

Inference of Stress-Strength Model for a Lomax Distribution

H. Panahi and S. Asadi

Abstract—In this paper, the estimation of the stress-strength parameter $R = P(Y < X)$, when X and Y are independent and both are Lomax distributions with the common scale parameters but different shape parameters is studied. The maximum likelihood estimator of R is derived. Assuming that the common scale parameter is known, the Bayes estimator and exact confidence interval of R are discussed. Simulation study to investigate performance of the different proposed methods has been carried out.

Keywords—Stress-Strength model; maximum likelihood estimator; Bayes estimator; Lomax distribution

I. INTRODUCTION

THE reliability of a system is the probability that when operating under stated environmental conditions, the system will perform its intended function adequately. For stress-strength models both the strength of the system, X , and the stress, Y , imposed on it by its operating environments are considered to be random variables. The reliability, R , of the system is the probability that the system is strong enough to overcome the stress imposed on it, that is to say, $R = P(Y < X)$. Several authors have studied the problem of estimating R , including, [1]-[7].

In this article, we consider the reliability, R , when X and Y are independent but not identically distributed Lomax random variables. The Lomax distribution has a position of importance in a field of life testing because of its uses to fit business failure data. This distribution has been used by several authors, see for example, [8]-[13].

The Lomax distribution has the following distribution function for $X > 0$:

$$F(x; \alpha, \lambda) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha} \quad \alpha, \lambda > 0 \quad (1)$$

Therefore, this distribution has the density function for $X > 0$ as

$$f(x; \alpha, \lambda) = \frac{\alpha}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)} \quad \alpha, \lambda > 0 \quad (2)$$

Here α and λ are the shape and scale parameters respectively. The purpose of this paper is to focus on the inference on $R = P(Y < X)$, where X and Y are independent

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Lomax distributions with the same scale parameter λ but different shape parameters α_1 and α_2 respectively. First, we provide the maximum likelihood estimator of reliability, R , and it is observed that the MLE can be obtained by solving a non-linear equation and we propose a simple iterative scheme to solve the non-linear equation. We also consider the case when the scale parameters are known. In this case, we compute the Bayes estimates with mean squared error loss functions corresponding to conjugate prior and exact confidence interval of R . This paper can be organized as follows. In Section II, we derive the MLE of R . Different estimation procedures of R if λ is known are considered in Section III. Monte Carlo simulation results are presented in Section IV. Finally, some concluding remarks are presented in section V.

II. MAXIMUM LIKELIHOOD ESTIMATOR OF R

Let X and Y are two independent Lomax random variables with parameters (α_1, λ) and (α_2, λ) respectively. Therefore

$$\begin{aligned} R &= \int_0^{\infty} \int_0^x \frac{\alpha_1}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha_1+1)} \frac{\alpha_2}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha_2+1)} dy dx \\ &= \int_0^{\infty} \left(1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha_2}\right) \cdot \frac{\alpha_1}{\lambda} \left(1 + \frac{x}{\lambda}\right)^{-(\alpha_1+1)} dx \\ &= \frac{\alpha_2}{\alpha_1 + \alpha_2} \end{aligned} \quad (3)$$

Now to compute the MLE of R , first we obtain the MLE's of α_1 and α_2 . Let X_1, \dots, X_n be a random sample from Lomax (α_1, λ) and Y_1, \dots, Y_m be a random sample from Lomax (α_2, λ) . Therefore the log-likelihood function of α_1 , α_2 and λ for the observed sample is

$$\begin{aligned} L(x, \alpha, \lambda) &= n \ln \alpha_1 + m \ln \alpha_2 - (n+m) \ln \lambda \\ &\quad - (\alpha_1 + 1) \sum_{i=1}^n \ln \left(1 + \frac{x_i}{\lambda}\right) - (\alpha_2 + 1) \sum_{j=1}^m \ln \left(1 + \frac{y_j}{\lambda}\right) \end{aligned} \quad (4)$$

