuzzy Subgroups of Groups with Operators

Definition 13. Let G be a $M-$ group and A be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup. If for $\forall x \in G,\ m \in M$,

$$\mu_A(mx) \vee \lambda \geq \mu_A(x) \wedge \mu,\ \nu_A(mx) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

Then A is called a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of a group G with operators, and denoted by a $(\lambda,\mu)-\ M-$ intuitionistic fuzzy subgroup.

Let $(\lambda,\mu)-\ IFMG[G]$ be the $(\lambda,\mu)-\ M-$ intuitionistic fuzzy subgroups of $M-$ group G.

Proposition 10. Let G be a $M-$ group, $A \in (\lambda,\mu)-$ $IFMG[G]$ and e be the identity element of G. Then

$$\mu_A(me) \vee \lambda \geq \mu_A(x) \wedge \mu,\ \nu_A(me) \wedge \mu \leq \nu_A(x) \vee \lambda.$$

Proof. For $\forall x \in G,\ m \in M$,

$$\mu_A(me) \vee \lambda \geq \mu_A(e) \wedge \mu \geq \mu_A(x) \wedge \mu,$$
$$\nu_A(me) \wedge \mu \leq \nu_A(e) \vee \lambda \leq \nu_A(x) \vee \lambda.$$

Proposition 11. Let A be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group G. Then $A \in (\lambda,\mu)-\ IFMG[G]$ iff for $\forall x, y \in G,\ m \in M$,

$$\mu_A\left(m\left(x^{-1}y\right)\right) \vee \lambda \geq \left(\mu_A(mx) \wedge \mu_A(my)\right) \wedge \mu,$$
$$\nu_A\left(m\left(x^{-1}y\right)\right) \wedge \mu \leq \left(\nu_A(mx) \vee \nu_A(my)\right) \vee \lambda.$$

Proof. For $\forall x, y \in G,\ m \in M$,

$$\begin{aligned}
\mu_A\left(m\left(x^{-1}y\right)\right) \vee \lambda &= \left(\mu_A\left(\left(mx^{-1}\right)(my)\right) \vee \lambda\right) \vee \lambda \\
&\geq \left(\left(\mu_A\left(mx^{-1}\right) \wedge \mu_A(my)\right) \wedge \mu\right) \vee \lambda \\
&= \left(\mu_A\left(mx^{-1}\right) \vee \lambda\right) \wedge \mu_A(my) \wedge \mu \\
&= \left(\mu_A(mx)^{-1} \vee \lambda\right) \wedge \mu_A(my) \wedge \mu \\
&\geq \left(\mu_A(mx) \wedge \mu\right) \wedge \mu_A(my) \wedge \mu \\
&= \left(\mu_A(mx) \wedge \mu_A(my)\right) \wedge \mu,
\end{aligned}$$

$$\begin{aligned}
\nu_A\left(m\left(x^{-1}y\right)\right) \wedge \mu &= \left(\nu_A\left(\left(mx^{-1}\right)(my)\right) \wedge \mu\right) \wedge \mu \\
&\leq \left(\left(\nu_A\left(mx^{-1}\right) \vee \nu_A(my)\right) \vee \lambda\right) \wedge \mu \\
&= \left(\nu_A\left(mx^{-1}\right) \wedge \mu\right) \vee \nu_A(my) \vee \lambda \\
&= \left(\nu_A(mx)^{-1} \wedge \mu\right) \vee \nu_A(my) \vee \lambda \\
&\leq \left(\nu_A(mx) \vee \lambda\right) \vee \nu_A(my) \vee \lambda \\
&= \left(\nu_A(mx) \vee \nu_A(my)\right) \vee \lambda.
\end{aligned}$$

