Impedance of an Encircling Coil due to a Cylindrical Tube with Varying Properties

Valentina Koliskina

***Abstract*—**Change in impedance of an encircling coil is obtained in the present paper for the case where the electric conductivity and magnetic permeability of a metal cylindrical tube depend on the radial coordinate. The system of equations for the vector potential is solved by means of the Fourier cosine transform. The solution is expressed in terms of improper integral containing modified Bessel functions of complex order.

***Keywords*—**Eddy currents, magnetic permeability, Bessel functions

I. Introduction

EDDY current methods are widely used in practice for quality control of products of a cylindrical shape [1] –[4]. Theoretical models are developed in [5], [6] for the case of planar multilayer media. Analytical solutions are obtained for the case of cylindrical geometry in [7]. The solution for a rod of finite length is given in [8]. The solutions obtained in [5]–[7] are found for the case of constant electric conductivity and magnetic permeability of the conducting medium.

In some industrial processes (examples include surface hardening and de-carbonization) the electric and magnetic properties of conducting layers can change with respect to geometrical coordinates (see [9], [10]). The assumption of constant electric conductivity and magnetic permeability is not valid anymore. Mathematical models describing the interaction of alternating current in a coil with objects of cylindrical shapes should be modified and variability of the parameters of the medium should be taken into account.

There are at least two possible solutions of the problem. First, one can still use analytical solutions [7] for the case of multilayer medium with constant properties in each cylindrical layer. In this case variability of the parameters of the medium is taken into account by using large number of layers with constant properties. In other words, electric conductivity and magnetic permeability are piecewise constant functions of the radial coordinate. Second, analytical solutions of the problem can be found in some cases where the electric conductivity and magnetic permeability are given functions of the radial coordinate.

The second approach is followed in the present paper. Two-parameter family of electric conductivity and magnetic permeability profiles are used in the paper in order to construct an analytical solution for the change in impedance of an encircling coil due to a two-layer cylindrical tube. The properties of the outer layer are assumed to vary with respect to the radial coordinate. The solution is found in terms of improper integral containing modified Bessel functions of complex order.

V. Koliskina is with the Department of Engineering Mathematics of the Riga Technical University, Riga, Latvia (phone: 371-6708-9528; fax: 371-6708-9694; e-mail: v.koliskina@ gmail.com).

II. Formulation of the Problem

Consider a coil of radius r_c with alternating current of the form $i(t)\vec{e}_\varphi = I\exp(j\omega t)\vec{e}_\varphi$ situated in a plane perpendicular to the axis of a two-layer infinitely long tube with external and internal radii $\tilde{r}_1$ and $\tilde{r}_2$, respectively, where I is the amplitude of the current in the coil, ω is the frequency and $\vec{e}_\varphi$ is a unit vector in the φ direction. The center of the coil lies on the axis of the tube. Let (r,φ,z) be a system of cylindrical polar coordinates centered at the origin O.

In this case the amplitude of the vector potential, $\vec{A}$, has only one non-zero component in the φ-direction which is the function of r and z only:

$$\vec{A} = A(r,z)\vec{e}_\varphi \tag{1}$$

We use $\tilde{r}_1$ as the measure of length, thus $r_0 = r_c/\tilde{r}_1, R = \tilde{r}_2/\tilde{r}_1$.
The amplitudes, $A_i(r,z),\ i = 0,1,2$, of the vector potential satisfy the following system of equations in regions R_0, R_1 and R_2:

$$\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{A_0}{r^2} + \frac{\partial^2 A_0}{\partial z^2} = -\mu_0 I\tilde{r}_1^2\delta(r-r_0)\delta(z), \tag{2}$$

$$\frac{\partial^2 A_1}{\partial r^2} + \left(\frac{1}{r} - \frac{1}{\mu_1}\frac{d\mu_1}{dr}\right)\frac{\partial A_1}{\partial r} - \left(\frac{1}{r^2} + \frac{1}{r\mu_1}\frac{d\mu_1}{dr} + \tilde{r}_1^2 j\omega\sigma_1\mu_0\mu_1\right)A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \tag{3}$$

$$\frac{\partial^2 A_2}{\partial r^2} + \frac{1}{r}\frac{\partial A_2}{\partial r} - \frac{A_2}{r^2} + \tilde{r}_1^2 j\omega\sigma_2\mu_0\mu_2 + \frac{\partial^2 A_1}{\partial z^2} = 0, \tag{4}$$

where $\delta(x)$ is the Dirac delta-function, μ_0 is the magnetic constant, σ_i and μ_i are the electric conductivity and magnetic permeability of region $R_i, i = 1,2,$ respectively.

