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Influence of Intermediate Principal Stress on Solution of Planar Stability Problems

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***Abstract*—**In this paper, von Mises and Drucker-Prager yield criteria, as typical ones that consider the effect of intermediate principal stress σ2, have been selected and employed for investigating the influence of σ2 on the solution of a typical stability problem. The bearing capacity factors have been calculated under plane strain condition (strip footing) and axisymmetric condition (circular footing) using the method of stress characteristics together with the criteria mentioned. Different levels of σ2 relative to the other two principal stresses have been considered. While a higher σ2 entry in yield criterion gives a higher bearing capacity; its entry in equilibrium equations (axisymmetric) causes substantial reduction.

***Keywords*—**Intermediate principal stress, plane strain, axisymmetric, yield criteria.

I. INTRODUCTION

MANY stability problems in geotechnical engineering are in plane strain or axisymmetric conditions. An example would be the strip and circular footings on the surface of ground. At the failure state in these problems; the soil flows in planes in which the displacements develop. In plane strain condition; these are planes in which the strains develop. In axisymmetric problems; they are redial planes (Fig. 1).

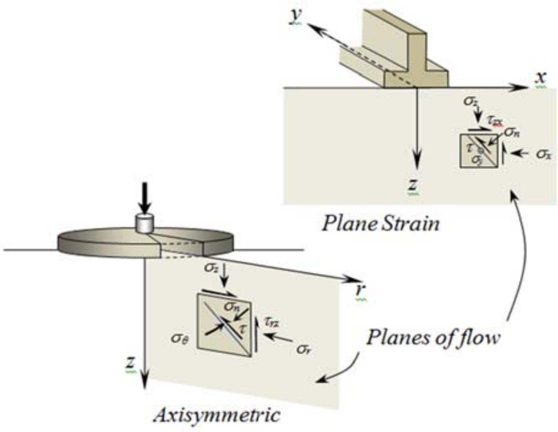

Fig. 1 Planes of flow in a) Plane strain b) axisymmetric cases

The failure state is represented by the Mohr circle having the largest diameter, i.e., σ_1-σ_3. The failure function is obtained by the envelope of such circles in τ-σ diagram as shown in Fig. 2. Any failure criterion suggested for soil, as applied to plane strain or axisymmetric conditions should appear as a functional relationship $\tau = f(\sigma_n)$ or $R = F(\sigma)$ in this diagram. Here, τ and σ_n are the shear and normal stresses on the failure plane at failure. R and σ are the radius of such circle and the distance of its center from the origin, respectively.

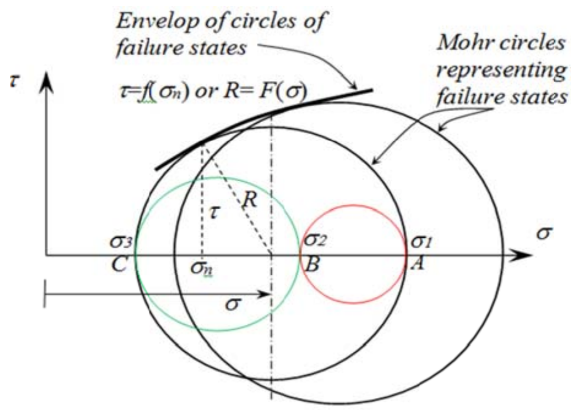

Fig. 2 Failure envelope obtained by pushing the greatest Mohr circle

As seen; the intermediate principal stress σ_2, can take any value between σ_3 and σ_1 when point B moves from C to A in Fig. 2. The effect of σ_2 and its variation on the solution have not been studied before. In triaxial test for example; σ_2 is either assumed to be equal to σ_3 (compression) or σ_1 (extension). The ambiguity about the value of σ_2 and its effects have made the researchers to develop the true triaxial device [1]. The ratio b is usually used to represent its effects. This ratio is defined as:

$$b = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} \tag{1}$$

Increase in σ_2 from σ_3 to σ_1 is equivalent to increase in b from 0 to 1. It is noteworthy that there is a unique relation between the Lode angle θ, and the stress ratio b, as [2]:

$$tan\theta = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{-\sqrt{3}(\sigma_2 - \sigma_3)} = \frac{2-b}{-\sqrt{3}b} \tag{2}$$

Related values of θ and b are typically given in Table I.

