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Investigation of Thermal and Mechanical Loading on Functional Graded Material Plates

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***Abstract*—**This paper interested in the mechanical deformation behavior of shear deformable functionally graded ceramic-metal (FGM) plates. Theoretical formulations are based on power law theory when build up functional graded material. The mechanical properties of the plate are graded in the thickness direction according to a power-law Displacement and stress is obtained using finite element method (FEM). The load is supposed to be a uniform distribution over the plate surface (XY plane) and varied in the thickness direction only. An FGM's gradation in material properties allows the designer to tailor material response to meet design criteria. An FGM made of ceramic and metal can provide the thermal protection and load carrying capability in one material thus eliminating the problem of thermo-mechanical deformation behavior. This thesis will explore analysis of FGM flat plates and shell panels, and their applications to r structural problems. FGMs are first characterized as flat plates under pressure in order to understand the effect variation of material properties has on structural response. In addition, results are compared to published results in order to show the accuracy of modeling FGMs using ABAQUS software.

***Keywords*—**Functionally graded material, finite element method, thermal and structural loading.

I. Introduction

THIS study, focus on functionally graded materials with mechanical loadings. The term "functionally graded materials (FGMs)" is now widely used by the materials community for a class of materials exhibiting spatially inhomogeneous microstructures and properties. Graded materials in themselves are not something new, but what is exciting about them is the realization that gradients can be designed at a micro structural level to tailor specific materials for their functional performance in particular applications.

Functionally Graded Materials (FGM) have continuous variation of material properties from one surface to another unlike a composite which has stepped (or discontinuous) material properties. The gradation of properties in an FGM reduces the thermal stresses, residual stresses, and stress concentrations found in traditional composites. An advanced composite materials known as functionally graded material (FGM) have received an appreciable consideration in structural engineering design, especially when the materials are subjected to extremely high thermal loading. The material property of the FGM can be tailored to accomplish the specific demands in different engineering utilization to achieve the advantage of the properties of individual material. This is possible due to the material composition of the FGM changes sequentially in a preferred direction. The applicability of this material is that it eliminates the interface problem due to proficient and continuous change of material properties from one surface to the other [1], [2].

The mechanical deformation of FGM structures have attracted the attention of many researchers in the past few years in different engineering applications which include design of nuclear power plants, heat engine components and aerospace structures etc. A number of plate theories are available to analyze the deformations of composite plates. The foremost constraint of using the classical Kirchhoff plate theory (CLPT) is that it ignores transverse shear effects and consequently provides reasonable results for relatively thin plates [3]. To abstain from the said complication, earlier attempts were made by Reissner [4] and Mindlin [5]. However, a shear-correction factor is needed to eliminate the problem of a constant transverse shear stress distribution. Several authors have used the first order shear deformation theory (FSDT) and higher order shear deformation theory HSDT kinematics to analyze the deformation characteristics of FGM plates. For example, Reddy [6] presented the mathematical formulation in conjunction with finite element model, based on third order shear deformation theory for static and dynamic analysis of the FGM plates.

"Functionally Graded Materials (FGM) are a class of composites that have continuous variation of material properties from one surface to another and thus alleviate the stress concentration found in laminated composites. The gradation in properties of the materiel reduces thermal stresses, residual stresses, and stress concentrations" [7].

A functionally graded structure is defined as, "those in which the volume fractions of two or more materials are varied continuously as a function of position along certain dimension(s) of the structure to achieve a require function" [6]. Because the properties change throughout the dimension (typically the thickness) of a material, FGMs can provide designers with tailored material response and exceptional performance in thermal and mechanical environments. For example, the Space Shuttle utilizes ceramic tiles as thermal protection from heat generated during re-entry into the Earth's atmosphere. However, these tiles are prone to cracking at the tile / superstructure interface due to differences in thermal expansion coefficients. Typically, FGMs are made from a mixture of metals and ceramics and are further characterized by a smooth and continuous change of the mechanical

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properties from one surface to another. The ceramic constituents of FGMs are able to withstand high temperature environments due to their better thermal resistance characteristics, while the metal constituents provide stronger mechanical performance and reduce the possibility of catastrophic fracture. FGMs are now developed for an increased use as face sheets of sandwich structures, and in such a case the face sheet may be modeled as an FGM plate resting on an elastic foundation.