The likelihood equations for parameters α_1 , α_2 and λ are given by

$$\frac{\partial L}{\partial \alpha_1} = 0 = \frac{n}{\alpha_1} - \sum_{i=1}^n \ln \left(1 + \frac{x_i}{\lambda}\right) \quad (5)$$

$$\frac{\partial L}{\partial \alpha_2} = 0 = \frac{m}{\alpha_2} - \sum_{j=1}^m \ln \left(1 + \frac{y_j}{\lambda}\right) \quad (6)$$

$$\frac{\partial L}{\partial \lambda} = 0 = \frac{n+m}{\lambda} + (\alpha_1 + 1) \sum_{i=1}^n \frac{x_i}{\lambda(\lambda + x_i)} + (\alpha_2 + 1) \sum_{j=1}^m \frac{y_j}{\lambda(\lambda + y_j)} \tag{7}$$

From (5) and (6), we obtain the maximum likelihood estimators of α_1 and α_2 as

$$\hat{\alpha}_1 = \frac{n}{\sum_{i=1}^n \ln\left(1 + \frac{x_i}{\hat{\lambda}}\right)} \tag{8}$$

$$\hat{\alpha}_2 = \frac{m}{\sum_{j=1}^m \ln\left(1 + \frac{y_j}{\hat{\lambda}}\right)} \tag{9}$$

If the scale parameter is known, the MLE of α_1 and α_2 can be obtained from (8) and (9). If all the parameters are unknown, we can first estimate the scale parameter by solving the non-linear equation

$$\frac{n+m}{\lambda} + \left(\frac{n}{\sum_{i=1}^n \ln\left(1 + \frac{x_i}{\hat{\lambda}}\right)} + 1 \right) \sum_{i=1}^n \frac{x_i}{\lambda(\lambda + x_i)} + \left(\frac{m}{\sum_{j=1}^m \ln\left(1 + \frac{y_j}{\hat{\lambda}}\right)} + 1 \right) \sum_{j=1}^m \frac{y_j}{\lambda(\lambda + y_j)} = 0 \tag{10}$$

Equation (10) can be written as

$$h(\lambda) = \lambda \tag{11}$$

Where

$$h(\lambda) = (n+m) \left[\left(\frac{n}{\sum_{i=1}^n \ln\left(1 + \frac{x_i}{\hat{\lambda}}\right)} + 1 \right) \sum_{i=1}^n \frac{x_i}{\lambda(\lambda + x_i)} + \left(\frac{m}{\sum_{j=1}^m \ln\left(1 + \frac{y_j}{\hat{\lambda}}\right)} + 1 \right) \sum_{j=1}^m \frac{y_j}{\lambda(\lambda + y_j)} \right]^{-1} \tag{12}$$

We propose a simple iterative scheme to solve for λ from (11). Start with an initial guess $\lambda_{(0)}$, obtain $\lambda_{(1)} = h(\lambda_{(0)})$, similarly, $\lambda_{(2)} = h(\lambda_{(1)})$ and so on. Stop the iterative procedure, when $|\lambda_{(i+1)} - \lambda_{(i)}| < \epsilon$, some pre-assigned tolerance limit. Once we obtain $\hat{\lambda}$, $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can be obtained

from (8) and (9) respectively. Therefore, the MLE of R becomes

$$\hat{R} = \frac{\hat{\alpha}_2}{\hat{\alpha}_1 + \hat{\alpha}_2} \tag{13}$$

III. ESTIMATION OF R IF λ IS KNOWN

A. Bayesian Inference

In this subsection, we obtain the Bayes estimator of R under the assumptions that the parameters α_1 and α_2 are random variables for both the populations. When the scale parameters are known, without loss of generality, we can assume that $\lambda = 1$. It is assumed that $\alpha_1 \sim \text{Gamma}(a_1, b_1)$ and $\alpha_2 \sim \text{Gamma}(a_2, b_2)$ respectively. Therefore the joint posterior density of α_1 and α_2 given the data as

$$\pi(\alpha_1, \alpha_2 | x, y; \lambda) = \frac{(T_1)^{n+a_1} (T_2)^{m+a_2}}{\Gamma(n+a_1) \Gamma(m+a_2)} \alpha_1^{n+a_1-1} \alpha_2^{m+a_2-1} \exp\{-\alpha_1 T_1 - \alpha_2 T_2\} \tag{14}$$

$$T_1 = b_1 + \sum_{i=1}^n \ln(1 + x_i) \quad \text{And} \quad T_2 = b_2 + \sum_{j=1}^m \ln(1 + y_j)$$