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TABLE I
APPROPRIATE VALUES OF *B* AND θ

θ	*b*
-30	1
-60	0.5
-90	0

II. Stress Level Independent Yield Criteria

We begin with criteria that do not depend on stress level first. This is helpful in realizing the elaboration that is added due to stress level dependency later. The simplest form is that suggested by Tresca [3] in 1864; which is relevant to undrained shearing of saturated clay soils. It can be considered as a special case of M-C criterion when $\phi = 0$. This is usually written as:

$$\sigma_1 - \sigma_3 = q_u \tag{3}$$

where q_u is the soil strength in uniaxial compression (unconfined compressive strength).

Writing this criterion in the form of $R = F(\sigma)$ will result in:

$$R = c_u = \frac{q_u}{2} = constant \tag{4}$$

where c_u is the undrained shear strength. As seen; there is no dependency on the intermediate principal stress σ_2, as is the case with its parent M-C criterion.

The other criterion is the one suggested by von Mises [4] in 1913. For saturated clay with strength q_u in undrained condition it would be in the following form:

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_3)^2 + (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2} = \frac{\sqrt{2}}{3}k \tag{5}$$

where k is the von Mises constant that takes different values depending on the condition of the problem.

In contrast to that of Tresca; the von Mises criterion depends on σ_2. Writing it in the form of $R = F(\sigma)$ using b defined above, we get:

$$R = \frac{k}{2\sqrt{1-b(1-b)}} \tag{6}$$

The von Mises constant k, is usually obtained from uniaxial compression test. In such a test; σ_2 and σ_3 are taken equal to zero. The value of b is zero and the constant k would be equal to the unconfined compressive strength, q_u. If these two criteria are set equal at uniaxial compression as it is usual (Fig. 3); envelops of both appear as straight lines parallel to σ-axis on the τ-σ diagram.

While the radius of Mohr circle at failure given by Tresca is independent of σ_2 and b; the radius given by von Mises is function of b. The ratio of the radii of these two failure criteria in Mohr diagram would be a function of b so that:

$$\frac{R_{vonMises}}{R_{Tresca}} = \frac{1}{\sqrt{1-b(1-b)}} \tag{7}$$

This function is drawn vs. b in Fig. 4.

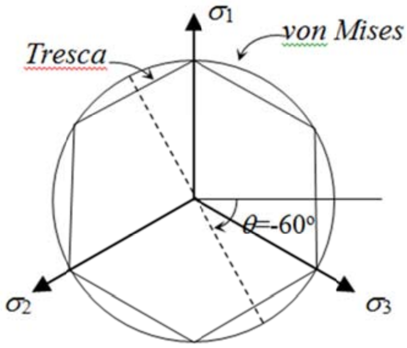


Fig. 3 Comparison between Tresca and von Mises at uniaxial compression

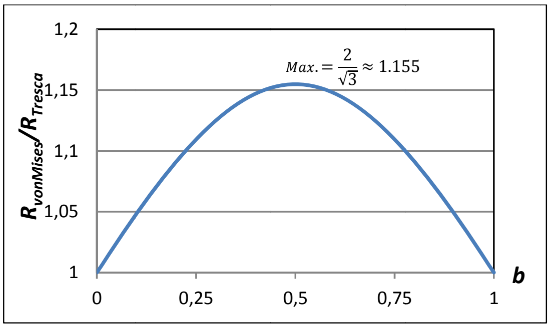


Fig. 4 The ratio of strengths by von Mises to that of Tresca as function of σ_2 and b.

