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Intuitionistic Fuzzy Positive Implicative Ideals with Thresholds (λ,μ) of BCI-Algebras

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***Abstract*—**The aim of this paper is to introduce the notion of intuitionistic fuzzy positive implicative ideals with thresholds (λ, μ) of BCI-algebras and to investigate its properties and characterizations.

***Keywords*—**BCI-algebra, intuitionistic fuzzy set, intuitionistic fuzzy ideal with thresholds (λ,μ), intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ).

I. Introduction

A BCI-algebra is an important class of logical algebra and was introduced by Iséki [1], [2]. K. Atannassov [3] introduced the concept of intuitionistic fuzzy sets. In 2003, K. Hur [4] applied the concept to the theory of rings, and introduced the concepts of intuitionistic fuzzy subgroups and subrings. M. Jiang and X.L. Xin [5] later introduced the concepts of (λ,μ) intuitionistic fuzzy subrings (ideals); some meaningful results are obtained. In [6], [7], we have given the concepts of intuitionistic fuzzy subalgebras (ideals) with thresholds (λ,μ) and intuitionistic fuzzy implicative ideals with thresholds (λ,μ) of BCI-algebras, in this paper, we introduce the notion of intuitionistic fuzzy positive implicative ideals with thresholds (λ,μ) of BCI-algebras and give several properties and characterizations of it.

II. Preliminaries

An algebra $(X;*,0)$ of type $(2,0)$ is called a BCI- algebra if it satisfies the following axioms:

- (BCI-1) $((x*y)*(x*z))*(z*y)=0,$
- (BCI-2) $(x*(x*y))*y=0,$
- (BCI-3) $x*x=0,$
- (BCI-4) $x*y=0$ and $y*x=0$ imply $x=y,$

for all $x,y,z\in X$. In a BCI-algebra X, we can define a partial ordering $\le$ by putting $x\le y$ if and only if $x*y=0.$

In any BCI-algebra X, the following hold:

1. $(x*y)*z=(x*z)*y,$
2. $x*0=x,$
3. $0*(x*y)=(0*x)*(0*y),$
4. $(x*z)*(y*z)\le x*y,$
5. $x*(x*(x*y))=x*y,$

for all $x,y,z\in X$.

In this paper, X always means a BCI-algebra unless otherwise specified.

A nonempty subset K of X is called an ideal of X if $(I_1): 0\in K, (I_2): x*y\in K$ and $y\in K$ imply $x\in K.$ A nonempty subset K of X is called a positive implicative ideal of X if it satisfies (I_1) and (I_3): $((x*z)*z)*(y*z)\in K$ and $y\in K$ imply $x*z\in K.$

Definition 1. [3] Let S be any set. An intuitionistic fuzzy subset A of S is an object of the following form

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\} \text{ where } \mu_A : S\to[0,1]$$

and $\nu_A : S\to[0,1]$ define the degree of membership and the degree of non-membership of the element $x\in S$ respectively and for every $x\in S$, $0\le\mu_A(x)+\nu_A(x)\le 1.$

Denote $\langle I\rangle=\{\langle a,b\rangle : a,b\in[0,1]\}$.

Definition 2. Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ be an intuitionistic fuzzy set in a set S. For $\langle\alpha,\beta\rangle\in\langle I\rangle$, the set $A_{\langle\alpha,\beta\rangle}=\{x\in S:\mu_A(x)\ge\alpha,\nu_A(x)\le\beta\}$ is called a cut set of A.

Definition 3. [6] Let $\lambda,\mu\in(0,1]$ and $\lambda<\mu.$

An intuitionistic fuzzy set A in X is said to be an intuitionistic fuzzy ideal with thresholds (λ,μ) of X if the following are satisfied:

$$(IF_1)\ \mu_A(0)\vee\lambda\ge\mu_A(x)\wedge\mu,$$
$$(IF_2)\ \nu_A(0)\wedge\mu\le\nu_A(x)\vee\lambda,$$
$$(IF_3)\ \mu_A(x)\vee\lambda\ge\mu_A(x*y)\wedge\mu_A(y)\wedge\mu,$$
$$(IF_4)\ \nu_A(x)\wedge\mu\le\nu_A(x*y)\vee\nu_A(y)\vee\lambda,$$

for all $x,y\in X$.