An FGM made of ceramic and metal can provide the thermal protection and load carrying capability in one material thus eliminating the problem of cracked tiles found on the Space Shuttle.

An FGM composed of ceramic on the outside surface and metal on the inside surface eliminates the abrupt transition between coefficients of thermal expansion, offers thermal/corrosion protection, and provides load carrying capability. This is possible because the material composition of an FGM changes gradually through-the-thickness; therefore, stress concentrations from abrupt changes in material properties (i.e., coefficients of thermal expansion) are eliminated. A real-world application example was given by [8].

Conventional aluminum was selected for the main structure and a layer of heat resistant material for protecting it. The properties of aluminum demand that the maximum temperature of the vehicle's structure be kept below 175°C in operation. But aero thermal heating during the re-entry process creates high surface temperature which is well above the melting point of aluminum (660°C). Thus, an effective insulator was needed. A silica-based insulation material was decided for the heat-resistant tiles and other coverings to protect the Shuttle's airframe. Fig. 1 shows seven different materials which cover the external surface of the Space Shuttle according to the temperature variation during the re-entry.

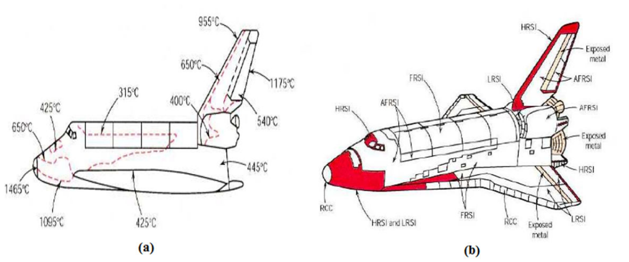


Fig. 1 Thermal protection system in Space Shuttle, (a) Temperature variation during re-entry (b) Location of different materials [8]

The materials were chosen by their weight efficiency and stability at high temperature. The areas of the highest surface temperature in the Shuttle, the forward nose cap and the leading edge of the wings, are made with Reinforced Carbon-Carbon (RCC). There are two main types of tiles, referred to as Low-temperature Reusable Surface Insulation (LRSI) and High-temperature Reusable Surface Insulation (HRSI). Relatively low temperature of surface where the maximum surface temperature runs between 370 and 650°C is covered by LRSI. HRSI covers the areas where the maximum surface temperature runs between 650 and 1,260°C. Metallic thermal production system (TPS) is considered as a much-needed alternative to the ceramic-based brittle tile and thermal-blanket surface insulation currently used on the Space Shuttle. Metallic thermal production system offers the significant advantages [9]. This study will explore FGM applications in severe thermal environments, such as aerospace and space vehicles. FGM performance is first characterized under thermal environments and mechanical loading in order to understand the unique characteristics of FGMs and to compare FGM structural response to traditional metal structure. The results for simply supported square and rectangular plates based on the finite element approach are compared with those obtained analytically for classical laminated plate theory [10] and the higher-order shear deformation theories [7], [11]. The results are completely similar to those analytical solutions.

II. Methodology

A. Research Focus and Developments

In this study, to establish the methodology used to model FGMs with finite elements. Coordinate systems, boundary conditions, theoretical formulation and details of the finite element analysis are discussed or defined. Units are as follows: length [m], pressure [N/m^2], temperature [°C], expansion [1/°C], density [kg/m^3], conductivity [W/m·ºC], and heat transfer film coefficient [W/m^2·°C].

Displacements in the x, y, and z directions are noted by u, v, and w, respectively. Rotation about the x and y axes are noted by dw/dx and dw/dy, respectively.

Because the material properties of the FGM change throughout the thickness, the numerical model must be broken up into various "slices" in order to capture the change in properties. These "slices" capture a finite portion of the thickness and are treated like isotropic materials. Material properties are calculated at the mid-plane of each of these "slices" using the power law equation previously outlined. The "slices" and their associated properties are then layered together to establish the through-the-thickness variation of material properties. Although the layered "slices" do not reflect the gradual change in material properties, a sufficient number of "slices" can reasonably approximate the material gradation. Fig. 2 shows how the thickness has been discretized into nine slices, that ply one is at the bottom of the plate, and that the origin of the z axis is at the mid-plane of the plate with +z in the direction of the top surface.