As mentioned; it is customary in the axisymmetric case to assume $\sigma_2 = \sigma_3$ when the flow of soil is away from the axis of symmetry and to assume $\sigma_2 = \sigma_1$ when it is toward it. These are known as von Karman regime. It may be rational to assume σ_2 to be the average of σ_3 and σ_1 in plane strain condition because, in this case, the flow neither can be considered outward nor inward. But let us assume it is a factor K of the average of the other two principal stresses so that:

$$\sigma_2 = K\frac{\sigma_1 + \sigma_3}{2} \tag{8}$$

Accordingly we can write:

$$b = \frac{0.5K(\sigma_1 + \sigma_3) - \sigma_3}{\sigma_1 - \sigma_3} \tag{9}$$

If the material is linear elastic; $K=2\nu$; and if the Poisson's ratio ν, is 0.5; K would be 1. The value of b relevant to this condition would be 0.5.

In general; we can say K_0 is a reasonable assumption for K in plane strain condition. This is consistent with the condition $\phi=0$ relevant to undrained behavior of saturated clay which gives $K=1$ and $b=0.5$; i.e.; the intermediate principal stress would be equal to the average of the other two principal stresses.

If we assume associated flow rule and take the plastic potential function g the same as von Mises yield function, we can investigate the plane strain condition $\varepsilon_2 = \varepsilon_y = 0$, by setting

$\partial g/\partial\sigma_2$ equal to zero [5], [6]. The following will result from this operation on (5):

$$\frac{2b-1}{\sqrt{2[1-b(1-b)]}}=0 \qquad (10)$$

This gives b=0.5 and θ= -60°; where there is maximum deviation of von Mises from Tresca (Fig. 3). Therefore; if the soil behavior obeys associated flow rule and the yield criterion is von Mises; b should be 0.5 (Fig. 5). Substituting this value of b in (6) gives $R=k/\sqrt{3}$. If we compare this with that of Tresca, we can conclude that if these two criteria are to be matched in plane strain condition; k should be equal to $\sqrt{3}c_u$. We should remember that this result has been obtained assuming associativity. But the soil behavior is usually not associated and the value of b in this case may be different from 0.5. It is interesting to see the effect of this deviation on the solution of typical plane strain problems. The bearing capacity of strip footing shown in Fig. 1 on a saturated clay in undrained condition can be considered as an example. The amount the value of b is different from 0.5 can be considered as the degree of nonassociativity of soil behavior. The solution is obtained using the rigorous method of characteristics [2], [7], [8]. For the associated case; the value of bearing capacity factor N_c is 5.14. The solution for nonassociative cases is obtained when different values of b are put in (6) using $k=\sqrt{3}c_u$. The result is shown in Fig. 6 and this figure indicates reduction in bearing capacity with increase in degree of nonassociativity of soil.

The axisymmetric case is not as straight forward as plane strain. The principal strain ε_2=ε_θ is not zero in this case to help finding the proper path. The b value can vary with distance from the axis of symmetry. Another important point is that the intermediate principal stress enters the equilibrium equations in this case. Therefore; in contrast to the plane strain case; the effect of σ_2 and b in this case is debatable even if the yield criterion dos not include σ_2. Fig. 7 shows the variation of N_c for a circular footing with b as predicted by Tresca and von Mises criteria. The starting point of all curves is approximately N_c=5.7. As (3) indicates; Tresca yield criterion does not depend on σ_2. Therefore; the curve related to Tresca in Fig. 7 indicates that entry of higher values of b in equilibrium equations causes reduction in bearing capacity. This is also clear when we compare the curves related to von Mises. The dashed line curve indicates that the entry of higher values of b in yield criterion has an increasing effect on N_c; but when their effect in equilibrium equations is also allowed; their increasing effect is substantially ceased (full line curve).