Proposition 1. [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ,μ) of X. If $x\le y$ holds in X, then

$$\mu_A(x)\vee\lambda\ge\mu_A(y)\wedge\mu,\ \nu_A(x)\wedge\mu\le\nu_A(y)\vee\lambda.$$

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Proposition 2. [6] Let A be an intuitionistic fuzzy ideal with thresholds (λ,μ) of X. If the inequality $x*y\le z$ holds in X, then for all $x,y,z\in X$,

$$\mu_A(x)\vee\lambda\ge\mu_A(y)\wedge\mu_A(z)\wedge\mu,$$
$$\nu_A(x)\wedge\mu\le\nu_A(y)\vee\nu_A(z)\vee\lambda.$$

III. Intuitionistic Fuzzy Positive Implicative Ideals with Thresholds (λ,μ) of BCI- Algebras

Definition 4. Let $\lambda,\mu\in(0,1]$ and $\lambda<\mu$. An intuitionistic fuzzy set A in X is called an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X if it satisfies $(IF_1),(IF_2)$ and

$$(IF_5)\ \mu_A(x*z)\vee\lambda\ge\mu_A(((x*z)*z)*(y*z))\wedge\mu_A(y)\wedge\mu,$$
$$(IF_6)\ \nu_A(x*z)\wedge\mu\le\nu_A(((x*z)*z)*(y*z))\vee\nu_A(y)\vee\lambda,$$

for all $x,y,z\in X$.

Example 1. Let $X=\{0,1,2\}$ with Cayley table given by

TABLE I
RESULT OF CALCULATION

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	2	0

Define $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ where $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$ by $\mu_A(0)=2/3$, $\mu_A(1)=\mu_A(2)=1/3$, $\nu_A(0)=1/4,\nu_A(1)=\nu_A(2)=1/2$. Let $\lambda=1/8$ and $\mu=3/4$. By routine calculations give that A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X.

The following proposition gives a relation between intuitionistic fuzzy positive implicative ideals with thresholds (λ,μ) and intuitionistic fuzzy ideals with thresholds (λ,μ) of X.

Proposition 3. Any intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X, but the converse does not hold.

Proof. Assume that A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X and put $z=0$ in (IF_5) and (IF_6), we get

$$\mu_A(x)\vee\lambda\ge\mu_A(x*y)\wedge\mu_A(y)\wedge\mu,$$
$$\nu_A(x)\wedge\mu\le\nu_A(x*y)\vee\nu_A(y)\vee\lambda.$$

This means that A satisfies (IF_3) and (IF_4). Combining (IF_1) and (IF_2), A is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X.

To show the last half part, we see the following example.

Example 2. Let $X=\{0,1,2\}$ with Cayley table given by

TABLE II
RESULT OF COMPUTATION

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Define $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ where $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$ by $\mu_A(0)=2/3$, $\mu_A(1)=\mu_A(2)=1/3$, $\nu_A(0)=1/4,\nu_A(1)=\nu_A(2)=1/2$. Let $\lambda=1/8$ and $\mu=3/4$. It is easy to verify that A is an intuitionistic fuzzy ideal with thresholds (λ,μ) of X. But it is not an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X since:

$$\mu_A(2*1)\vee\lambda<\mu_A(((2*1)*1)*(0*1))\wedge\mu_A(0)\wedge\mu.$$

Next, we give characterizations of intuitionistic fuzzy positive implicative ideals with thresholds (λ,μ) of X.