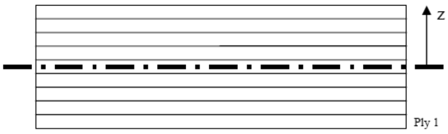


Fig. 2 Origin of z-axis

An FGM is defined to be a material which has a continuous gradation through the- thickness (h). One side of the material is typically ceramic and the other side is typically metal. A mixture of the two materials composes the through-the-thickness characteristics. This material variation is dictated by a parameter, "n." At n = 0 the plate is a fully ceramic plate while at n = ∞ the plate is fully metal. Material properties are dependent on the n value and the position in the plate and vary according to a power law. "The typical material property P is varied through the plate thickness according to the expressions (a power law).

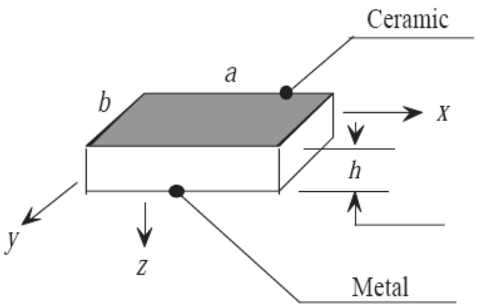


Fig. 3 Schematic diagram and dimension of plate

Material properties are given Table I. Ceramics and metal combinations are Zircon (ZrO_2) - Aluminum, Aluminum Oxide (Al_2O_3) - Steel (SUS304), Silicon Nitride (Si_3N_4) - Nickel (Ti-6Al-4V) in general. In this study, zircon and aluminum are used when functional graded material is created. Material property varied though the plate thickness.

TABLE I
MATERIAL PROPERTIES FOR FGM

	Material	E	ν	ρ
Ceramics	Zircon (ZrO_2)	151 GPa	0.29	3000 kg/m^3
	Aluminum Oxide (Al_2O_3)	320.24 GPa	0.26	3750 kg/m^3
	Silicon Nitride (Si_3N_4)	322.27 GPa	0.24	2370 kg/m^3
Metal	Aluminum	70 GPa	0.26	2707 kg/m^3
	Steel (SUS304)	207 GPa	0.31	8166 kg/m^3
	Nickel (Ti-6Al-4V)	105.70 GPa	0.29	4429 kg/m^3

III. PROBLEM FORMULATION

A. Theoretical Formulation of FGM

A schematic diagram of the problem studied and the rectangular Cartesian coordinate system used describe the mechanical deformations of the FGM plate are shown in Fig. 3. It is assumed that the material properties of FGM plate varies in the thickness direction only, such that the top surface (z = h/2) of the plate is ceramic rich, whereas the bottom surface (z = −h/2) is metal rich. The effective material properties at an arbitrary point within the structural domain, like Young's modulus E, Poisson's ratio ν, mass density ρ, thermal expansion coefficient α, of the functionally graded plate are the effective material properties P. These properties are position dependent and can be expressed as,

$$P = P_t x V_t(z) + P_b x V_b(z) \tag{1}$$

where P_t and P_b denote the particular property being considered at the top and bottom faces of the plate, respectively, and n is the parameter that dictates the material variation profile through-the-thickness [7]. The variation of the volume fraction, V_f, of ceramic through-the-thickness of the plate is an indicator of the volumetric fraction of ceramic at a given location. This volume fraction is based on the mixture of metal and ceramic and is an indicator of the material composition (volumetric wise) at any given location in the thickness. If the volume fraction of ceramic is defined as Vf then the volume fraction of metal is the remainder of the material, or 1- Vf. E, G, ρ, α, and k vary according to the power law and their calculated values are entered into ABAQUS accordingly.