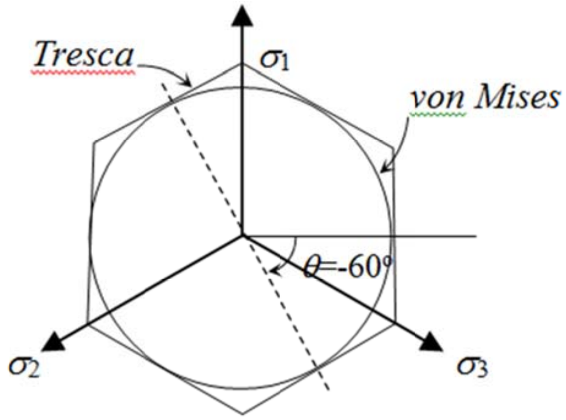


Fig. 5 Comparison between Tresca and von Mises at plane strain condition

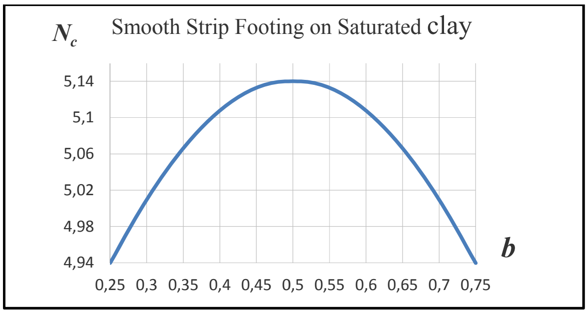


Fig. 6 Effect of variation of b on N_c factor of a saturated clay

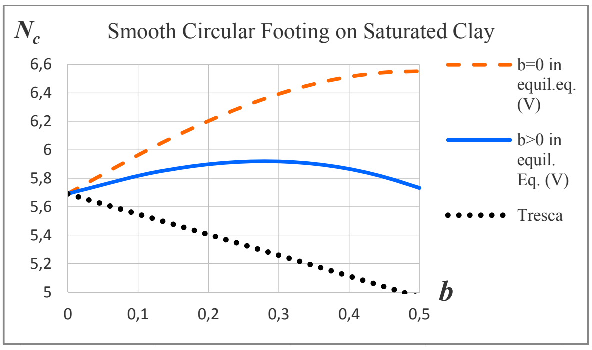


Fig. 7 Effect of b on N_c via yield and equilibrium eqs.

III. Stress Level Dependent Yield Criteria

We have selected Mohr-Coulomb (M-C) and Drucker-Prager (D-P) yield criteria for our investigation in this part [9]; but the discussion can be extended to Lade-Duncan, Masuoka-Nakai and others as well. Here, the M-C can be considered as an extension of Tresca because it does not depend on σ_2. D-P is stress level dependent form of von Mises and considers σ_2. For the sake of clarity; we limit our discussion to frictional soil. The M-C in this case can be written as:

$$\sigma_1-\sigma_3=sin\phi(\sigma_1+\sigma_3) \qquad (11)$$

which can be easily written in $R=F(\sigma)$ form as:

$$R = \sigma sin\phi$$

The D-P criterion in the absence of cohesion can be written as [9], [10]:

$$\sqrt{(\sigma_1-\sigma_3)^2+(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2} = \alpha I_1 \tag{12}$$

where I_1 is the first invariant of stress tensor and α is the D-P parameter. It can be easily shown that the $R = F(\sigma)$ form of D-P in this case, is:

$$R = \frac{3\alpha}{2\sqrt{2}\sqrt{1-b(1-b)}-\alpha(2b-1)}\sigma \tag{13}$$

Comparison with similar form of M-C, indicates that:

$$\alpha = \frac{2\sqrt{2}sin\phi\sqrt{b^2-b+1}}{3+(2b-1)sin\phi} \tag{14}$$

which gives the D-P parameter α, for matching with M-C in general.

Assuming the behavior to be associated; we again set $\partial g/\partial\sigma_2$ equal to zero [3], [4] to get the matching in plane strain condition. This operation on (12) will result in:

$$\frac{2b-1}{\sqrt{2[1-b(1-b)]}} = \alpha \tag{15}$$

Applying the requirement for general matching expressed by (14) to this equation results in:

$$b = \frac{1+sin\phi}{2} \tag{16}$$

which gives the value of b in terms of ϕ for matching of D-P with M-C in plane strain condition. In other words; both criteria predict the same strength for the soil if the value of b is assumed to be that given by (16). Fig. 8 shows the matching of D-P and M-C in plane strain condition.