Proposition 4. Let A be an intuitionistic fuzzy ideal with thresholds (λ,μ) of X. Then the following are equivalent:

(i). A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X,

(ii). $\mu_A((x*y)*z)\vee\lambda\ge\mu_A(((x*z)*z)*(y*z))\wedge\mu,$
$\nu_A((x*y)*z)\wedge\mu\le\nu_A(((x*z)*z)*(y*z))\vee\lambda,$ for all $x,y,z\in X$,

(iii). $\mu_A(x*y)\vee\lambda\ge\mu_A(((x*y)*y)*(0*y))\wedge\mu,$
$\nu_A(x*y)\wedge\mu\le\nu_A(((x*y)*y)*(0*y))\vee\lambda,$ for all $x,y\in X$.

Proof. (i) $\Rightarrow$ (ii) Suppose that A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X. Since

$$(((x*y)*z)*z)*(0*z)=(((x*y)*z)*z)*((y*y)*z)$$
$$=(((x*z)*z)*y)*((y*z)*y)\le((x*z)*z)*(y*z),$$

by $(IF_5),(IF_6),(IF_1),(IF_2)$ and Proposition 1, we have

$$\mu_A((x*y)*z)\vee\lambda=(\mu_A((x*y)*z)\vee\lambda)\vee\lambda$$
$$\ge(\mu_A((((x*y)*z)*z)*(0*z))\wedge\mu_A(0)\wedge\mu)\vee\lambda$$
$$=(\mu_A((((x*y)*z)*z)*(0*z))\vee\lambda)\wedge(\mu_A(0)\vee\lambda)\wedge(\mu\vee\lambda)$$
$$\ge(\mu_A(((x*z)*z)*(y*z))\wedge\mu)$$
$$\wedge(\mu_A(((x*z)*z)*(y*z))\wedge\mu)\wedge\mu=\mu_A(((x*z)*z)*(y*z))\wedge\mu,$$
$$\nu_A((x*y)*z)\wedge\mu=(\nu_A((x*y)*z)\wedge\mu)\wedge\mu$$

$$\leq\left(\nu_A\left((((x*y)*z)*z)*(0*z)\right)\vee\nu_A(0)\vee\lambda\right)\wedge\mu$$
$$=\left(\nu_A\left((((x*y)*z)*z)*(0*z)\right)\wedge\mu\right)\vee\left(\nu_A(0)\wedge\mu\right)\vee(\lambda\wedge\mu)$$
$$\leq\left(\nu_A\left(((x*z)*z)*(y*z)\right)\vee\lambda\right)$$
$$\vee\left(\nu_A\left(((x*z)*z)*(y*z)\right)\vee\lambda\right)\vee\lambda=\nu_A\left(((x*z)*z)*(y*z)\right)\vee\lambda.$$

Hence

$$\mu_A\left((x*y)*z\right)\vee\lambda\geq\mu_A\left(((x*z)*z)*(y*z)\right)\wedge\mu,$$
$$\nu_A\left((x*y)*z\right)\wedge\mu\leq\nu_A\left(((x*z)*z)*(y*z)\right)\vee\lambda$$

and (ii) holds.

- (ii) $\Rightarrow$ (iii) Substituting 0 for y and y for z in (ii), respectively, we have (iii).
- (iii) $\Rightarrow$ (i) Since

$$\left(((x*y)*y)*(0*y)\right)*\left(((x*y)*y)*(z*y)\right)\leq(z*y)*(0*y)\leq z,$$

by Proposition 2, we obtain

$$\mu_A\left(((x*y)*y)*(0*y)\right)\vee\lambda\geq\mu_A\left(((x*y)*y)*(z*y)\right)\wedge\mu_A(z)\wedge\mu,$$
$$\nu_A\left(((x*y)*y)*(0*y)\right)\wedge\mu\leq\nu_A\left(((x*y)*y)*(z*y)\right)\vee\nu_A(z)\vee\lambda.$$