Therefore the Bayes estimator of R with respect to the mean squared error loss function is

$$\hat{R}_{BS} = E(R | x_1, \dots, x_n, y_1, \dots, y_m; \lambda) \tag{15}$$

After making suitable transformations and simplifications we get

$$\hat{R}_{BS} = \frac{(T_1)^{n+a_1} (T_2)^{m+a_2}}{B(n+a_1, m+a_2)} \int_0^1 \frac{u^{n+a_1}}{(1-u)^{m+a_2+1}} \times \left(T_1 \frac{u}{1-u} + T_2 \right)^{-(n+a_1+m+a_2)} du \tag{16}$$

Which, can be rewritten as

$$\hat{R}_{BS} = \frac{(T_1)^{n+a_1}}{(T_2)^{n+a_1} B(n+a_1, m+a_2)} \int_0^1 u^{n+a_1} (1-u)^{n+a_1+1} \times \left(1 - \left(1 - \frac{T_1}{T_2} \right) u \right)^{-(n+a_1+m+a_2)} du \tag{17}$$

$$= \left(\frac{T_1}{T_2} \right)^{n+a_1} \left(\frac{n+a_1}{n+m+a_1+a_2} \right) F(n+m+a_1+a_2; n+a_1+1; n+m+a_1+a_2+1; 1 - \frac{T_1}{T_2}); T_1 < T_2$$

And, using the transformation $w = 1 - u$, we obtain

$$\hat{R}_{BS} = \left(\frac{T_2}{T_1}\right)^{m+a_2} \left(\frac{n+a_1}{n+m+a_1+a_2}\right) F(n+m+a_1+a_2; m+a_2; n+m+a_1+a_2+1; 1-\frac{T_2}{T_1}); \text{ if } T_2 < T_1 \quad (18)$$

and

$$\hat{R}_{BS} = \frac{n+a_1}{n+a_1+m+a_2}; \text{ if } T_2 = T_1$$

Combining the above equations, we get [6]

$$\hat{R}_{BS} = \begin{cases} \left(\frac{T_2}{T_1}\right)^{m+a_2} \left(\frac{n+a_1}{n+m+a_1+a_2}\right) F(n+m+a_1+a_2; m+a_2; n+m+a_1+a_2+1; 1-\frac{T_2}{T_1}); \text{ if } T_2 \leq T_1 \\ \left(\frac{T_1}{T_2}\right)^{n+a_1} \left(\frac{n+a_1}{n+m+a_1+a_2}\right) F(n+m+a_1+a_2; n+a_1+1; n+m+a_1+a_2+1; 1-\frac{T_1}{T_2}); \text{ if } T_1 < T_2 \end{cases} \quad (19)$$

Where $F(\alpha, \beta, \lambda, \delta)$ is Gauss hyper geometric function given by:

$$F(\alpha, \beta, \lambda, \delta) = 1 + \frac{\alpha\beta}{\lambda.1} \delta + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\lambda(\lambda+1).1.2} \delta^2 + \dots \quad (20)$$

Notice that the Bayes estimator \hat{R}_{BS} depends on the parameters of the prior distributions of α_1 and α_2 . These parameters could be estimated by means of an empirical Bayes procedure. (See, Lindely [13] and Awad and Gharraf [4]).

B. Exact Confidence Interval of R

Let X_1, \dots, X_n and Y_1, \dots, Y_m be random samples from Lomax distribution with parameters $(\alpha_1, 1)$ and $(\alpha_2, 1)$, respectively.

Since $U = \sum_{i=1}^n \ln(1+x_i)$ and $V = \sum_{j=1}^m \ln(1+y_j)$ are independent

with gamma distribution with parameters (n, α_1) and (m, α_2)

respectively, $2\alpha_1 \sum_{i=1}^n \ln(1+x_i)$ and $2\alpha_2 \sum_{j=1}^m \ln(1+y_j)$ are

independent with chi square distributions with degree of freedom $2n$ and $2m$. Therefore, $\hat{R}_{MLE} = \frac{1}{1 + \frac{\alpha_1}{\alpha_2} F_1}$, this

equation can be written as follow

$$\frac{R}{1-R} \times \frac{1-\hat{R}_{MLE}}{\hat{R}_{MLE}} = F_1 \quad (21)$$

Using F_1 as a pivotal quantity, we obtain a $100(1-\gamma)\%$ confidence interval for R as

$$\left[\frac{1}{1 + F_{2n, 2m; 1-\frac{\gamma}{2}} \left(\frac{1}{\hat{R}} - 1\right)}, \frac{1}{1 + F_{2n, 2m; \frac{\gamma}{2}} \left(\frac{1}{\hat{R}} - 1\right)} \right] \quad (22)$$