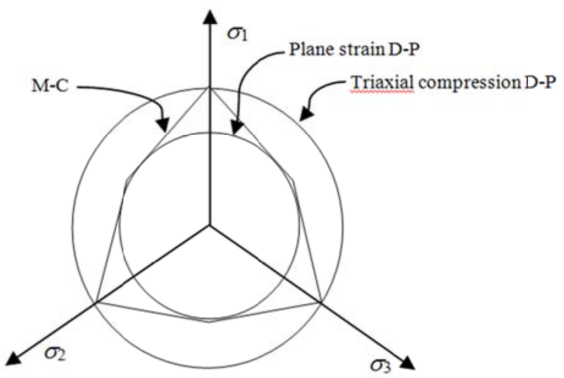
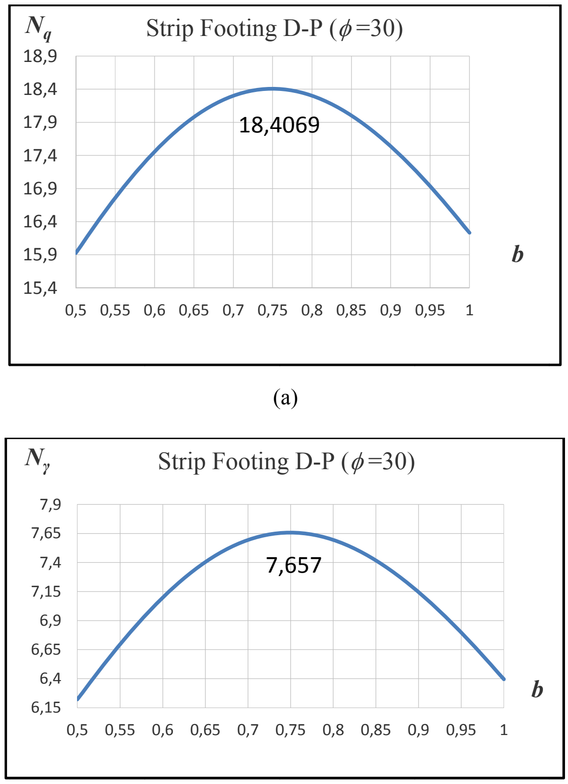


Fig. 8 Matching of Drucker-Prager and Mohr-Coulomb criteria

The actual value of b can be different from what is given by (16) due to nonassociativity in soil behavior. To investigate the effect of variation of b on the solution of plane strain problems like bearing capacity; we can assume we have found the D-P parameter α, on the basis of (15). We then substitute this value of α in (14) and find an equivalent value for ϕ in terms of b. In this way; the variation of b affects the material equivalent strength parameter ϕ. Values of b different from what is given by (16) will result in equivalent friction angles less than the original. This is clear from Fig. 8 as we see that the D-P circle lies inside of M-C hexagon. Fig. 9 shows the variation of bearing capacity factors N_q and N_γ for smooth strip footing with b when ϕ=30. The value of b relevant to associative behavior in this case is 0.75 and the figure indicates lower bearing capacity factors when the behavior of soil is nonassociated; i.e.; when b is different from 0.75. Therefore; it is not conservative to go with the usual bearing capacity factors if the soil behavior is really nonassociated.