From (iii), we have

$$\mu_A(x*y)\vee\lambda=\left(\mu_A(x*y)\vee\lambda\right)\vee\lambda$$
$$\geq\left(\mu_A\left(((x*y)*y)*(0*y)\right)\wedge\mu\right)\vee\lambda$$
$$=\left(\mu_A\left(((x*y)*y)*(0*y)\right)\vee\lambda\right)\wedge(\mu\vee\lambda)$$
$$\geq\mu_A\left(((x*y)*y)*(z*y)\right)\wedge\mu_A(z)\wedge\mu,$$
$$\nu_A(x*y)\wedge\mu$$
$$\left(\nu_A(x*y)\wedge\mu\right)\wedge\mu\leq\left(\nu_A\left(((x*y)*y)*(0*y)\right)\vee\lambda\right)\wedge\mu$$
$$=\left(\nu_A\left(((x*y)*y)*(0*y)\right)\wedge\mu\right)\vee(\lambda\wedge\mu)$$
$$\leq\nu_A\left(((x*y)*y)*(z*y)\right)\vee\nu_A(z)\vee\lambda.$$

Hence, A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X.

Proposition 5. An intuitionistic fuzzy set A of X is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X if and only if, for all $\alpha,\beta\in(\lambda,\mu]$, $A_{\langle\alpha,\beta\rangle}$ is either empty or a positive implicative ideal of X.

Proof. Let A be an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X and $A_{\langle\alpha,\beta\rangle}\neq\varnothing$ for some $\alpha,\beta\in(\lambda,\mu]$. It is clear that $0\in A_{\langle\alpha,\beta\rangle}$. Let $((x*z)*z)*(y*z)\in A_{\langle\alpha,\beta\rangle}$ and $y\in A_{\langle\alpha,\beta\rangle}$, then

$$\mu_A\left(((x*z)*z)*(y*z)\right)\geq\alpha,\ \mu_A(y)\geq\alpha,$$
$$\nu_A\left(((x*z)*z)*(y*z)\right)\leq\beta,\ \nu_A(y)\leq\beta.$$

It follows from (IF_5) and (IF_6),

$$\mu_A(x*z)\vee\lambda\geq\mu_A\left(((x*z)*z)*(y*z)\right)\wedge\mu_A(y)\wedge\mu\geq\alpha,$$
$$\nu_A(x*z)\wedge\mu\leq\nu_A\left(((x*z)*z)*(y*z)\right)\vee\nu_A(y)\vee\lambda\leq\beta.$$

Namely, $\mu_A(x*z)\geq\alpha$, $\nu_A(x*z)\leq\beta$ and $x*z\in A_{\langle\alpha,\beta\rangle}$. This shows that $A_{\langle\alpha,\beta\rangle}$ is a positive implicative ideal of X. Conversely, suppose that for each $\alpha,\beta\in(\lambda,\mu]$, $A_{\langle\alpha,\beta\rangle}$ is either empty or a positive implicative ideal of X. For any $x\in X$, let $\alpha=\mu_A(x)\wedge\mu,\beta=\nu_A(x)\vee\lambda.$ Then $\mu_A(x)\geq\alpha,\nu_A(x)\leq\beta,$ hence $x\in A_{\langle\alpha,\beta\rangle}$ and $A_{\langle\alpha,\beta\rangle}$ is a positive implicative ideal of X, therefore $0\in A_{\langle\alpha,\beta\rangle}$, i.e., $\mu_A(0)\geq\alpha$ and $\nu_A(0)\leq\beta.$ We get

$$\mu_A(0)\vee\lambda\geq\mu_A(0)\geq\alpha=\mu_A(x)\wedge\mu,$$
$$\nu_A(0)\wedge\mu\leq\nu_A(0)\leq\beta=\nu_A(x)\vee\lambda,$$

i.e., $\mu_A(0)\vee\lambda\geq\mu_A(x)\wedge\mu$ and $\nu_A(0)\wedge\mu\leq\nu_A(x)\vee\lambda,$ for all $x\in X$. Now we only need to show that A satisfies (IF_5) and (IF_6). Let

$$\alpha=\mu_A\left(((x*z)*z)*(y*z)\right)\wedge\mu_A(y)\wedge\mu,$$
$$\beta=\nu_A\left(((x*z)*z)*(y*z)\right)\vee\nu_A(y)\vee\lambda.$$

