International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:2, 2013

Hybrid function method for solving nonlinear Fredholm integral equations of the second kind

Jianhua Hou, Changqing Yang, and Beibo Qin,

Abstract—A numerical method for solving nonlinear Fredholm integral equations of second kind is proposed. The Fredholm type equations which have many applications in mathematical physics are then considered. The method is based on hybrid function approximations. The properties of hybrid of block-pulse functions and Chebyshev polynomials are presented and are utilized to reduce the computation of nonlinear Fredholm integral equations to a system of nonlinear. Some numerical examples are selected to illustrate the effectiveness and simplicity of the method.

Keywords—Hybrid functions, Fredholm integral equation, Block-pulse, Chebyshev polynomials, Product operational matrix.

I. Introduction

INTEGRAL equations are often involved in the mathematical formulation of physical phenomena. Integral equations can be encountered in various fields such as physics [1], biology [2] and engineering. But we can also use it in numerous applications, such as biomechanics, control, economics, elasticity, electrical engineering, fluid dynamics, heat and mass transfer, oscillation theory, queuing theory, etc. Fredholm and Volterra integral equations of the second kind show up in studies that include airfoil theory, elastic contact problems, fracture mechanics, combined infrared radiation and molecular conduction [3] and so on.

The problem of finding numerical solutions for Fredholm integral equations of the second kind is one of the oldest problems in the applied mathematics literature and many computational methods are introduced in this field. One may find in the references [4], [5], [6], [7], a collection of the best numerical methods for solving Fredholm integral equations appeared after 1960. Also, a functional analysis framework for these methods can be found in [8]. The classical methods for finding approximate solutions, dependent on the definition of the approximate solution, are mostly classified into two types, collocation methods and Galerkin methods.

In this study, we are concerned with the application of hybrid block-pulse function and Chebyshev polynomials to the numerical solution of Fredholm integral equation of the form

$$y(x) = f(x) + \int_0^1 k(x,t)[y(t)]^m dt, \quad 0 \leq x \leq 1. \tag{1}$$

The function $f(x)$ and $k(x,t)$ are known. $y(x)$ is unknown function to be determined and $m \leq 1$ is a positive integer. For $m = 1$, (1) is a linear and $m \geq 2$ is a nonlinear Fredholm integral equation.

J. Hou is with the Department of Science, Huaihai Institute of Technology, Lianyungang, Jiangsu, 222005 China e-mail: (young@hhit.edu.cn).
C. Yang and B. Qin are with the Department of Science, Huaihai Institute of Technology.

The article is organized as follows: In Section 2, we describe the basic formulation of hybrid block-pulse function and Chebyshev polynomials required for our subsequent. Section 3 is devoted to the solution of equation (1) by using hybrid functions. In Section 4, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

II. Properties of hybrid functions

A. Hybrid functions of block-pulse and Chebyshev polynomials

Hybrid function $b_{nm}(x), n = 1, 2, \cdots, N, \ m = 0, 1, \cdots, M-1$, are defined on the interval $[0, 1)$ as

$$b_{nm}(x) = \begin{cases} T_m(2Nx - 2n + 1), & x \in [\frac{n-1}{N}, \frac{n}{N}]; \\ 0, & otherwise. \end{cases} \tag{2}$$

where n and m are the orders of block-pulse functions and Chebyshev polynomials, respectively. Here $T_m(x)$ are the well-known Chebyshev polynomials which are orthogonal in the interval $[0, 1]$ with respect to the weight function $\omega(x) = 1/\sqrt{1-x^2}$ and satisfy the following recursive formula:

$$\begin{aligned} T_0 =& 1, \\ T_1 =& x, \\ T_{m+1} =& 2xT_m(x) - T_{m-1}(x), \ m = 1, 2, \cdots. \end{aligned}$$

Since b_{nm} consists of block-pulse functions and Chebyshev polynomials, which are both complete and orthogonal, the set of hybrid functions is complete orthogonal set.

B. Function approximation

A function $y(x)$ defined over the interval $[0, 1]$ may be expanded as

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} b_{nm}, \tag{3}$$

where

$$c_{nm} = (f(x), b_{nm}(x)),$$

in which $(.,.)$ denotes the inner product.