$V_t(z)$ and $V_b(z)$ are defined as the volume fractions of the constituent of the top and bottom faces of the plates, respectively, and are related by

$$V_t(z) + V_b(z) = 1 \tag{2}$$

The effective properties of functionally graded material are obtained according to a simple power−law. The volume fraction of the constituent of the top surface of the plate follows a simple power−law as,

$$V_t = \left(\frac{2z + h}{2h} \right)^n \tag{3}$$

where n is the non-negative volume fraction index which prescribes the material variation profile through the thickness of the plate and may be adjusted to obtain the optimum distribution of the constituent material. It is ascertained that the effective Young's modulus E and thermal expansion coefficient α are the temperature dependent. However, the mass density ρ and the thermal conductivity κ are independent of the temperature.

Poisson's ratio ν is assumed to be constant as it weakly depends on the temperature changes. From (1) and (3), the effective material properties with two constituents for graded plates can be expressed as,

$$E(z)=(E_t-E_b)\left(\frac{2z+h}{2h}\right)^n+E_b$$
$$v(z)=(v_t-v_b)\left(\frac{2z+h}{2h}\right)^n+v_b \quad (4)$$
$$p(z)=(p_t-p_b)\left(\frac{2z+h}{2h}\right)^n+p_b$$

The “n” value is of significance because it is an exponent of the volume fraction equation. "n" essentially dictates the amount and distribution of ceramic in the plate. With higher values of "n" the plate tends toward metal (the lower surface) while lower values of "n" tend toward ceramic (the upper surface). Designers can vary the "n" value to tailor the FGM to specific applications. This thesis will characterize "n" for each of the models studied in order to provide designers with general value of "n" that will best suit their needs. Fig. 4 details the change in volume fraction through-the-thickness for the values of "n" studied (n=0.0 (fully ceramic), n=0.2, n=0.5, n=1.0, n=2.0, n=∞ (fully metal)).

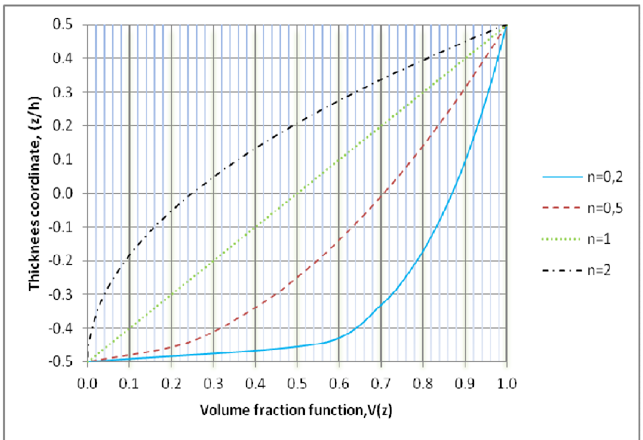
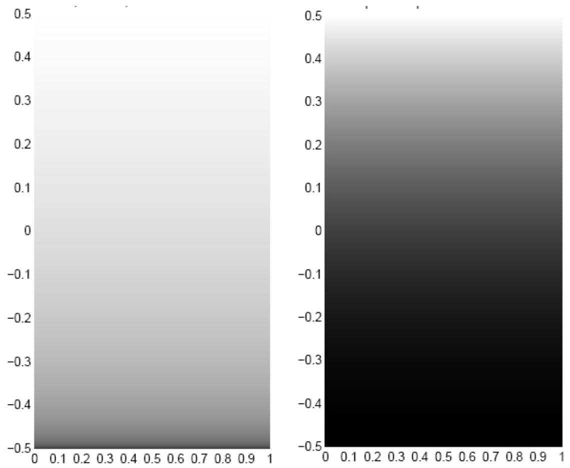


Fig. 4 Variation of volume fraction through the plane thickness

Note that lower values of "n" push the curve toward the right, or toward a fully ceramic plate (bottom surface material). In fact, at n=0, the curve would actually be a vertical line corresponding to a volume fraction of ceramic equal to 1. Additionally, higher values of "n" push the curve toward the left, or toward a metal plate (top surface material). At n=∞, a straight line would exist at a ceramic volume fraction of 0 indicating a fully metal material.

A more detailed representation of “n” is shown in Fig. 5 (a) and Fig. 5 (b) where shaded graphs show the reader the gradual change in material properties. Note: black indicates metal material and white indicates ceramic material. The gray shaded region between black and white is a visual indicator of the mixture of metal and ceramic. Dark gray is a metal rich mixture while light gray is a ceramic rich mixture.



