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Integrable Heisenberg Ferromagnet Equations with Self-Consistent Potentials

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Abstract—In this paper, we consider some integrable Heisenberg Ferromagnet Equations with self-consistent potentials. We study their Lax representations. In particular we derive their equivalent counterparts in the form of nonlinear Schrödinger type equations. We present the integrable reductions of the Heisenberg Ferromagnet Equations with self-consistent potentials. These integrable Heisenberg Ferromagnet Equations with self-consistent potentials describe nonlinear waves in ferromagnets with some additional physical fields.

Keywords—Spin systems, equivalent counterparts, integrable reductions, self-consistent potentials.

I. Introduction

NONLINEAR effects play fundamental role in many phenomena in different branches of sciences. Such nonlinear effects are modelled by nonlinear differential equations (NDE). Some of this equations are integrable, and are known as soliton equations. Integrable spin systems (SS) are one of main sectors of integrable NDE and are important in mathematics, in particular in the geometry of curves and surfaces. On the other hand, integrable SS play crucial role in the description of nonlinear phenomena in magnets.

In this paper, we study some integrable Myrzakulov equations with self-consistent potentials. We investigate their Lax representations as well as their reductions. Finally, we give their equivalent counterparts which have nonlinear Schrödinger equation type form.

The paper is organized as follows. In Sec. II, we give the basic formalism for the theory of the Heisenberg ferromagnet equation. In Sec. III, we investigate the (1+1)-dimensional M-XCIX equation. Sec. IV is denoted to the study of the (1+1)-dimensional M-LXIV equation. In Sec. V we consider the (1+1)-dimensional M-XCIV equation. Finally, conclusions are given in Sec. VI.

II. Preliminaries

First example of integrable SS is the so-called Heisenberg ferromagnetic model (HFM) which reads as [1], [2]

$$\mathbf{S}_t = \mathbf{S} \wedge \mathbf{S}_{xx}, \tag{1}$$

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where $\wedge$ denotes a vector product and

$$\mathbf{S} = (S_1, S_2, S_3), \quad \mathbf{S}^2 = 1. \tag{2}$$

The matrix form of the HFM looks like

$$iS_t = \frac{1}{2}[S, S_{xx}], \tag{3}$$

where

$$S = S_i\sigma_i = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}. \tag{4}$$

Here $S^2 = I$, $S^{\pm} = S_1 \pm iS_2$, $[A, B] = AB - BA$ and σ_i are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5}$$

Note that the HFM (1) is Lakshmanan equivalent [1] to the nonlinear Schrödinger equation (NSE)

$$i\varphi_t + \varphi_{xx} + 2|\varphi|^2\varphi = 0. \tag{6}$$

Also we recall that between the HFM (1) and NSE (6) takes place the gauge equivalence [2]. In literature different types integrable and nonintegrable SS have been proposed (see e.g. [3]-[14]).

III. The (1+1)-Dimensional M-XCIX Equation

The (1+1)-dimensional Myrzakulov-XCIX equation (or shortly M-XCIX equation) reads as [3]

$$\mathbf{S}_t + 0.5\epsilon_1\mathbf{S} \wedge \mathbf{S}_{xx} + \frac{2}{\omega}\mathbf{S} \wedge \mathbf{W} = 0, \tag{7}$$

$$\mathbf{W}_x + 2\omega\mathbf{S} \wedge \mathbf{W} = 0, \tag{8}$$

where $\wedge$ denotes a vector product and

$$\mathbf{S} = (S_1, S_2, S_3), \quad \mathbf{W} = (W_1, W_2, W_3), \tag{9}$$

Here α is a real function, $\mathbf{S}^2 = S_1^2 + S_2^2 + S_3^2 = 1$, S_i and W_i are some real functions, ω and ϵ_i are real constants. In terms of components the M-XCIX equation (7)-(8) takes the form

$$S_{1t} + 0.5\epsilon_1(S_2S_{3xx} - S_3S_{2xx}) + \frac{2}{\omega}(S_2W_3 - S_3W_2) = 0, \tag{10}$$

$$S_{2t} + 0.5\epsilon_1(S_3S_{1xx} - S_1S_{3xx}) + \frac{2}{\omega}(S_3W_1 - S_1W_3) = 0, \tag{11}$$

