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Induced Acyclic Path Decomposition in Graphs

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Abstract—A decomposition of a graph G is a collection ψ of graphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. If each H_i is either an induced path in G, then ψ is called an induced acyclic path decomposition of G and if each H_i is a (induced) cycle in G then ψ is called a (induced) cycle decomposition of G. The minimum cardinality of an induced acyclic path decomposition of G is called the induced acyclic path decomposition number of G and is denoted by $\pi_{ia}(G)$. Similarly the cyclic decomposition number $\pi_c(G)$ is defined. In this paper we begin an investigation of these parameters.

Keywords—Cycle decomposition, Induced acyclic path decomposition, Induced acyclic path decomposition number.

I. Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected graph without loops or multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [8]. The order and size of a graph are denoted by n and m respectively.

Let $P = (v_1, v_2, \ldots, v_r)$ be a path in a graph $G = (V, E)$. The vertices $v_2, \ldots, v_{r-1}$ are called internal vertices of P and v_1 and v_r are called external vertices of P. Two paths P and Q of a graph G are said to be internally disjoint if no vertex of G is an internal vertex of both P and Q. If $P = (v_0, v_1, v_2, \ldots, v_r)$ and $Q = (v_r = w_0, w_1, w_2, \ldots, w_s)$ are two paths in G, then the walk obtained by concatenating P and Q at v_r is denoted by $P \circ Q$ and the path $(v_n, v_{n-1}, \ldots, v_0)$ is denoted by P^{-1}.

A decomposition of a graph G is a collection of subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i. Various types of decompositions and corresponding parameters have been studied by several authors by imposing conditions on the members of the decomposition. Some such decomposition parameters are path decomposition number, acyclic path decomposition number and simple acyclic path decomposition number which are defined as follows.

Let $\psi = \{H_1, H_2, \ldots, H_r\}$ be a decomposition of a graph G. If each H_i is a path or a cycle, then ψ is called a path decomposition of G. If each H_i is a path, then ψ is called an *acyclic path decomposition* of G. Further, an acyclic path decomposition in which any two paths have at most one vertex in common is called a *simple acyclic path decomposition* of G. The minimum cardinality of a path decomposition (acyclic path decomposition, simple acyclic path decomposition) of G is called the *path decomposition number (acyclic path decomposition number, simple acyclic path decomposition number)* of G and is denoted by $\pi(G), (\pi_a(G), \pi_{as}(G))$.

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The parameter π_a was introduced by Harary [9] and further studied by Harary and Schwenk [10], Peroche [11], Stanton *et al.* [13] and Arumugam and Suresh Suseela [7] who used the notation π for the acyclic path decomposition number of G and called an acyclic path decomposition as path cover. The parameter π_{as} was introduced by Arumugam and Sahul Hamid [5] who used π_s for simple acyclic path decomposition number and called a simple acyclic path decomposition as a simple path cover and the parameter π was introduced by Arumugam *et al.* [6].

Further, by imposing on each of the decomposition defined above the condition that every vertex of G is an internal vertex of at most one member of the decomposition, we get an another set of path covering parameters, namely, graphoidal covering number $\eta(G)$, acyclic graphoidal covering number $\eta_a(G)$, simple graphoidal covering number $\eta_s(G)$ and simple acyclic graphoidal covering number $\eta_{as}(G)$ and all these parameters can be found respectively in [1], [7], [4] and [3].

In [5] it has been observed that every member of a simple acyclic path decomposition of a graph G is an induced path in G. However, a collection ψ of induced paths such that every edge of G is exactly one path in ψ need not be a simple acyclic path decomposition of G. Motivated by this observation, Arumugam [2] introduced the concept of induced path decomposition and induced path decomposition number of a graph and Sahul Hamid and Abraham [12] initiated the study of this parameter.

In this paper we introduce two more decomposition parameters, namely, induced acyclic path decomposition number, cycle decomposition number and we begin an investigation of these parameters.