Some analytical solutions for the case where only the magnetic permeability of region R_1 is a function of the radial coordinate are given in [11]. In this paper we generalize the solution presented in [11] for the case of encircling coil where both the electric conductivity and magnetic permeability of region R_1 are functions of the radial coordinate of the form

$$\mu_1 = \mu_* r^\alpha, \sigma_1 = \sigma_* r^\beta, \tag{5}$$

where α and β are real numbers.

Substituting (5) into (3) we obtain the following equation in region R_1:

$$\frac{\partial^2 A_1}{\partial r^2} + \frac{(1-\alpha)}{r}\frac{\partial A_1}{\partial r} - \left(\frac{1+\alpha}{r^2} + p_1{}^2 r^{\alpha+\beta}\right) A_1 + \frac{\partial^2 A_1}{\partial z^2} = 0, \tag{6}$$

where $p_1 = \eta_1 \sqrt{j}, \eta_1 = \tilde{r}_1 \sqrt{\omega \sigma_* \mu_* \mu_0}$.

The boundary conditions have the form

$$A_0 |_{r=1} = A_1 |_{r=1}, \ \frac{\partial A_0}{\partial r}|_{r=1} = \frac{1}{\mu_*}\frac{\partial A_1}{\partial r}|_{r=1}, \tag{7}$$

$$A_1 |_{r=R} = A_2 |_{r=R}, \ \frac{1}{\tilde{\mu}}\frac{\partial A_1}{\partial r}|_{r=R} = \frac{1}{\mu_2}\frac{\partial A_2}{\partial r}|_{r=R}, \tag{8}$$

$$A_i \to 0 \text{ as } z \to \pm\infty, \ i = 0,1,2, \tag{9}$$

$$A_0 \to 0, \text{ as } r \to \infty. \tag{10}$$

III. Problem Solution

Applying the Fourier cosine transform

$$\tilde{A}_i(r,\lambda) = \int_0^\infty A_i(r,z)\cos\lambda z dz, \ i = 0,1,2 \tag{11}$$

to the solution of (2), (4), (6), (7) – (10) we obtain

$$\frac{d^2\tilde{A}_0}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_0}{dr} - \frac{\tilde{A}_0}{r^2} - \lambda^2 \tilde{A}_0 = -\frac{1}{2}\mu_0 I \tilde{r}_1{}^2 \delta(r - r_0), \tag{12}$$

$$\frac{d^2\tilde{A}_1}{dr^2} + \frac{(1-\alpha)}{r}\frac{d\tilde{A}_1}{dr} - \left(\frac{1+\alpha}{r^2} + p^2 r^{\alpha+\beta} + \lambda^2\right) A_1 = 0, \tag{13}$$

$$\frac{d^2\tilde{A}_2}{dr^2} + \frac{1}{r}\frac{d\tilde{A}_2}{dr} - \frac{\tilde{A}_2}{r^2} - q^2\tilde{A}_2 = 0, \tag{14}$$

where

$q = \sqrt{\lambda^2 + p_2^2}$, $p_2 = \eta_2\sqrt{j}, \eta_2 = \tilde{r}_1\sqrt{\omega\sigma_2\mu_2\mu_0}$.

Using the notation $s = \sqrt{\sigma_2\mu_2/(\sigma_*\mu_*)}$

we obtain $\eta_2 = s\eta_1$.

The boundary conditions are

$$\tilde{A}_0 |_{r=1} = \tilde{A}_1 |_{r=1}, \ \frac{d\tilde{A}_0}{dr}|_{r=1} = \frac{1}{\mu_*}\frac{d\tilde{A}_1}{dr}|_{r=1}, \tag{15}$$

$$\tilde{A}_1 |_{r=R} = \tilde{A}_2 |_{r=R}, \ \frac{1}{\tilde{\mu}}\frac{d\tilde{A}_1}{dr}|_{r=R} = \frac{1}{\mu_2}\frac{d\tilde{A}_2}{dr}|_{r=R}, \tag{16}$$

$$\tilde{A}_0 \to 0 \text{ as } r \to \infty, \tag{17}$$

where $\tilde{\mu} = \mu_* R^\alpha$.

It is convenient to consider two sub-regions of region R_0, namely, $1 < r < r_0$ and $r > r_0$. The solutions in regions $1 < r < r_0$ and $r > r_0$ are denoted by $\tilde{A}_{00}(r,\lambda)$ and $\tilde{A}_{01}(r,\lambda)$, respectively. The general solution to (12) in region $1 < r < r_0$ is

$$\tilde{A}_{00}(r,\lambda) = C_1 I_1(\lambda r) + C_2 K_1(\lambda r), \tag{18}$$

where $I_1(\lambda r)$ and $K_1(\lambda r)$ are the modified Bessel functions of order one of the first and second kind, respectively.