Then

$$\mu_A\left(((x*z)*z)*(y*z)\right)\geq\alpha,\ \mu_A(y)\geq\alpha,$$
$$\nu_A\left(((x*z)*z)*(y*z)\right)\leq\beta,\ \nu_A(y)\leq\beta.$$

Hence $((x*z)*z)*(y*z)\in A_{\langle\alpha,\beta\rangle}$ and $y\in A_{\langle\alpha,\beta\rangle}$. Since $A_{\langle\alpha,\beta\rangle}$ is a positive implicative ideal of X, thus $x*z\in A_{\langle\alpha,\beta\rangle}$, i.e., $\mu_A(x*z)\geq\alpha$, $\nu_A(x*z)\leq\beta.$ We get

$$\mu_A(x*z)\vee\lambda\geq\mu_A(x*z)\geq\alpha=\mu_A\left(((x*z)*z)*(y*z)\right)\wedge\mu_A(y)\wedge\mu,$$
$$\nu_A(x*z)\wedge\mu\leq\nu_A(x*z)\leq\beta=\nu_A\left(((x*z)*z)*(y*z)\right)\vee\nu_A(y)\vee\lambda.$$

Namely,

$$\mu_A(x*z)\vee\lambda\geq\mu_A\left(((x*z)*z)*(y*z)\right)\wedge\mu_A(y)\wedge\mu,$$
$$\nu_A(x*z)\wedge\mu\leq\nu_A\left(((x*z)*z)*(y*z)\right)\vee\nu_A(y)\vee\lambda.$$

Summarizing the above arguments, A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X.

Proposition 6 Let J be a positive implicative ideal of X. Then there exists an intuitionistic fuzzy positive implicative ideal A with thresholds (λ,μ) of X such that $A_{\langle\alpha,\beta\rangle}=J$ for some $\alpha,\beta\in(\lambda,\mu]$.

Proof. Define $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}$ by

$$\mu_A(x)=\begin{cases}\alpha & if\ x\in J,\\ \lambda & if\ x\notin J,\end{cases}$$
$$\nu_A(x)=\begin{cases}\beta & if\ x\in J,\\ \mu & if\ x\notin J,\end{cases}$$

where α,β are two fixed numbers in $(\lambda,\mu]$. Since J is a positive implicative ideal of X, if $((x*z)*z)*(y*z)\in J$ and $y\in J$ then $x*z\in J$. Hence

$$\mu_A(((x*z)*z)*(y*z))=\mu_A(y)=\mu_A(x*z)=\alpha,$$
$$\nu_A(((x*z)*z)*(y*z))=\nu_A(y)=\nu_A(x*z)=\beta,$$

Thus,

$$\mu_A(x*z)\vee\lambda\geq\mu_A(((x*z)*z)*(y*z))\wedge\mu_A(y)\wedge\mu,$$
$$\nu_A(x*z)\wedge\mu\leq\nu_A(((x*z)*z)*(y*z))\vee\nu_A(y)\vee\lambda.$$

If at least one of $((x*z)*z)*(y*z)$ and y is not in J, then at least one of $\mu_A(((x*z)*z)*(y*z))$ and $\mu_A(y)$ is λ, and at least one of $\nu_A(((x*z)*z)*(y*z))$ and $\nu_A(y)$ is μ. Therefore,

$$\mu_A(x*z)\vee\lambda\geq\mu_A(((x*z)*z)*(y*z))\wedge\mu_A(y)\wedge\mu,$$
$$\nu_A(x*z)\wedge\mu\leq\nu_A(((x*z)*z)*(y*z))\vee\nu_A(y)\vee\lambda.$$

This means that A satisfies (IF_5) and (IF_6). Since $0\in J$, $\mu_A(0)\vee\lambda=\alpha\geq\mu_A(x)\wedge\mu$, $\nu_A(0)\wedge\mu=\beta\leq\nu_A(x)\vee\lambda$, for all $x\in X$ and so A satisfies (IF_1) and (IF_2). Thus, A is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X. It is clear that $A_{\langle\alpha,\beta\rangle}=J$.