II. Induced acyclic path decomposition

In this section we introduce the notion of induced acyclic path decomposition and induced acyclic path decomposition number and determine the value of this parameter for several families of graphs such as complete graphs, complete bipartite graphs, wheels, trees and unicyclic graphs. Further we exhibit the relation between this parameter with some existing decomposition parameters.

Definition 2.1. *An acyclic path cover ψ of G such that every path in ψ is an induced path is called an induced acyclic path cover. The minimum cardinality of an induced acyclic path cover of G is called by the induced acyclic path covering number of G and is denoted by $\pi_{ia}(G)$.*

Example 2.2. *Consider the graph G given in Figure 1.*

Here $\psi = \{(v_1, v_2, v_4, v_6), (v_3, v_4, v_5), (v_5, v_2, v_3)\}$ is a minimum induced acyclic path cover of G so that $\pi_{ia}(G) = 3$.

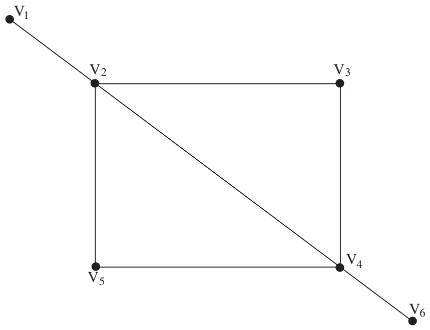

Fig. 1. Induced acyclic path decomposition in a graph.

Remark 2.3. *Since the edges are the only induced path of a complete graph K_n it follows that $\pi_{ia}(K_n) = \frac{n}{2}$. Since every acyclic path decomposition of a tree T is induced and since it has been proved that $\pi_{ia}(T) = \frac{k}{2}$ where k is the number of vertices of odd degree.*

Theorem 2.4. *For the complete bipartite graph $K_{r,s}$ with $r \leq s$ we have $\pi_{ia}(K_{r,s}) = \lceil \frac{rs}{2} \rceil$.*

Proof: Let

$$X = \{x_1, x_2, \ldots, x_r\} \quad \text{and} \quad Y = \{y_1, y_2, \ldots, y_s\}$$

be the bipartition of $K_{r,s}$. We observe that every member of an induced acyclic path cover of $K_{r,s}$ is either a path of length two or an edge. Let

$$\begin{aligned} P_{ij} &= (x_{2i-1}, y_j, x_{2i}), \quad \forall i = 1, 2, \ldots, \left\lfloor \frac{r}{2} \right\rfloor, \\ &\qquad j = 1, 2, \ldots, s, \\ Q_i &= (y_{2i-1}, x_r, y_{2i}), \quad \forall i = 1, 2, \ldots, \left\lceil \frac{s}{2} \right\rceil - 1, \\ Q_{\lceil \frac{s}{2} \rceil} &= \begin{cases} (x_r, y_s) & \text{if } s \text{ is odd,} \\ (y_{s-1}, x_r, y_s) & \text{if } s \text{ is even.} \end{cases} \end{aligned}$$

Now let $\psi = \bigcup_{i=1}^{\frac{r}{2}} (\bigcup_{j=1}^{s} P_{ij})$ if r is even and let

$$\psi = \left(\bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \left(\bigcup_{j=1}^{s} P_{ij} \right) \right) \cup \{Q_1, Q_2, \ldots, Q_{\lceil \frac{s}{2} \rceil}\}, \text{ otherwise.}$$

Then ψ is an induced acyclic path cover of $K_{r,s}$. Now, if r is even, then $|\psi| = \frac{rs}{2}$. Suppose r is odd. Then

$$\begin{aligned} |\psi| &= \begin{cases} \frac{(r-1)s}{2} + \frac{s}{2} & \text{if } s \text{ is even} \\ \frac{(r-1)s}{2} + \frac{s+1}{2} & \text{if } s \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{rs}{2} & \text{if } s \text{ is even} \\ \frac{rs+1}{2} & \text{if } s \text{ is odd} \end{cases} \\ &= \left\lceil \frac{rs}{2} \right\rceil. \end{aligned}$$

Hence $\pi_{ia}(K_{r,s}) \leq |\psi| = \lceil \frac{rs}{2} \rceil$.