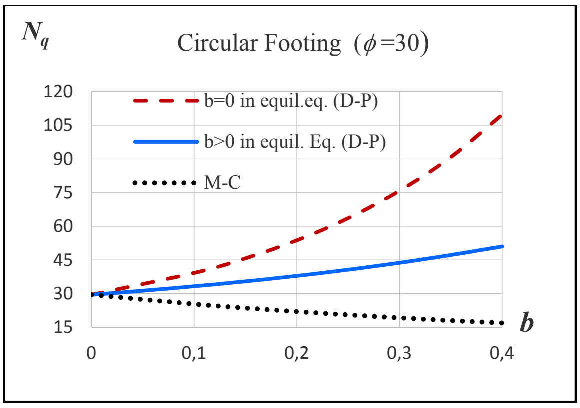


(a)

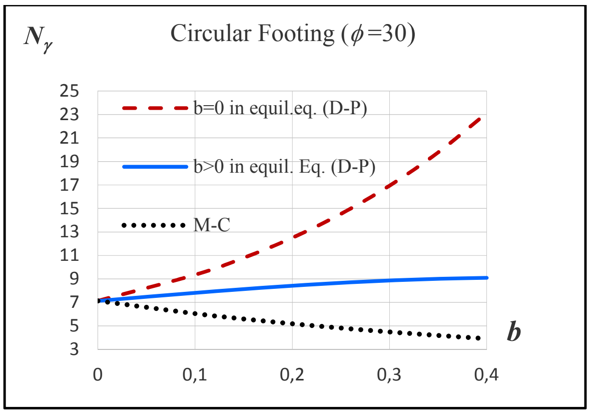
(b)

Fig. 9 Effect of variation of b on (a) N_q and (b) N_γ for strip footing (ϕ =30°)

As mentioned before; the intermediate principal stress participates in the equilibrium equations in case of axial symmetry. In order to see the effect of its entry in equilibrium equations on bearing capacity calculations; we have chosen the M-C criterion because it does not include σ_2. It is usual to assume σ_2=σ_3 in this condition i.e.; b=0. But this is merely an assumption and it is interesting to see if σ_2 is really greater than σ_3; how are the values of N_q and N_γ influenced by this matter. The result of investigation for ϕ = 30° is demonstrated by the curve drawn for M-C in Fig. 10. The curve indicates entry of higher values of b in equilibrium equations results in lower values for bearing capacity factors, N_q and N_γ. Matching of other yield criteria like D-P with M-C under axial compression relevant to bearing capacity problem of circular footings is made at the top point of Fig. 8, where b is zero. For other points; the D-P circles lies outside of the M-C hexagon. This indicates higher strength with increase in b. This effect is demonstrated by the dashed line curve in Fig. 10. When the effect of σ_2 being greater than σ_3 is considered both in equilibrium and yield equations; the full line curve of Fig. 10 is obtained. Comparison of these two curves indicates again, the reduction in N_q and N_γ due to participation of σ_2 in equilibrium equations. The full line curve indicates the net effect has been increase in bearing capacity factors in case of $\phi = 30^o$.



(a)



(b)

Fig. 10 Effect of variation of b on (a) N_q and (b) N_γ for circular footing

IV. Conclusion

Tresca and Mohr-Coulomb yield criteria have long been criticized for not considering the effect of intermediate principal stress. As typical stress level- dependent and independent yield criteria that consider the effect of σ_2; von Mises and Drucker-Prager were selected for investigating the effect of σ_2 on solution of bearing capacity problem under plane strain and axisymmetric conditions. It was found that in plane strain condition; the value of σ_2 may be different from what is usually assumed and this may be due to non-associativity in soil behavior. It was shown that under such a condition, a lower bearing capacity is obtained. Therefore; the usual solution to bearing capacity of strip footings would not be conservative.

In the axisymmetric conditions however; σ_2 enters the equilibrium equations as well; and if the Mohr-Coulomb criterion is employed for bearing capacity calculation of circular footings; increase in σ_2 relative to other principal stresses results in decrease in bearing capacity. Therefore; if in reality, σ_2 is greater than σ_3; this effect should be considered in calculations. If other criteria that include σ_2 are used instead of Mohr-Coulomb; the calculated bearing capacity may be less or more than what is usually calculated, depending on the criterion used, and the amount σ_2 has been taken more than σ_3. In case of Drucker-Prager for example; the net effect has been shown here to be a little increase in the calculated bearing capacity.

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