Definition 5. Let S be any set. If

$$A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in S\}, B=\{\langle x,\mu_B(x),\nu_B(x)\rangle : x\in S\}$$

be any two intuitionistic fuzzy subsets of S **,** then

$$A\cap B=\{\langle x,(\mu_A\cap\mu_B)(x),(\nu_A\cup\nu_B)(x)\rangle : x\in S\}$$
$$=\{\langle x,\mu_A(x)\wedge\mu_B(x),\nu_A(x)\vee\nu_B(x)\rangle : x\in S\}$$

Proposition 7 Let A and B be two intuitionistic fuzzy positive implicative ideals with thresholds (λ,μ) of X. Then $A\cap B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X.

Proof. For all $x,y,z\in X$, by Definition 4, we have

$$\mu_{A\cap B}(0)\vee\lambda=(\mu_A(0)\wedge\mu_B(0))\vee\lambda=(\mu_A(0)\vee\lambda)\wedge(\mu_B(0)\vee\lambda)$$
$$\geq(\mu_A(x)\wedge\mu)\wedge(\mu_B(x)\wedge\mu)=(\mu_A(x)\wedge\mu_B(x))\wedge\mu=\mu_{A\cap B}(x)\wedge\mu,$$

$$\nu_{A\cap B}(0)\wedge\mu=(\nu_A(0)\vee\nu_B(0))\wedge\mu=(\nu_A(0)\wedge\mu)\vee(\nu_B(0)\wedge\mu)$$
$$\leq(\nu_A(x)\vee\lambda)\vee(\nu_B(x)\vee\lambda)=(\nu_A(x)\vee\nu_B(x))\vee\lambda=\nu_{A\cap B}(x)\vee\lambda,$$

$$\mu_{A\cap B}(x*z)\vee\lambda=(\mu_A(x*z)\wedge\mu_B(x*z))\vee\lambda$$
$$=(\mu_A(x*z)\vee\lambda)\wedge(\mu_B(x*z)\vee\lambda)$$
$$\geq(\mu_A(((x*z)*z)*(y*z))\wedge\mu_A(y)\wedge\mu)$$
$$\wedge(\mu_B(((x*z)*z)*(y*z))\wedge\mu_B(y)\wedge\mu)$$
$$=(\mu_A(((x*z)*z)*(y*z))\wedge\mu_B(((x*z)*z)*(y*z)))$$
$$\wedge(\mu_A(y)\wedge\mu_B(y))\wedge\mu=\mu_{A\cap B}(((x*z)*z)*(y*z))\wedge\mu_{A\cap B}(y)\wedge\mu.$$

$$\nu_{A\cap B}(x*z)\wedge\mu=(\nu_A(x*z)\vee\nu_B(x*z))\wedge\mu$$
$$=(\nu_A(x*z)\wedge\mu)\vee(\nu_B(x*z)\wedge\mu)$$
$$\leq(\nu_A(((x*z)*z)*(y*z))\vee\nu_A(y)\vee\lambda)$$
$$\vee(\nu_B(((x*z)*z)*(y*z))\vee\nu_B(y)\vee\lambda)$$
$$=(\nu_A(((x*z)*z)*(y*z))\vee\nu_B(((x*z)*z)*(y*z)))$$
$$\vee(\nu_A(y)\vee\nu_B(y))\vee\lambda=\nu_{A\cap B}(((x*z)*z)*(y*z))\vee\nu_{A\cap B}(y)\vee\lambda.$$

Hence $A\cap B$ is an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of X.