Further, since every member of any induced acylic path covers of ψ covers at most two edges, we have $|\psi| \geq \lceil \frac{rs}{2} \rceil$ so that $\pi_{ia}(K_{r,s}) \geq \lceil \frac{rs}{2} \rceil$.

Thus $\pi_{ia}(K_{r,s}) = \lceil \frac{rs}{2} \rceil$. ■

Theorem 2.5. *For the wheel $W_n = C_{n-1} + K_1$, we have*

$$\pi_{ia}(W_n) = \begin{cases} 6 & \text{if } n = 4, \\ \lceil \frac{n-1}{2} \rceil + 2 & \text{otherwise.} \end{cases}$$

Proof: If $n = 4$, then $W_n = K_4$ so that $\pi_{ia}(W_n) = 6$. Assume $n \geq 5$. Let

$$\begin{aligned} V(W_n) &= \{v_0, v_1, v_2, \ldots, v_{n-1}\} \text{ and} \\ E(W_n) &= \{v_0 v_i \colon 1 \leq i \leq n-1\} \\ &\qquad \cup \{v_i v_{i+1} \colon 1 \leq i \leq n-2\} \cup \{v_{n-1}, v_1\}. \end{aligned}$$

Let

$$\begin{aligned} P_i &= \begin{cases} v_i, v_0, v_{\frac{n-1}{2}+i} & \text{if } n \text{ is odd,} \\ v_i, v_0, v_{\frac{n-2}{2}+i} & \text{if } n \text{ is even,} \end{cases} \\ &\qquad \forall i = 1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil - 1 \\ P_{\lceil \frac{n-1}{2} \rceil} &= \begin{cases} (v_{\frac{n-1}{2}}, v_0, v_{n-1}) & \text{if } n \text{ is odd,} \\ (v_0, v_{n-1}) & \text{if } n \text{ is even,} \end{cases} \\ Q_1 &= (v_1, v_2, v_3) \text{ and} \\ Q_2 &= (v_3, v_4, \ldots, v_{n-1}, v_1). \end{aligned}$$

Then $\psi = \{P_1, P_2, \ldots, P_{\lceil \frac{n-1}{2} \rceil}, Q_1, Q_2\}$ is an induced path cover of W_n so that

$$\pi_{ia}(W_n) \leq |\psi| = \left\lceil \frac{n-1}{2} \right\rceil + 2$$

Now, the minimum number of paths required to cover the $n-1$ spokes of the wheel in $\lceil \frac{n-1}{2} \rceil$ and hence for any induced acyclic path cover ψ we have $|\psi| \geq \lceil \frac{n-1}{2} \rceil + 2$ so that $\pi_{ia}(W_n) \geq \lceil \frac{n-1}{2} \rceil + 2$. Thus $\pi_{ia}(W_n) = \lceil \frac{n-1}{2} \rceil + 2$. ■

Theorem 2.6. *Let G be a unicyclic graph with the cycle $C = (v_1, v_2, \ldots, v_n, v_1)$. Let m denote the number of vertices of degree greater than two on C. Let k be the number of vertices of odd degree. Then*

$$\pi_{ia}(G) = \begin{cases} \frac{k}{2} + 2 & \text{if } n = 3 \text{ and } m = 1, \\ \pi_i & \text{otherwise.} \end{cases}$$

Proof: If $n \geq 4$ or $n = 3$ and $m \geq 2$, then the minimum induced path covers constructed in all the cases of Theorem 3.10 are minimum induced acyclic path covers of G so that $\pi_{ia}(G) = \pi_i(G)$.

Suppose $n = 3$ and $m = 1$. Then $C = (v_1, v_2, v_3, v_1)$. Let v_1 be the vertex of degree greater than two on C. Let T be the sub tree rooted at v_1 and let ψ_1 be a minimum induced acyclic path cover of T so that $|\psi_1| = \frac{k}{2}$.