White = Ceramic; Black = Metal

Fig. 5 (a) Graphical representation of material (n=0.2) (b) graphical representation of material (n=2)

It is noted that at n=0.2 there is very little metal in the material and a sharp transition from majority metal to majority ceramic is located at z/h = -0.4. As the "n" value increases the material composition tends more toward metal. It is apparent that structural designers requiring significant thermal protection should consider low values of "n" which will yield a ceramic rich plate. Designers that desire corrosion protection with high load carrying capability should consider higher values on “n” which yield a metal rich plate.

IV. Solutions Methodology

A. Finite Element Model

The goal of analyzing flat plates under thermal loading is to characterize the effect "n" has on the structural response to thermal loading. Following work published by J.N. Reddy [7], a flat Aluminum-Zircon with sides a=0.2 m and thickness h = 0.01 m is exposed to various surface temperatures. The top surface is exposed to isothermal temperatures in a range 0ºC to 600ºC and the bottom temperature is exposed to a constant temperature of 20ºC. Note: each top surface temperature examined is treated as an independent model. "n" values of 0 (ceramic), 0.2, 0.5, 1.0, 2.0, and ∞ (metal) are examined. Material properties for the bottom and top surface are listed below:

Aluminum (Bottom surface)
E = 70 GPa; ν = 0.3; ρ = 2,707 Kg/m3; $\alpha = 23\ x10^{-6}$·ºC;
k = 204 W/m·K
Zircon (Top surface)
E = 151 GPa; ν = 0.3; ρ = 3,000 Kg/m3; $\alpha = 10x10^{-6}$·ºC;
k = 2.09 W/m·K

A 2D shell analysis with an 8x8 mesh and 9 slices is used to solve the problem. A steady state heat transfer analysis is first performed to obtain nodal temperatures.

Since ABAQUS software is utilized as the FEA processor, ABAQUS CAE is obviously desired as the pre-processor and post-processor. However, due to the unique techniques required to models FGMS, Matlab is used as a pre-processing tool to create input files more efficiently. Matlab is used to automatically generate ABAQUS input files. This powerful tool allows one to focus on the study of FGMs rather than the mundane task of entering properties by hand.

The second step was to analyze the structural response to the applied temperature profile created in the heat transfer analysis. The goal of analyzing flat plates under distributed pressure loading (q_0a^4/ E_bh^4) is to characterize the effect “n” has on the structural response to mechanical loading. Again following work published by J.N. Reddy [3], the same flat plate previously analyzed was exposed to a distributed pressure load applied to the elements and in the –z direction. Nodal temperatures are applied to a structural problem with simply support boundary conditions. The difference in applied temperature and stress free reference temperature causes the material to expand, thus creating stress and out of plane displacements. This analysis was performed for both linear and non-linear geometries and for all simply supported boundary conditions.

B. Non-Linearity Background and Analysis

Knowing that plates and shells subject to thermal loading will likely experience deformation that is over 50% of the thickness, it is of interest to compare linear and nonlinear solutions when characterizing FGMs. Major sources of non-linearity are geometric stiffening and material non-linearity (yielding). Geometric stiffening is caused by structural deformation which affects the structural stiffness. "An example of geometric nonlinearity is a thin flat disk, clamped around its circular boundary, and loaded by lateral pressure. If lateral deflection at the center is more than about half the thickness, the disk develops membrane stretching forces that carry a considerable portion of the load" [12].

In the examples, the stiffness matrix and reaction matrix are dependent on the displacement. Both matrices must be updated after every iteration in order to account for these stress stiffening effects. In other words, as the structure undergoes large displacements, the structural stiffness is affected by the displacement, as shown in the equation below:

$$\left[K + K(\delta, \delta^2)\right]\{\delta\} = \{F\} \tag{5}$$

The non-linear stiffness matrix (Greens’ strain) is proportional to the displacement and the square of the displacement, as shown below for strain in the x direction; [12]

$$\varepsilon_{11} = \frac{\delta u_1}{\delta x_1} + \frac{1}{2}\left(\left[\frac{\delta u_1}{\delta x_1}\right]^2 + \left[\frac{\delta u_2}{\delta x_1}\right]^2 + \left[\frac{\delta u_3}{\delta x_1}\right]^2\right) \tag{6}$$

In order to evaluate the non-linear portion of the stiffness matrix, displacements are required. However, these displacements are affected by the non-linear portion of the stiffness matrix therefore the non-linear analysis becomes an interactive process. In ABAQUS, non-linear geometry is turned on using "NLGEOM = YES" in the "Step" card.