Conversely, for $\forall x \in G,\ m \in M$, let $y = e$,

$$\begin{aligned}
\mu_A(m(xe)) \vee \lambda &= \left(\mu_A\left(m\left(\left(x^{-1}\right)^{-1} e\right)\right) \vee \lambda\right) \vee \lambda \\
&\geq \left(\left(\mu_A\left(mx^{-1}\right) \wedge \mu_A(me)\right) \wedge \mu\right) \vee \lambda \\
&= \left(\mu_A(me) \vee \lambda\right) \wedge \left(\mu_A\left(mx^{-1}\right) \wedge \mu\right) \\
&\geq \left(\mu_A(e) \wedge \mu\right) \wedge \left(\mu_A\left(x^{-1}\right) \wedge \mu\right) \\
&\geq \left(\mu_A(e) \wedge \mu\right) \wedge \left(\mu_A(x) \wedge \mu\right) \\
&= \mu_A(x) \wedge \mu,
\end{aligned}$$

$$\begin{aligned}\nu_A\left(m\left(xe\right)\right)\wedge\mu &= \left(\nu_A\left(m\left(\left(x^{-1}\right)^{-1}e\right)\right)\wedge\mu\right)\wedge\mu \\ &\le \left(\left(\nu_A\left(mx^{-1}\right)\wedge\nu_A\left(me\right)\right)\vee\lambda\right)\wedge\mu \\ &= \left(\nu_A\left(me\right)\wedge\mu\right)\vee\left(\nu_A\left(mx^{-1}\right)\vee\lambda\right) \\ &\le \left(\nu_A\left(e\right)\vee\lambda\right)\vee\left(\nu_A\left(x^{-1}\right)\vee\lambda\right) \\ &\le \left(\nu_A\left(e\right)\vee\lambda\right)\vee\left(\nu_A\left(x\right)\vee\lambda\right) \\ &= \nu_A\left(x\right)\vee\lambda.\end{aligned}$$

Thus, $A\in(\lambda,\mu)-IFMG[G]$.

Proposition 12. Let A be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group G. Then $A\in(\lambda,\mu)-IFMG[G]$ iff for $\forall\alpha,\beta\in(\lambda,\mu)$, $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of G when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$, where $\langle\alpha,\beta\rangle\in\langle I\rangle$.

Proof. For $\forall\alpha,\beta\in(\lambda,\mu)$, $x\in A_{\langle\alpha,\beta\rangle}$ when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$. Therefore, $\mu_A\left(x\right)\ge\alpha$, $\nu_A\left(x\right)\le\beta$. Then,

$$\mu_A\left(mx\right)\vee\lambda\ge\mu_A\left(x\right)\wedge\mu\ge\alpha>\lambda,$$
$$\nu_A\left(mx\right)\wedge\mu\le\nu_A\left(x\right)\vee\lambda\le\beta<\mu.$$

We have $\mu_A\left(mx\right)\ge\alpha$, $\nu_A\left(mx\right)\le\beta$. Therefore, $mx\in A_{\langle\alpha,\beta\rangle}$. Thus, $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of G when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$.

Conversely, for $\forall\alpha,\beta\in(\lambda,\mu)$, we get the information from Proposition 4 that A is a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of group G. Besides, for $\forall x\in G$, let $\alpha=\mu_A\left(x\right)\wedge\mu$, $\beta=\nu_A\left(x\right)\vee\lambda$. Therefore, $\mu_A\left(x\right)\ge\alpha$, $\nu_A\left(x\right)\le\beta$, and $x\in A_{\langle\alpha,\beta\rangle}$. And $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of G when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$. Thus, $mx\in A_{\langle\alpha,\beta\rangle}$. We have $\mu_A\left(mx\right)\ge\alpha$, $\nu_A\left(mx\right)\le\beta$. And

$$\mu_A\left(mx\right)\vee\lambda\ge\alpha=\mu_A\left(x\right)\wedge\mu,$$
$$\nu_A\left(mx\right)\wedge\mu\le\beta=\nu_A\left(x\right)\vee\lambda.$$

Thus, $A\in(\lambda,\mu)-IFMG[G]$.

Proposition 13. Let G be a $M-$ group and $A,B\in(\lambda,\mu)-IFMG[G]$. Then $A\cap B\in(\lambda,\mu)-IFMG\left[G\right]$.