If $\deg v_1$ is odd, let P_1 be the path in ψ_1 having v_1 as a terminal vertex. Now let $Q_1 = P_1 \circ (v_1, v_2), Q_2 = (v_2, v_3)$ and $Q_3 = (v_3, v_1)$.

If $\deg v_1$ is even, then let $P_1 = (u_1, u_2, \ldots, u_n, v_1, u_{r+1}, \ldots, u_s)$ be a path in ψ_1 having v_1 as an internal vertex. Now let $Q_1 = (u_1, u_2, \ldots, u_r, v_1, v_2), Q_2 = (v_2, v_3)$ and $Q_3 = (u_s, u_{s-1}, \ldots, u_{r+1}, v_1, v_3)$.

Then $\psi = (\psi_1 - \{P_1\}) \cup \{Q_1, Q_2, Q_3\}$ is an induced acyclic path cover so that $\pi_{ia}(G) \leq |\psi| = |\psi_1| + 2 = \frac{k}{2} + 2$. Further for any induced acyclic path cover ψ of G, the vertices v_2 and v_3 are exterior to ψ so that $|\psi| \geq \frac{k}{2} + 2$ and hence $\pi_{ia}(G) \geq \frac{k}{2} + 2$. Thus $\pi_{ia}(G) = \frac{k}{2} + 2$. ■

Theorem 2.7. *For any induced path cover ψ of a graph G, let $t_\psi = \sum_{P \in \psi} t(P)$ where $t(P)$ denotes the number of internal vertices of P and let $t = \max t_\psi$ where the maximum is taken over all induced path cover ψ of G. Then $\pi_{ia} = m - t$.*

Proof: Let ψ be an induced path decomposition of G. Then

$$\begin{aligned} m &= \sum_{P \in \psi} |E(P)| \\ &= \sum_{P \in \psi} (t(P) + 1) \\ &= \sum_{P \in \psi} t(P) + |\psi| \\ &= t_\psi + |\psi| \end{aligned}$$

Hence $|\psi| = m - t_\psi$ so that $\pi_{ia} = m - t$. ■

Corollary 2.8. *For any graph G with k vertices of odd degree, we have $\pi_{ia}(G) = \frac{k}{2} + \sum_{v \in V(G)} \lfloor \frac{\deg v}{2} \rfloor - t$.*

Proof: Since $m = \frac{k}{2} + \sum_{v \in V(G)} \lfloor \frac{\deg v}{2} \rfloor$, the result follows. ■

Corollary 2.9. *For any graph $G, \pi_{ia}(G) \geq \frac{k}{2}$. Further $\pi_{ia}(G) = \frac{k}{2}$ if and only if there exists an induced path cover of ψ of G such that every vertex v of G is an internal vertex of $\lfloor \frac{\deg v}{2} \rfloor$ paths in ψ.*

Theorem 2.10. *For any graph $G, \pi_{ia}(G) \leq m$. Further equality holds if and only if G is complete.*

Proof: Suppose there exists two vertices u and v in G which are not adjacent. Let P be a shortest u–v path in G so that P is an induced path and $|E(P)| > 1$. Then $\psi = \{P\} \cup (E(G) - E(P))$ is an induced acyclic path cover of G such that $|\psi| < m$ and hence $\pi_{ia}(G) < m$.

Conversely, suppose G is complete. Then every induced acyclic path of G is of length one and hence $\pi_{ia}(G) = m$. ■

Remark 2.11. *Since every induced acyclic path cover of a graph G is an induced path cover of G and every induced path cover of G is a path cover we have $\pi_{ia} \geq \pi_i \geq \pi$. These inequalities can be strict. For example, for the complete graph K_4, $\pi = 2, \pi_i = 4$ and $\pi_{ia} = 6$.*

Theorem 2.12. *For any graph G, $\pi_{ia}(G) \geq \binom{\omega}{2}$ where ω is the clique number of G.*

Proof: Let H be a maximum clique in G. Let ψ be any induced acyclic path cover of G. Then every path in ψ covers at most one edge of H so that $|\psi| \geq \binom{\omega}{2}$. Hence $\pi_{ia}(G) \geq \binom{\omega}{2}$. ■

III. Cycle decomposition

In this section we introduce the concept of cycle decomposition of a graph.