Definition 6. Let A and B be two intuitionistic fuzzy sets of a set X. The Cartesian product of A and B is defined by

$$A\times B=\{\langle\mu_{A\times B}(x,y),\nu_{A\times B}(x,y)\rangle : x,y\in X\}$$

where

$$\mu_{A\times B}(x,y)=\mu_A(x)\wedge\mu_B(y),\ \nu_{A\times B}(x,y)=\nu_A(x)\vee\nu_B(y).$$

Proposition 8. Let A and B be two intuitionistic fuzzy positive implicative ideals with thresholds (λ,μ) of X. Then $A\times B$ is also an intuitionistic fuzzy positive implicative ideal with thresholds (λ,μ) of $X\times X$.

Proof. For all $(x,y)\in X\times X$, by Definition 4, we get

$$\mu_{A\times B}(0,0)\vee\lambda=(\mu_A(0)\wedge\mu_B(0))\vee\lambda=(\mu_A(0)\vee\lambda)\wedge(\mu_B(0)\vee\lambda)$$
$$\geq(\mu_A(x)\wedge\mu)\wedge(\mu_B(y)\wedge\mu)=\mu_{A\times B}(x,y)\wedge\mu,$$

$$\nu_{A\times B}(0,0)\wedge\mu=(\nu_A(0)\vee\nu_B(0))\wedge\mu=(\nu_A(0)\wedge\mu)\vee(\nu_B(0)\wedge\mu)$$
$$\leq(\nu_A(x)\vee\lambda)\vee(\nu_B(y)\vee\lambda)=\nu_{A\times B}(x,y)\vee\lambda,$$

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have

$$\begin{aligned}
\mu_{A\times B}(x_1 * z_1, x_2 * z_2) \vee \lambda &= (\mu_A(x_1 * z_1) \wedge \mu_B(x_2 * z_2)) \vee \lambda \\
&= (\mu_A(x_1 * z_1) \vee \lambda) \wedge (\mu_B(x_2 * z_2) \vee \lambda) \\
&\geq (\mu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \wedge \mu_A(y_1) \wedge \mu) \\
&\quad \wedge (\mu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \wedge \mu_A(y_2) \wedge \mu) \\
&= \mu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \wedge \mu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
&\quad \wedge \mu_A(y_1) \wedge \mu_B(y_2) \wedge \mu \\
&= \mu_{A\times B}(((x_1 * z_1) * z_1) * (y_1 * z_1), ((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
&\quad \wedge \mu_{A\times B}(y_1, y_2) \wedge \mu,
\end{aligned}$$

$$\begin{aligned}
\nu_{A\times B}(x_1 * z_1, x_2 * z_2) \wedge \mu &= (\nu_A(x_1 * z_1) \vee \nu_B(x_2 * z_2)) \wedge \mu \\
&= (\nu_A(x_1 * z_1) \wedge \mu) \vee (\nu_B(x_2 * z_2) \wedge \mu) \\
&\leq (\nu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \vee \nu_A(y_1) \vee \lambda) \\
&\quad \vee (\nu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \vee \nu_A(y_2) \vee \lambda) \\
&= \nu_A(((x_1 * z_1) * z_1) * (y_1 * z_1)) \vee \nu_B(((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
&\quad \vee \nu_A(y_1) \vee \nu_B(y_2) \vee \lambda \\
&= \nu_{A\times B}(((x_1 * z_1) * z_1) * (y_1 * z_1), ((x_2 * z_2) * z_2) * (y_2 * z_2)) \\
&\quad \vee \nu_{A\times B}(y_1, y_2) \vee \lambda,
\end{aligned}$$

Hence $A \times B$ is an intuitionistic fuzzy positive implicative ideal with thresholds (λ, μ) of $X \times X$.

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