Proof. Let

$$A=\left\{\left\langle x,\mu_A\left(x\right),\nu_A\left(x\right)\right\rangle\middle|x\in G\right\},$$
$$B=\left\{\left\langle x,\mu_B\left(x\right),\nu_B\left(x\right)\right\rangle\middle|x\in G\right\}.$$

Then

$$A\cap B=\left\{\left\langle x,\mu_A\left(x\right)\wedge\mu_B\left(x\right),\nu_A\left(x\right)\vee\nu_B\left(x\right)\right\rangle\middle|x\in G\right\}.$$

Let

$\mu_{A\cap B}\left(x\right)=\mu_A\left(x\right)\wedge\mu_B\left(x\right)$, $\nu_{A\cap B}\left(x\right)=\nu_A\left(x\right)\vee\nu_B\left(x\right)$.

First, we provide that $A\cap B$ is $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group G. For $\forall x,y\in G$, on the one hand,

$$\begin{aligned}\mu_{A\cap B}\left(xy\right)\vee\lambda &= \left(\mu_A\left(xy\right)\wedge\mu_B\left(xy\right)\right)\vee\lambda \\ &= \left(\left(\mu_A\left(xy\right)\wedge\mu_B\left(xy\right)\right)\vee\lambda\right)\vee\lambda \\ &= \left(\mu_A\left(xy\right)\vee\lambda\right)\wedge\left(\mu_B\left(xy\right)\vee\lambda\right) \\ &\ge \left(\left(\mu_A\left(x\right)\wedge\mu_A\left(y\right)\right)\wedge\mu\right)\wedge\left(\left(\mu_B\left(x\right)\wedge\mu_B\left(y\right)\right)\wedge\mu\right) \\ &= \left(\mu_A\left(x\right)\wedge\mu_B\left(x\right)\right)\wedge\left(\mu_A\left(y\right)\wedge\mu_B\left(y\right)\right)\wedge\mu \\ &= \left(\mu_{A\cap B}\left(x\right)\wedge\mu_{A\cap B}\left(y\right)\right)\wedge\mu.\end{aligned}$$

Similarly, $\nu_{A\cap B}\left(xy\right)\wedge\mu\le\left(\nu_{A\cap B}\left(x\right)\vee\nu_{A\cap B}\left(y\right)\right)\vee\lambda.$ On the other hand,

$$\begin{aligned}\mu_{A\cap B}\left(x^{-1}\right)\vee\lambda &= \left(\mu_A\left(x^{-1}\right)\wedge\mu_B\left(x^{-1}\right)\right)\vee\lambda \\ &= \left(\left(\mu_A\left(x^{-1}\right)\wedge\mu_B\left(x^{-1}\right)\right)\vee\lambda\right)\vee\lambda \\ &= \left(\mu_A\left(x^{-1}\right)\vee\lambda\right)\wedge\left(\mu_B\left(x^{-1}\right)\vee\lambda\right) \\ &\ge \left(\mu_A\left(x\right)\wedge\mu\right)\wedge\left(\mu_B\left(x\right)\wedge\mu\right) \\ &= \left(\mu_A\left(x\right)\wedge\mu_B\left(x\right)\right)\wedge\mu \\ &= \mu_{A\cap B}\left(x\right)\wedge\mu.\end{aligned}$$

Similarly, $\nu_{A\cap B}\left(x^{-1}\right)\wedge\mu\le\nu_{A\cap B}\left(x\right)\vee\lambda$. Thus, $A\cap B$ is $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group G.Then for $\forall x\in G$, $m\in M$, $A,B\in(\lambda,\mu)-IFMG\left[G\right]$.

The following can be obtained from Definition 15. On the one hand,

$$\begin{aligned}\mu_{A\cap B}\left(mx\right)\vee\lambda &= \left(\left(\mu_A\left(mx\right)\wedge\mu_B\left(mx\right)\right)\vee\lambda\right)\vee\lambda \\ &= \left(\mu_A\left(mx\right)\vee\lambda\right)\wedge\left(\mu_B\left(mx\right)\vee\lambda\right) \\ &\ge \left(\mu_A\left(x\right)\wedge\mu\right)\wedge\left(\mu_B\left(x\right)\wedge\mu\right) \\ &= \left(\mu_A\left(x\right)\wedge\mu_B\left(x\right)\right)\wedge\mu \\ &= \mu_{A\cap B}\left(x\right)\wedge\mu.\end{aligned}$$