Definition 3.1. *Let G be an eulerian graph. The minimum number of cycles required to decompose G is called the cycle decomposition number and is denoted by $\pi_c(G)$ or π_c.*

Theorem 3.2. *[14] There exists a decomposition of the complete graph K_n of order $n = 2$ into Hamiltonian path (cycles) if and only if n is even (odd).*

According to this theorem K_n can be decomposed into $\frac{n}{2}$ copies of Hamiltonian paths of length $n - 1$, if n is even, and $\frac{n-1}{2}$ copies of Hamiltonian cycles of length n, if n is odd.

Remark 3.3. *If G is Hamiltonian cycle decomposable then $\pi = \pi_c$. In particular if n is odd $\pi(K_n) = \pi_c(K_n)$.*

Example 3.4. *For the graph given in Figure 2, we have $\pi = \pi_a = \pi_c = 2$.*

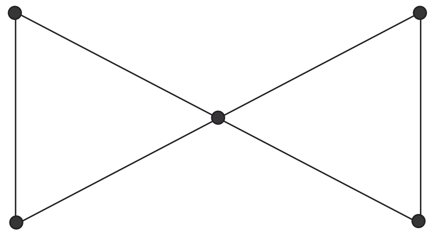

Fig. 2. A graph showing $\pi = \pi_a = \pi_c = 2$.

Problem 3.5. *Characterize Eulerian graph for which $\pi = \pi_a = \pi_c$.*

Remark 3.6. *If G is graph having exactly one cut vertex in which each block is a cycle, then $\pi = \pi_a = \pi_c = r$ where r is the number of blocks.*

Remark 3.7. *Since every cycle cover of an eulerian graph G is a path cover we have $\pi \leq \pi_c$. This inequality can be strict. For example, the graph G in Figure 3, we have $\pi = 2, \pi_c = 3$.*

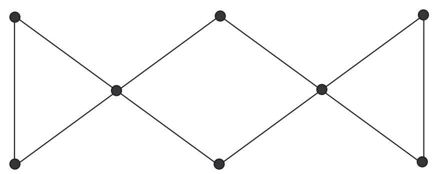

Fig. 3. Demonstration of $\pi \leq \pi_c$.

Question 3.8. *For a given positive integer n does there exist a graph G for which $\pi_c = \pi = n$.*

Problem 3.9. *Characterize eulerian graphs for which $\pi = \pi_c$.*

Remark 3.10. *For any graph G if a π_c-cover contains cycles C_1, C_2 and C_3 as in Figure 5.3 we have $\pi_c > \pi$.*

Remark 3.11. *Let G_1, G_2 be two eulerian graphs and G be the graph obtained by identifying G_1 and G_2 at a vertex. Then* $\pi_c(G) = \pi_c(G_1) + \pi_c(G_2)$.

IV. Conclusion and Scope

A decomposition of a graph G is a collection of edged is joint subgraphs of G whose union is G. Various types of decomposition and corresponding parameters have been studied by imposing certain condition on the members of the decomposition. The key condition that we impose here is "inducedness" and arrived at the concept of acyclic induced path decomposition and the acyclic induced path decomposition number $\pi_{ia}(G)$. Here, we first determined $\pi_{ia}(G)$ for several families of graphs and obtained some bounds for π_{ia} together with the characterization of graphs attaining these bounds and finally discuss the relation of π_{ia} with some well-known related parameters.

Even if this paper is just an initiation of the concept of induced acyclic path decomposition, numerous problems can be identified for further investigation and here are some interesting problems.

(a) Characterize graphs for which $\pi_{ia} = \left(\begin{array}{c} \omega \\ 2 \end{array} \right)$.
(b) For a tree T we have $\pi = \pi_a = \pi_i = \pi_{ia}$. So characterize graphs for which $\pi = \pi_a = \pi_i = \pi_{ia}$.

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