$$S_{3t} + 0.5\epsilon_1(S_1S_{2xx} - S_2S_{1xx}) + \frac{2}{\omega}(S_1W_2 - S_2W_1) = 0, \tag{12}$$

$$W_{1x} + 2\omega(S_2W_3 - S_3W_2) = 0, \tag{13}$$

$$W_{2x} + 2\omega(S_3W_1 - S_1W_3) = 0, \tag{14}$$

$$W_{3x} + 2\omega(S_1W_2 - S_2W_1) = 0. \tag{15}$$

On the other hand, the M-XCIX equation (7)-(8) can be rewritten as

$$iS_t + 0.25\epsilon_1[S, S_{xx}] + \frac{1}{\omega}[S, W] = 0, \quad (16)$$
$$iW_x + \omega[S, W] = 0, \quad (17)$$

where

$$S = S_i\sigma_i = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad (18)$$

$$W = W_i\sigma_i = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \quad (19)$$

Here $S^{\pm} = S_1 \pm iS_2, W^{\pm} = W_1 \pm iW_2$, $[A, B] = AB - BA$, σ_i are Pauli matrices.

A. Lax Representation

Let us consider the system of the linear equations

$$\Phi_x = U\Phi, \quad (20)$$
$$\Phi_t = V\Phi. \quad (21)$$

Let the Lax pair $U - V$ has the form [3]-[14]

$$U = -i\lambda S, \quad (22)$$
$$V = \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda+\omega}V_{-1} - \frac{i}{\omega}V_0, \quad (23)$$

where

$$V_2 = -i\epsilon_1 S, \quad (24)$$
$$V_1 = 0.25\epsilon_1[S, S_x], \quad (25)$$
$$V_{-1} = V_0 = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \quad (26)$$

With such U, V matrices, the equation

$$U_t - V_x + [U, V] = 0 \quad (27)$$

is equivalent to the M-XCIX equation (7)-(8). It means that the M-XCIX equation (7)-(8) is integrable by the Inverse Tranform Method (ITM).

B. Shcrödinger-type Equivalent Counterpart

Our aim in this section is to find the Shcrödinger-type equivalent counterpart of the M-XCIX equation. To do is, let us we introduce the 3 new functions φ, p and η as

$$\varphi = \alpha e^{i\beta}, \quad (28)$$
$$p = -\left[2S^-W_3 - (S_3+1)W^- + \frac{S^{-2}W^+}{S_3+1}\right]e^{i\varsigma}, \quad (29)$$
$$\eta = 2S_3W_3 + S^-W^+ + S^+W^-, \quad (30)$$

where

$$\alpha = 0.5(S_{1x}^2 + S_{2x}^2 + S_{3x}^2)^{0.5}, \quad (31)$$
$$\beta = -i\partial_x^{-1}\left[\frac{tr(S_xSS_{xx})}{tr(S_x^2)}\right], \quad (32)$$
$$\varsigma = \exp\left[i\theta - \frac{1}{2}\partial_x^{-1}\left(\frac{S^+S_x^- - S_x^+S^-}{1+S_3}\right)\right] \quad (33)$$

and $\theta = const$. It is not difficult to verify that these 3 new functions satisfy the following equations

$$i\varphi_t + \epsilon_1(0.5\varphi_{xx} + |\varphi|^2\varphi) - 2ip = 0, \quad (34)$$
$$p_x - 2i\omega p - 2\eta\varphi = 0, \quad (35)$$
$$\eta_x + \varphi^*p + \varphi p^* = 0, \quad (36)$$

It is nothing but the nonlinear Schrödinger-Maxwell-Bloch equation (NSMBE). It is well-known that the SMBE is integrable by IST. Its Lax representation reads as [15]-[16]

$$\Psi_x = A\Psi, \quad (37)$$
$$\Psi_t = B\Psi, \quad (38)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (39)$$
$$B = \lambda^2 B_2 + \lambda B_1 + B_0 + \frac{i}{\lambda+\omega}B_{-1}. \quad (40)$$

Here

$$A_0 = \begin{pmatrix} 0 & \varphi \\ -\varphi^* & 0 \end{pmatrix}, \quad (41)$$
$$B_2 = -i\epsilon_1\sigma_3, \quad (42)$$
$$B_1 = \epsilon_1 A_0, \quad (43)$$
$$B_0 = 0.5i\epsilon_1\alpha^2\sigma_3 + 0.5i\epsilon_1\sigma_3 A_{0x}, \quad (44)$$
$$B_{-1} = \begin{pmatrix} \eta & -p \\ -p^* & -\eta \end{pmatrix}. \quad (45)$$