Using (17) we conclude that the bounded general solution to (12) in region $r > r_0$ is

$$\tilde{A}_{01}(r,\lambda) = C_3 K_1(\lambda r). \tag{19}$$

Equation (13) can be solved in terms of different special functions for different values of α and β. In this paper we consider one particular case, namely, $\alpha = -1$ and $\beta = -1$. Then the solution to (13) is (see [12]):

$$\tilde{A}_1(r,\lambda) = C_4 \frac{I_\nu(\lambda r)}{\sqrt{r}} + C_5 \frac{K_\nu(\lambda r)}{\sqrt{r}}, \tag{20}$$

where $\nu = \sqrt{p_1^2 + 1/4}$.

The bounded solution to (14) has the form

$$\widetilde{A}_2(r,\lambda) = C_6 I_1(qr). \qquad (21)$$

There are six unknown constants in (18)-(21) and only four boundary conditions (15) and (16). The remaining two conditions are obtained at $r = r_0$. First, the functions $\widetilde{A}_{00}(r,\lambda)$ and $\widetilde{A}_{01}(r,\lambda)$ are continuous at $r = r_0$:

$$\widetilde{A}_{00}\,|_{r=r_0} = \widetilde{A}_{01}\,|_{r=r_0} \ . \qquad (22)$$

Integrating (12) with respect to r from $r = r_0 - \varepsilon$ to $r = r_0 + \varepsilon$ and considering the limit in the resulting equation as $\varepsilon \to +0$ we obtain the last boundary condition in the form

$$\frac{d\widetilde{A}_{01}}{dr}\,|_{r=r_0} - \frac{d\widetilde{A}_{00}}{dr}\,|_{r=r_0} = -\frac{\mu_0 I\widetilde{r}_1^{\,2}}{2}. \qquad (23)$$

Using conditions (15), (16), (22) and (23) we obtain all unknown constants $C_1, C_2, ..., C_6$ in (18)-(21). The solution is

$$C_1 = \frac{\mu_0 I\widetilde{r}_1^{\,2}}{2} r_0 K_1(\lambda r_0),$$
$$C_2 = -\frac{\mu_0 I\widetilde{r}_1^{\,2}}{2} r_0 K_1(\lambda r_0)\frac{\gamma_1}{\gamma_2},$$
$$C_3 = C_2 + \frac{\mu_0 I\widetilde{r}_1^{\,2}}{2} r_0 I_1(\lambda r_0),$$
$$C_4 = \gamma_3 C_5,$$
$$C_5 = \frac{K_1(\lambda)}{K_\nu(\lambda) + \gamma_3 I_\nu(\lambda)} C_2 + \frac{\mu_0 I\widetilde{r}_1^{\,2} r_0 K_1(\lambda r_0) I_1(\lambda)}{2[K_\nu(\lambda) + \gamma_3 I_\nu(\lambda)]},$$
$$C_6 = \frac{1}{I_1(qR)}\left[C_4 \frac{I_\nu(\lambda R)}{\sqrt{R}} + C_5 \frac{K_\nu(\lambda R)}{\sqrt{R}}\right].$$

Here we used the notations

$$\gamma_1 = -\mu_* \lambda I_1^{\prime}(\lambda)[K_\nu(\lambda) + \gamma_3 I_\nu(\lambda)]$$
$$+ I_1(\lambda)[\lambda K_\nu^{\prime}(\lambda) - 0.5\cdot K_\nu(\lambda)$$
$$+ \gamma_3 \lambda I_\nu^{\prime}(\lambda) - 0.5\cdot\gamma_3 I_\nu(\lambda)],$$
$$\gamma_2 = -\mu_* \lambda K_1^{\prime}(\lambda)[K_\nu(\lambda) + \gamma_3 I_\nu(\lambda)]$$
$$+ K_1(\lambda)[\lambda K_\nu^{\prime}(\lambda) - 0.5\cdot K_\nu(\lambda)$$
$$+ \gamma_3 \lambda I_\nu^{\prime}(\lambda) - 0.5\cdot\gamma_3 I_\nu(\lambda)],$$
$$\gamma_3 = -\frac{D_1}{D_2},$$
$$D_1 = 2R\widetilde{\mu} q I_1^{\prime}(qR) K_\nu(\lambda R)$$
$$- \mu_2 I_1(qR)[2R\lambda K_\nu^{\prime}(\lambda R) - K_\nu(\lambda R)],$$
$$D_2 = 2R\widetilde{\mu} q I_1^{\prime}(qR) I_\nu(\lambda R)$$
$$- \mu_2 I_1(qR)[2R\lambda I_\nu^{\prime}(\lambda R) - I_\nu(\lambda R)].$$