This tells ABAQUS to check the stiffness matrix to ensure that the internal and external forces balance as the load is incremented toward the desired load. If the internal / external forces and displacement do not balance with the stiffness matrix, as outlined above, then ABAQUS updates the stiffness matrix using the incremental displacement and tries again. This increment control is handled automatically by ABAQUS; however the user can provide a means for direct control. ABAQUS/Standard generally uses Newton's method as a numerical technique for solving the nonlinear equilibrium equations because convergence is much faster than with other solvers (usually modified Newton or quasi-Newton methods) for the types of nonlinear problems most often studied with ABAQUS [13]. ABAQUS/Standard uses a scheme based predominantly on the maximum force residuals following each iteration. By comparing consecutive values of these quantities, ABAQUS/Standard determines whether convergence is likely in a reasonable number of iterations.

If convergence is deemed unlikely, ABAQUS/Standard adjusts the load increment; if convergence is deemed likely, ABAQUS/Standard continues with the iteration process. In this way excessive iteration is eliminated in cases where convergence is unlikely, and an increment that appears to be converging is not aborted because it needed a few more iterations. One other ingredient in this algorithm is that a minimum increment size is specified (0.01), which prevents excessive computation in cases where buckling, limit load, or some modeling error causes the solution to stall. This control is handled internally, with user override if needed. Several other controls are built into the algorithm; for example, it will cut back the increment size if an element inverts due to excessively large geometry changes. These detailed controls are based on empirical testing [13].

V. Results and Discussions

This part will present results from finite element models for flat plates under thermal loading and pressure loading.

Linear and non-linear plots of non-dimensional deflections (w/h) vs. load parameter (q_0a^4/ E_bh^4) are shown in Figs. 6 and 7.

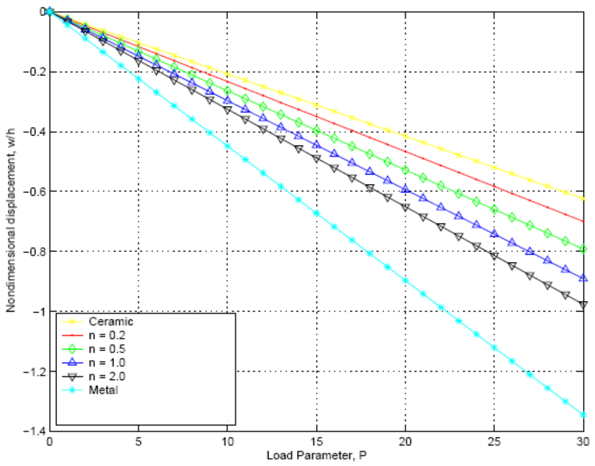


Fig. 6 Linear central deflection for pressure loaded Aluminum-zircon plate (linear)

Ceramic provides the lowest deflection because it is the stiffest material while aluminum provides the highest deflection because it is the softest material. FGM plate deflections fall in the middles with n=0.2 providing the minimum deflection. Lowering “n” tends to lower the center deflection. Of particular importance is the drastic reduction of deflections by functionally graded materials. For all “n” values, FGMs reduced deflections by approximately 50% over a metal plate. The trend of lowering “n” to lower deflection follows the same trend found in thermally loaded plates. In addition, n=0.2 provides the lowest deflection on both thermally loaded plates and pressure loaded plates.