Where, $F_{2n, 2m; \frac{\gamma}{2}}$ and $F_{2n, 2m; 1-\frac{\gamma}{2}}$ are the lower and upper $\frac{\gamma}{2}$ percentile points of a F distribution with $2n$ and $2m$ degrees of freedom.

IV. SIMULATION RESULTS

In this section we mainly perform some simulation experiments to observe the behavior of the different methods for different sample sizes and for different parameter values. We mainly compare the performances of the MLEs and the Bayes estimates with respect to the squared error loss function in terms of biases and mean squares errors (MSEs). We also obtain confidence interval obtained by using (22) in terms of the average confidence lengths and coverage percentages. We consider the following small sample sizes; $n=m= 5, 10, 20$ and 30 and we take $\alpha_1=1.5$, and $\alpha_2=1.0, 1.5, 2.0$ respectively. Without loss of generality, we take $\lambda=1$. All the results are based on 1000 replications. We obtain the estimates of R by MLE and by using the Bayesian procedure under squared error loss function. We do not have any prior information on R, and therefore, we prefer to use the non-informative prior to compute Bayes estimates. The results are reported in Tables I and II respectively.

V. CONCLUDING REMARKS

In this paper we consider the estimation of the stress-strength parameter of Lomax distribution. It is assumed that the two populations have the same scale parameters, but different shape parameters. It is observed that the maximum likelihood estimators of the unknown parameters can be obtained by solving one non-linear equation. We provide one simple iterative procedure to compute the MLEs of the unknown parameters and in turn to compute the MLE of R. When the scale parameter is known we obtain Bayes estimates and exact confidence interval of R. It is observed that the Bayes estimators with non-informative priors behave quite similarly with the MLEs. From a simulation study, we also observe the following:

- 1- For a fixed (m, n) , as parameter α_2 increases the average relative MSE of the estimates decrease quite rapidly.
- 2- We observed that when $m; n$ increase then MSEs of all the estimators decrease.
- 3- The performance of the Bayes estimators also are quite satisfactory. The MSEs of the Bayes estimators are smaller than the MSEs of the MLEs.
- 4- The average lengths of all intervals decrease as (m, n) increases.

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TABLE I
BIASES AND MSEs OF THE MLEs AND BAYES ESTIMATORS OF R, WHEN
 $\alpha_1 = 1.5$ AND FOR DIFFERENT VALUES OF α_2 .

(n, m)	Methods → α_2 ↓	MLE	BAYES
(5,5)	1.0	-0.00097(0.03459)	-0.04168(0.02826)
	1.5	-0.01758(0.03137)	-0.09076(0.02743)
	2.0	-0.02203(0.02031)	-0.01335(0.01764)
(10,10)	1.0	-0.00651(0.01002)	-0.03593(0.00981)
	1.5	-0.01215(0.00833)	-0.15217(0.00780)
	2.0	-0.01302(0.00632)	-0.08524(0.00591)
(20,20)	1.0	-0.00131(0.00716)	0.01530(0.00722)
	1.5	-0.00466(0.00709)	-0.05368(0.00712)
	2.0	-0.00651(0.00591)	-0.08788(0.00563)
(30,30)	1.0	-0.00065(0.00558)	-0.02388(0.00527)
	1.5	-0.00531(0.00464)	-0.03961(0.00443)
	2.0	-0.00662(0.00316)	-0.01682(0.00358)

TABLE II
AVERAGE LENGTH AND COVERAGE PERCENTAGE OF THE INTERVALS

α_2 → (n, m) ↓	1.0	1.5	2.0
(5,5)	0.43267(94.56)	0.41956(95.11)	0.34743(95.24)
(10,10)	0.31741(94.88)	0.30621(94.63)	0.24768(94.44)
(20,20)	0.28233(95.02)	0.27066(94.81)	0.21893(94.68)
(30,30)	0.22891(95.22)	0.22049(95.03)	0.17485(95.01)

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