The Fourier cosine transform of the induced vector potential in region R_0 due to the presence of electrically conducting magnetic two-layer tube is

$$\widetilde{A}_0^{\,ind}(r,\lambda) = C_2 K_1(\lambda r). \qquad (24)$$

Applying the inverse Fourier cosine transform of the form

$$A_i(r,z) = \frac{2}{\pi}\int_0^\infty \widetilde{A}_i(r,\lambda)\cos\lambda z d\lambda, \; i = 0,1,2, \qquad (25)$$

we obtain the amplitude of the induced vector potential in region R_0 in the form

$$A_0^{\,ind}(r,z) = \frac{\mu_0 I\widetilde{r}_1^{\,2} r_0}{\pi}\int_0^\infty K_1(\lambda r_0)\frac{\gamma_1}{\gamma_2}K_1(\lambda r)\cos\lambda z d\lambda.$$

The induced change in impedance in the coil due to the presence of the conducting tube can be computed as follows

$$Z^{\,ind} = \frac{j\omega}{I}\oint_L A_0^{\,ind}(r,z)dl, \qquad (26)$$

where L is the contour of the coil. Calculating the integral in (26) we obtain the induced change in impedance of the form

$$Z^{\,ind} = 2\omega\widetilde{r}_1^{\,2} r_0^2 \mu_0 Z,$$

where

$$Z = j\int_0^\infty \frac{\gamma_1}{\gamma_2} K_1^2(\lambda r_0) d\lambda \cdot \qquad (27)$$

IV. Numerical results

Formula (27) is used to compute the change in impedance of a coil for different values of the parameters of the problem. Calculations are done with “Mathematica”. The change in impedance, Z, is plotted in Fig. 1 for three values of r_0, namely, $r_0 = 1.1, 1.2$ and 1.3 (from right to left). The other parameters are as follows: $\mu_1 = 1, \mu_2 = 4, R = 0.8, s = 1.5.$ The points on the curves correspond to the following values of $\eta_1 = 2,3,...,10$ (from top to bottom).

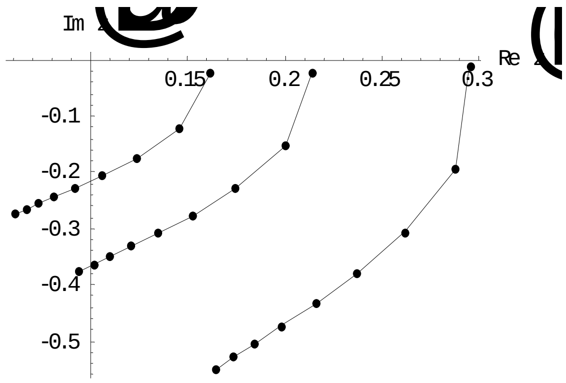

Fig. 1 The change in impedance calculated by formula (27) for three different values of r_0

It follows from Fig. 1 that the increase in frequency (larger values of η_1) leads to smaller values of the real part of the change in impedance. On the other hand, the imaginary part of the change in impedance increases as η_1 grows.

The values of Z for three different values of μ_2, namely, $\mu_2 = 2,4,6$ (from left to right) are plotted in Fig. 2. The other parameters are set at $\mu_1 = 1, r_0 = 1.1, R = 0.8, s = 1.5.$ The points on each curve correspond to the following values of $\eta_1 = 3,3,...,10$ (from top to bottom).

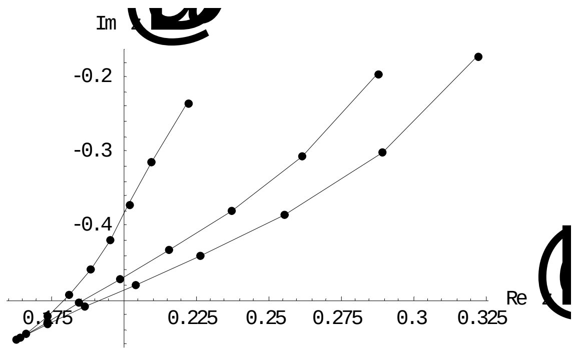

Fig. 2 The change in impedance calculated by formula (27) for three different values of μ_2

It is seen from Fig. 2 that for large frequencies (large values of η_1) the change in impedance is almost independent on μ_2 since all the three curves are very close to one another.

V. Conclusions

Closed-form solution for the change in impedance of an encircling coil located outside a two-layer metal tube is found in the present paper. The electric conductivity and magnetic permeability of the outer layer of the tube are power functions of the radial coordinate. The solution is obtained by the method of Fourier cosine integral transform. Computational results show that for large frequencies the change in impedance is almost independent on the properties of the inner layer.

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