On the other hand,

$$\begin{aligned}\nu_{A\cap B}(mx)\wedge\mu &= \left(\left(\nu_A(mx)\vee\nu_B(mx)\right)\wedge\mu\right)\wedge\mu\\ &= \left(\nu_A(mx)\wedge\mu\right)\vee\left(\nu_B(mx)\wedge\mu\right)\\ &\le \left(\nu_A(x)\vee\lambda\right)\vee\left(\nu_B(x)\vee\lambda\right)\\ &= \left(\nu_A(x)\vee\nu_B(x)\right)\vee\lambda\\ &= \nu_{A\cap B}(x)\vee\lambda.\end{aligned}$$

Thus, $A\cap B\in(\lambda,\mu)-IFMG[G]$.

Proposition 14. Let G_1, G_2 be $M-$ group, $f:G_1\to G_2$ be a $M-$ surjective homomorphism of groups, and $A\in(\lambda,\mu)-IFMG[G_1]$. Then $f(A)\in(\lambda,\mu)-IFMG[G_2]$.

Proof. Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle | x\in G_1\}$. We get the information from Definition 4 that

$$f(A)=\left\{<y,\mu_{f(A)}(y),\nu_{f(A)}(y)>y\in G_2\right\}.$$

It is can be obtained from Proposition 5 that $f(A)$ is $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of G_2.

Because f is $M-$ surjective homomorphism, we have $f^{-1}(y)\neq\varnothing$ for $\forall y\in G_2$, $m\in M$. For $\forall x\in f^{-1}(y)$, then

$$x^{-1}\in f^{-1}(y^{-1}),\quad f(mx)=mf(x)=my,\quad x\in f^{-1}(y).$$

The following can be obtained from Definition 15. On the one hand,

$$\mu_A(mx)\vee\lambda\ge\mu_A(x)\wedge\mu,$$

and

$$\begin{aligned}\mu_{f(A)}(my)\vee\lambda &= \sup_{x\in f^{-1}(my)}\mu_A(x)\vee\lambda\\ &= \sup_{f(x)=my}\mu_A(x)\vee\lambda\\ &\ge \sup_{f(mx')=my}\mu_A(mx')\vee\lambda\\ &= \sup_{\substack{mf(x')=my\\ mx'\in G_1}}\mu_A(mx')\vee\lambda\\ &\ge \sup_{\substack{f(x')=y\\ x'\in G_1}}\mu_A(x')\wedge\mu\\ &= \mu_{f(A)}(y)\wedge\mu.\end{aligned}$$

On the other hand,

$$\nu_A(mx)\wedge\mu\le\nu_A(x)\vee\lambda,$$

and

$$\begin{aligned}\nu_{f(A)}(my)\wedge\mu &= \inf_{x\in f^{-1}(my)}\nu_A(x)\wedge\mu\\ &= \inf_{f(x)=my}\nu_A(x)\wedge\mu\\ &\le \inf_{f(mx')=my}\nu_A(mx')\wedge\mu\\ &= \inf_{\substack{mf(x')=my\\ mx'\in G_1}}\nu_A(mx')\wedge\mu\\ &\le \inf_{\substack{f(x')=y\\ x'\in G_1}}\nu_A(x')\vee\lambda\\ &= \nu_{f(A)}(y)\vee\lambda.\end{aligned}$$

Thus, $f(A)\in(\lambda,\mu)-IFMG[G_2]$.

Proposition 15. Let G_1, G_2 be $M-$ group, $f:G_1\to G_2$ be a $M-$ surjective homomorphism of groups, and $B\in(\lambda,\mu)-IFMG[G_2]$. Then $f^{-1}(B)\in(\lambda,\mu)-IFMG[G_1]$.

Proof. Let $B=\{\langle y,\mu_B(y),\nu_B(y)\rangle | y\in G_2\}$. We get the information from Definition 5 that

$$f^{-1}(B)=\left\{\left\langle x,\mu_{f^{-1}(B)}(x),\nu_{f^{-1}(B)}(x)\right\rangle \middle| x\in G_1\right\}.$$

It is can be obtained from Proposition 6 that $f^{-1}(B)$ is $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of G_1.