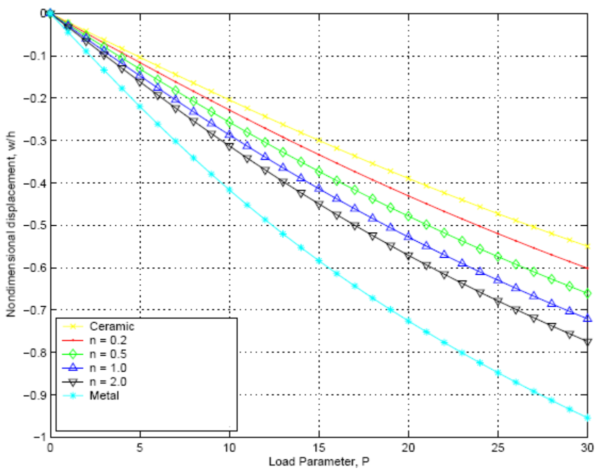


Fig. 7 Linear central deflection for pressure loaded Aluminum-zircon plate (non-linear)

It is noted that a non-linear analysis provides lower displacements than a linear analysis. This is due to the geometric stiffening effect provided by a non-linear analysis and as discussed in the previous section. It is important to note that a linear analysis provides non-dimensional displacements greater than 0.5, which is the general cutoff for deciding between a linear and non-linear analysis. Finally, it is noted that FGMs exhibit non-linearity much earlier than homogeneous materials.

It is assumed that the dimension of FGM plate is plate width and b is plate length. Displacement (w_{max}) is on x=a/2 and y=b/2 varies in the value of "'n" (Table II).

TABLE II
DISPLACEMENT OF CENTER POINT ACCORDING TO N

Displacement (z)	n=2	n=1	n=0.5
a=b	5.88E-06	5.38E-06	4.79E-06
a=b/2	1.51E-05	1.38E-05	1.23E-05
a=b/4	2.08E-05	1.91E-05	1.70E-05

When value of "n" is increased, both functional graded material's density and material hardness increase for these reason displacements of the center point decreases.

Simple support boundary conditional and under the uniform load ($q=10^5$N) FGM plate displacement is investigated. Fig .8 is show that displacement is according to distance of plate edge on x axis and also variant 'n' value. Centre of the plate displacement maximum is seen Fig. 8.

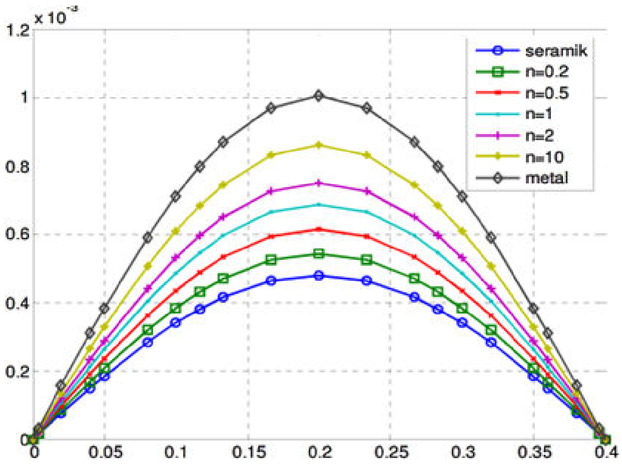


Fig. 8 Displacement is according to distance of plate edge on x axis

VI. CONCLUSION

FGMs provide designers the ability to customize material response to meet their design objectives. In addition, FGMs provide thermal protection and load carrying capability, all while alleviating the stress concentrations found in thermal protection systems. In this study, rectangular FGM plate which is loaded both temperature and pressure loading.

Rectangular Zi-Al FGM flat plates with a temperature gradient of 580°C it was found that a plate with n=0.2 provides the lowest deflection and n=0.5 provides the lowest mid-plane stress (only 2.4% lower than mid-plane stress for n=0.2). These displacements and stresses are 68% and 54%, respectively, lower than a metal plate. It can be concluded that a non-linear analysis with a range of “n” from 0.2 to 0.5 provides the optimum solution to thermally loaded plates studied in this thesis (n=0.2 corresponds to a plate that is 83.3% by volume Zi).

For rectangular Zi-Al FGM flat plates under a non-dimensional pressure loading of 30 it was found that a plate with n=0.2 provides the lowest deflection and lowest mid plane stress; the displacements and stresses are 50% and 4%, respectively, lower than a metal plate. It can be concluded that a non-linear analysis with n=0.2 provides the optimum solution to pressure loaded plates studied in this thesis.

It is important to note that n=0.2 performed best in nearly every FGM plate and panel analyzed in this thesis, both under pressure and thermal loading.

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