C. Reductions

1) Principal Chiral Equation: Let us we set $\epsilon_1 = 0$. Then the M-XCIX equation reduces to the equation

$$iS_t + \frac{1}{\omega}[S, W] = 0, \quad (46)$$
$$iW_x + \omega[S, W] = 0. \quad (47)$$

It is nothing but the principal chiral equation. As is well-known that it is integrable by ITM. The corresponding Lax pair is given by

$$U = -i\lambda S, \quad (48)$$
$$V = -\frac{i\lambda}{\omega(\lambda+\omega)}W. \quad (49)$$

2) Heisenberg Ferromagnetic Equation: Now let us we assume that $W = 0$. Then the M-XCIX equation reduces to the equation

$$iS_t + 0.25\epsilon_1[S, S_{xx}] = 0. \quad (50)$$

It is the HFM (1) within to the simplest scale transformations.

IV. The (1+1)-Dimensional M-LXIV Equation

The (1+1)-dimensional M-LXIV equation (or shortly M-LXIV equation) reads as [3]:

$$iS_t + \epsilon_2 i[S_{xxx} + 6(\beta S)_x] + \frac{1}{\omega}[S, W] = 0, \quad (51)$$
$$iW_x + \omega[S, W] = 0. \quad (52)$$

The corresponding Lax pair is given by

$$U = -i\lambda S, \quad (53)$$
$$V = \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda+\omega}V_{-1} - \frac{i}{\omega}V_{-1}, \quad (54)$$

where [3]

$$V_3 = -4i\epsilon_2 S, \quad (55)$$
$$V_2 = 2\epsilon_2 SS_x, \quad (56)$$
$$V_1 = \epsilon_2 i(S_{xx} + 6\beta S), \quad (57)$$
$$V_{-1} = W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix} \quad (58)$$

with $\beta = rq = 0.125tr[(S_x)^2]$. The functions φ, p and η as (28)-(30) give us the Schrodinger equivalent of the (1+1)-dimensional M-XCIV equation. It has the form (see e.g. [17], [18])

$$iq_t + i\epsilon_2(q_{xxx} + 6rqq_x) - 2ip = 0, \quad (59)$$
$$ir_t + i\epsilon_2(r_{xxx} + 6rqr_x) - 2ik = 0, \quad (60)$$
$$p_x - 2i\omega p - 2\eta q = 0, \quad (61)$$
$$k_x + 2i\omega k - 2\eta r = 0, \quad (62)$$
$$\eta_x + rp + kq = 0. \quad (63)$$

This system is nothing but the Hirota-Maxwell-Bloch equation. Its Lax representation reads as

$$\Psi_x = A\Psi, \quad (64)$$
$$\Psi_t = [-4i\epsilon_2\lambda^3\sigma_3 + B]\Psi, \quad (65)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (66)$$
$$B = \lambda^2 B_2 + \lambda B_1 + B_0 + \frac{i}{\lambda+\omega}B_{-1}. \quad (67)$$

Here

$$B_2 = 4\epsilon_2 A_0, \quad (68)$$
$$B_1 = 2i\epsilon_2 rq\sigma_3 + 2i\epsilon_2\sigma_3 A_{0x}, \quad (69)$$
$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \quad (70)$$
$$B_0 = \epsilon_2(r_x q - rq_x)\sigma_3 + B_{01}, \quad (71)$$
$$B_{01} = \begin{pmatrix} 0 & -\epsilon_2 q_{xx} - 2\epsilon_2 rq^2 \\ \epsilon_2 r_{xx} + 2\epsilon_2 qr^2 & 0 \end{pmatrix}, \quad (72)$$
$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \quad (73)$$

This system we can reduce to the form

$$iq_t + i\epsilon_2(q_{xxx} + 6\delta|q|^2 q_x) - 2ip = 0, \quad (74)$$
$$p_x - 2i\omega p - 2\eta q = 0, \quad (75)$$
$$\eta_x + \delta(q^* p + p^* q) = 0. \quad (76)$$