Because $B\in(\lambda,\mu)-IFMG[G_2]$, we have for $\forall x\in G_1$, $m\in M$,

$$\begin{aligned}\mu_{f^{-1}(B)}(mx)\vee\lambda &= \mu_B(f(mx))\vee\lambda\\ &= \mu_B(mf(x))\vee\lambda\\ &\ge \mu_B(f(x))\wedge\mu\\ &= \mu_{f^{-1}(B)}(x)\wedge\mu.\end{aligned}$$

$$\begin{aligned}\nu_{f^{-1}(B)}(mx)\wedge\mu &= \nu_B(f(mx))\wedge\mu\\ &= \nu_B(mf(x))\wedge\mu\\ &\le \nu_B(f(x))\vee\lambda\\ &= \nu_{f^{-1}(B)}(x)\vee\lambda.\end{aligned}$$

Thus, $f^{-1}\left(B\right)\in(\lambda,\mu)-IFMG\left[G_1\right]$.

IV. $(\lambda,\mu)-$ Intuitionistic Fuzzy Normal Subgroups of Groups with Operators

Definition 14. Let G be a $M-$ group, $A\in(\lambda,\mu)-$ $IFMG[G]$ and A be a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup. Then A is called a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of a group G with operators, denoted by a $(\lambda,\mu)-$ $M-$ intuitionistic fuzzy normal subgroup.

Let $(\lambda,\mu)-$ $IFMNG[G]$ be the $(\lambda,\mu)-$ $M-$ intuitionistic fuzzy normal subgroups of $M-$ group G.

Proposition 16. Let G be a $M-$ group and $A\in(\lambda,\mu)-$ $IFMG[G]$. Then $A\in(\lambda,\mu)-$ $IFMNG[G]$ iff for $\forall x,y\in G$, $m\in M$,

$$\mu_A\left(m\left(xy\right)\right)\vee\lambda\geq\mu_A\left(m\left(yx\right)\right)\wedge\mu,$$
$$\nu_A\left(m\left(xy\right)\right)\wedge\mu\leq\nu_A\left(m\left(yx\right)\right)\vee\lambda.$$

Proof. For $\forall x,y\in G$, $m\in M$,

$$\begin{aligned}\mu_A\left(m\left(xy\right)\right)\vee\lambda&=\mu_A\left(m\left(x(yx)x^{-1}\right)\right)\vee\lambda\\&=\mu_A\left((mx)(m(yx))(mx)^{-1}\right)\vee\lambda\\&\geq\mu_A\left(m\left(yx\right)\right)\wedge\mu,\end{aligned}$$

$$\begin{aligned}\nu_A\left(m\left(xy\right)\right)\wedge\mu&=\nu_A\left(m\left(x(yx)x^{-1}\right)\right)\wedge\mu\\&=\nu_A\left((mx)(m(yx))(mx)^{-1}\right)\wedge\mu\\&\leq\nu_A\left(m\left(yx\right)\right)\vee\lambda.\end{aligned}$$

Conversely, for $\forall x\in G$, $m\in M$, let $y=e$. We get the information from Proposition 7 that A is a $(\lambda,\mu)-$ intuitionistic fuzzy normal subgroup of G. $A\in(\lambda,\mu)-$ $IFMG[G]$, thus $A\in(\lambda,\mu)-$ $IFMNG[G]$.

Proposition 17. Let A be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group G. Then $A\in(\lambda,\mu)-$ $IFMNG[G]$ iff for $\forall x,y\in G$, $m\in M$,

1. $\mu_A\left(m\left(x^{-1}y\right)\right)\vee\lambda\geq\left(\mu_A\left(mx\right)\wedge\mu_A\left(my\right)\right)\wedge\mu,$
   $\nu_A\left(m\left(x^{-1}y\right)\right)\wedge\mu\leq\left(\nu_A\left(mx\right)\vee\nu_A\left(my\right)\right)\vee\lambda.$
2. $\mu_A\left(m\left(xy\right)\right)\vee\lambda\geq\mu_A\left(m\left(yx\right)\right)\wedge\mu,$
   $\nu_A\left(m\left(xy\right)\right)\wedge\mu\leq\nu_A\left(m\left(yx\right)\right)\vee\lambda.$

Proposition 18. Let A be a $(\lambda,\mu)-$ intuitionistic fuzzy subgroup of $M-$ group G. Then $A\in(\lambda,\mu)-$ $IFMNG[G]$ iff for $\forall\alpha,\beta\in(\lambda,\mu)$, $A_{\langle\alpha,\beta\rangle}$ is $M-$ normal subgroup of G when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$, where $\langle\alpha,\beta\rangle\in\langle I\rangle$.

Proof. We get the information from Proposition 12 that for $\forall\alpha,\beta\in(\lambda,\mu)$, $A_{\langle\alpha,\beta\rangle}$ is $M-$ subgroup of G when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$. For $\forall x\in G$, $\forall y\in A_{\langle\alpha,\beta\rangle}$, we have $\mu_A\left(y\right)\geq\alpha$, $\nu_A\left(y\right)\leq\beta$. Therefore,

$$\mu_A\left(xyx^{-1}\right)\vee\lambda\geq\mu_A\left(y\right)\wedge\mu\geq\alpha\wedge\mu>\lambda,$$
$$\nu_A\left(xyx^{-1}\right)\wedge\mu\leq\nu_A\left(y\right)\vee\lambda\leq\beta<\mu.$$

Then, $\mu_A\left(xyx^{-1}\right)\geq\alpha$, $\nu_A\left(xyx^{-1}\right)\leq\beta$, and $xyx^{-1}\in A_{\langle\alpha,\beta\rangle}$. Thus, $A_{\langle\alpha,\beta\rangle}$ is $M-$ normal subgroup of G when $A_{\langle\alpha,\beta\rangle}\neq\varnothing$.

Conversely, we get the information from Proposition 12 that $A\in(\lambda,\mu)-$ $IFMG[G]$. And for $\forall x,y\in G$, $m\in M$, we have

$$\mu_A\left(m\left(x^{-1}y\right)\right)\vee\lambda\geq\left(\mu_A\left(mx\right)\wedge\mu_A\left(my\right)\right)\wedge\mu,$$
$$\nu_A\left(m\left(x^{-1}y\right)\right)\wedge\mu\leq\left(\nu_A\left(mx\right)\vee\nu_A\left(my\right)\right)\vee\lambda.$$

If exist $x_0,y_0\in G$, satisfying

$$\mu_A\left(x_0y_0x_0^{-1}\right)\vee\lambda\leq\mu_A\left(y_0\right)\wedge\mu,$$
$$\nu_A\left(x_0y_0x_0^{-1}\right)\wedge\mu\geq\nu_A\left(y_0\right)\vee\lambda.$$

Let $\alpha=\mu_A\left(y_0\right)\wedge\mu$, $\beta=\nu_A\left(y_0\right)\vee\lambda$. Then $y_0\in A_{\langle\alpha,\beta\rangle}$. But $x_0y_0x_0^{-1}\notin A_{\langle\alpha,\beta\rangle}$, it is in contradiction with $A_{\langle\alpha,\beta\rangle}$ is $M-$ normal subgroup of G. Therefore, for $\forall x,y\in G$,

$$\mu_A\left(xyx^{-1}\right)\vee\lambda\geq\mu_A\left(y\right)\wedge\mu,$$
$$\nu_A\left(xyx^{-1}\right)\wedge\mu\leq\nu_A\left(y\right)\vee\lambda.$$

Thus, $A\in(\lambda,\mu)-$ $IFMNG[G]$.

The following proposition can be easily proved.

Proposition 19. Let G_1,G_2 be $M-$ group, $f:G_1\to G_2$ be a $M-$ surjective homomorphism of groups, and $A\in(\lambda,\mu)-$ $IFMNG[G_1]$. Then $f\left(A\right)\in(\lambda,\mu)-IFMNG\left[G_2\right]$.

Proposition 20. Let G_1,G_2 be $M-$ group, $f:G_1\to G_2$ be a $M-$ surjective homomorphism of groups, and $B\in(\lambda,\mu)-$ $IFMNG[G_2]$. Then $f^{-1}\left(B\right)\in(\lambda,\mu)-IFMNG\left[G_1\right]$.

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