V. The (1+1)-Dimensional M-XCIV Equation

The Myrzakulov-XCIV equation or shortly M-XCIV equation reads as [3]:

$$iS_t + 0.5\epsilon_1[S, S_{xx}] + \epsilon_2 i[S_{xxx} + 6(\beta S)_x] + \frac{1}{\omega}[S, W] = 0, \quad (77)$$

$$iW_x + \omega[S, W] = 0. \quad (78)$$

A. Lax Representation

The Lax pair of the M-XCIV equation (77)-(78) is given by

$$U = -i\lambda S, \quad (79)$$
$$V = \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda+\omega}V_{-1} - \frac{i}{\omega}V_{-1}, \quad (80)$$

where [3]

$$V_3 = -4i\epsilon_2 S, \quad (81)$$
$$V_2 = -2i\epsilon_1 S + 2\epsilon_2 SS_x, \quad (82)$$
$$V_1 = \epsilon_1 SS_x + \epsilon_2 i(S_{xx} + 6\beta S), \quad (83)$$
$$V_{-1} = W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix} \quad (84)$$

with $\beta = rq = 0.125tr[(S_x)^2]$.

B. Reductions

The M-XCIV equation admits some integrable reductions. For example, it has the following integrable reductions.

1) The M-XCIX Equation: Let $\epsilon_2 = 0$. Then the M-XCIV equation takes the form

$$iS_t + 0.5\epsilon_1[S, S_{xx}] + \frac{1}{\omega}[S, W] = 0, \quad (85)$$
$$iW_x + \omega[S, W] = 0. \quad (86)$$

It has the Lax pair of the form

$$U = -i\lambda S, \quad (87)$$
$$V = \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda+\omega}W - \frac{i}{\omega}W, \quad (88)$$

where [3]

$$V_2 = -2i\epsilon_1 S, \quad (89)$$
$$V_1 = \epsilon_1 SS_x, \quad (90)$$
$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \quad (91)$$

2) The M-LXIV Equation: Now let us consider the case $\epsilon_1 = 0$. In this case the M-XCIV equation transforms to the equation

$$iS_t + \epsilon_2 i[S_{xxx} + 6(\beta S)_x] + \frac{1}{\omega}[S, W] = 0, \quad (92)$$
$$iW_x + \omega[S, W] = 0. \quad (93)$$

The corresponding Lax pair reads as

$$U = -i\lambda S, \quad (94)$$
$$V = \lambda^3 V_3 + \lambda^2 V_2 + \lambda V_1 + \frac{i}{\lambda+\omega}V_{-1} - \frac{i}{\omega}V_{-1}, \quad (95)$$

where [3]

$$V_3 = -4i\epsilon_2 S, \quad (96)$$
$$V_2 = 2\epsilon_2 SS_x, \quad (97)$$
$$V_1 = \epsilon_2 i(S_{xx} + 6\beta S), \quad (98)$$
$$V_{-1} = W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix} \quad (99)$$

with $\beta = rq = 0.125tr[(S_x)^2]$.

C. Equivalent Counterpart

To find the Schrodinger equivalent, we again us the functions φ, p and η as (28)-(30). Finally the Schrodinger equivalent of the (1+1)-dimensional M-XCIV equation has the form (see e.g. [17], [18])

$$iq_t + \epsilon_1(q_{xx} + 2rq^2) + i\epsilon_2(q_{xxx} + 6rqq_x) - 2ip = 0, \quad (100)$$
$$ir_t - \epsilon_1(r_{xx} + 2r^2 q) + i\epsilon_2(r_{xxx} + 6rqr_x) - 2ik = 0, \quad (101)$$
$$p_x - 2i\omega p - 2\eta q = 0, \quad (102)$$
$$k_x + 2i\omega k - 2\eta r = 0, \quad (103)$$
$$\eta_x + rp + kq = 0. \quad (104)$$

This system is nothing but the Hirota-Maxwell-Bloch equation. Its Lax representation reads as

$$\Psi_x = A\Psi, \quad (105)$$
$$\Psi_t = [-4i\epsilon_2\lambda^3\sigma_3 + B]\Psi, \quad (106)$$

where

$$A = -i\lambda\sigma_3 + A_0, \quad (107)$$
$$B = \lambda^2 B_2 + \lambda B_1 + B_0 + \frac{i}{\lambda+\omega}B_{-1}. \quad (108)$$

Here

$$B_2 = -2i\epsilon_1\sigma_3 + 4\epsilon_2 A_0, \tag{109}$$

$$B_1 = 2i\epsilon_2 rq\sigma_3 + 2i\epsilon_2\sigma_3 A_{0x} + 2\epsilon_1 A_0, \tag{110}$$

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{111}$$

$$B_0 = (i\epsilon_1 rq + \epsilon_2(r_x q - rq_x))\sigma_3 + B_{01}, \tag{112}$$

$$B_{01} = \begin{pmatrix} 0 & i\epsilon_1 q_x - \epsilon_2 q_{xx} - 2\epsilon_2 rq^2 \\ i\epsilon_1 r_x + \epsilon_2 r_{xx} + 2\epsilon_2 qr^2 & 0 \end{pmatrix}, \tag{113}$$

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{114}$$

If $p = \delta k^*, r = \delta q^*$, this system we can reduce to the form

$$iq_t + \epsilon_1(q_{xx} + 2\delta|q|^2 q) + i\epsilon_2(q_{xxx} + 6\delta|q|^2 q_x) - 2ip = 0, \tag{115}$$

$$p_x - 2i\omega p - 2\eta q = 0, \tag{116}$$

$$\eta_x + \delta(q^* p + p^* q) = 0. \tag{117}$$

Note that the (1+1)-dimensional HMBE (115)-(117) admits the following integrable reductions.

i) The NSLE as $\epsilon_1 - 1 = \epsilon_2 = p = \eta = 0$:

$$iq_t + q_{xx} + 2\delta|q|^2 q = 0. \tag{118}$$

ii) The (1+1)-dimensional complex mKdV eqation as $\epsilon_1 = \epsilon_2 - 1 = p = \eta = 0$:

$$q_t + q_{xxx} + 6\delta|q|^2 q_x = 0. \tag{119}$$

iii) The (1+1)-dimensional Schrodinger-Maxwell-Bloch equation as $\epsilon_1 - 1 = \epsilon_2 = 0$:

$$\begin{aligned} iq_t + q_{xx} + 2\delta|q|^2 q - 2ip &= 0, && (120) \\ p_x - 2i\omega p - 2\eta q &= 0, && (121) \\ \eta_x + \delta(q^* p + p^* q) &= 0. && (122) \end{aligned}$$

iv) The (1+1)-dimensional complex mKdV-Maxwell-Bloch equation as $\epsilon_1 = \epsilon_2 - 1 = 0$:

$$\begin{aligned} q_t + q_{xxx} + 6\delta|q|^2 q_x - 2p &= 0, && (123) \\ p_x - 2i\omega p - 2\eta q &= 0, && (124) \\ \eta_x + \delta(q^* p + p^* q) &= 0. && (125) \end{aligned}$$

v) The following (1+1)-dimensional equation as $\epsilon_1 = \epsilon_2 = 0$:

$$\begin{aligned} q_t - 2p &= 0, && (126) \\ p_x - 2i\omega p - 2\eta q &= 0, && (127) \\ \eta_x + \delta(q^* p + p^* q) &= 0. && (128) \end{aligned}$$

or

$$\begin{aligned} q_{xt} - 2i\omega q_t - 4\eta q &= 0, && (129) \\ 2\eta_x + \delta(|q|^2)_t &= 0. && (130) \end{aligned}$$

vi) The following (1+1)-dimensional equation as $\delta = 0$:

$$\begin{aligned} iq_t + \epsilon_1 q_{xx} + i\epsilon_2 q_{xxx} - 2ip &= 0, && (131) \\ p_x - 2i\omega p - 2\eta_0 q &= 0, && (132) \end{aligned}$$

where $\eta_0 = 0$. Again we note that all these reductions are integrable by IST. The corresponding Lax representations we get from the Lax representation (105)-(106) as the corresponding reductions.

VI. Conclusion

Heisenberg ferromagnet models play an important role in modern theory of magnets. They are based on nonlinear partial differential equations. Some of these models are integrable by using the Inverse Scattaring Method, and namely their equations are soliton equations. In this paper, we have studied some Heisenberg ferromagnet equations (models) with self-consistent potentials. We have investigated their Lax representations. Also we have found their Schrödinger type equivalent counterparts.

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