If $y(x)$ in (3) is truncated, then (3) can be written as

$$y(x) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm} = C^T B(x) = B^T(x) C, \tag{4}$$

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:2, 2013

where

$$C = [c_{10}, c_{11}, \cdots, c_{1M-1}, c_{20}, \cdots, c_{2M-1}, \cdots, c_{NM-1}]^T \tag{5}$$

and

$$\begin{aligned} B(x) =& [b_{10}(x), b_{11}(x), \cdots, b_{1M-1}(x), c_{20}(x), \cdots, \\ & b_{2M-1}(x), \cdots, b_{N0}(x), \cdots, b_{NM-1}(x)]^T \end{aligned} \tag{6}$$

In (5) and (6), $c_{nm}, n = 1, 2, \cdots, N, m = 0, 1, 2, \cdots, M-1$, are the coefficients expansions of the function $y(x)$ in the subinterval $[(n-1)/N, n/N]$ and $b_{nm}(x)$ are defined in (2).

C. The product operational of matrix the hybrid of block-pulse and Chebyshev polynomials

The following property of the product of two hybrid function vectors will also be used. Let

$$B(x)B^T(x)C \approx \widetilde{C}B(x) \tag{7}$$

where $\widetilde{C} = diag(\widetilde{C}_1, \widetilde{C}_2, \cdots, \widetilde{C}_N)$ is a $MN \times MN$ product operational matrix. And $\widetilde{C}_i, i = 1, 2, \cdots, N$ are $M \times M$ matrices given in [9]. We also define the matrix D as follows

$$D = \int_0^1 B(x)B^T(x)dx \tag{8}$$

For the hybrid functions of block-pulse and Chebyshev polynomials, D has the following form:

$$D = \begin{pmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{pmatrix}$$

where L is $M \times M$ nonsingular symmetric matrix given in [10].

III. Nonlinear Fredholm integral equations

Consider the following integral equation

$$y(x) = f(x) + \int_0^1 k(x,t)[y(t)]^m dt, \quad 0 \le x \le 1. \tag{9}$$

We approximate $f(x), y(x), k(x,t)$ and $[y(t)]^m$ by the way mentioned in Section 2 as

$$\begin{aligned} f(x) =& B^T(x)F, \\ y(x) =& B^T(x)C, \\ k(x,t) =& B^T(x)KB(t), \end{aligned}$$

and

$$[y(t)]^m = [B^T(t)C]^m = C^T B(t) \cdot B^T(t)C[B^T(t)C]^{m-2}. \tag{10}$$

Applying (7),equation (10) becomes

$$[y(t)]^m = C^T \widetilde{C}^{m-1} B(t) = C^* B(t).$$

With substituting in (9) we have

$$B^T(x)C = B^T(x)F + B^T(x)K \left(\int_0^1 B(t)B^T(t)dt \right) C^{*T}.$$

Applying (8), then we get

$$C = F + KDC^{*T}$$

which is a nonlinear system of equations. By solving this equation we can find the vector C.

IV. Convergence and error analysis

New we discuss the convergence of the hybrid functions method for the nonlinear integral equation (1). The following theorem is fundamental for the convergence analysis.

Theorem 1: Let $y(x) \in H^K(-1,1)$ (Sobolev space), $y_N(x) = \sum_{i=0}^N a_i T_i(x)$ be the best approximation polynomials of $y(x)$ in $L^2_\omega - norm$, then

$$\| y(x) - y_N(x) \|_{L^2_\omega[-1,1]} \le C_0 N^{-K} \| y(x) \|_{H^K_\omega[-1,1]}$$

where C_0 is a positive constant, which depend on selected norm and independent with $y(x)$ and N [11].

Theorem 2: Let $y(x) \in H^K_\omega(0,1), \ I_n = [(n-1)/N, n/N]$ then

$$\| y(x) - y_{NM}(x) \|_{L^2_\omega[0,1]} \le C_0 M^{-K} \underset{0 \le n \le N}{max} \| y(x) \|_{H^K_\omega(I_n)}$$

By using of Theorem 1, it is obvious[12].

We can easily verify the accuracy of the method. Given that the truncated hybrid function in (6) is an approximate solution of (1), it must have approximately satisfied these equations. Thus, for each $x_i \in [0,1]$

$$E(x_i) = B^T(x_i)C - \int_0^1 k(x_i,t)C^{*T}B(t)dt - f(x_i) \approx 0$$

If max $E(x_i) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N, M is increase until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}.

V. Numerical examples

In this section, we applied the method presented in this paper for solving integral equation of the form (1) and solved some examples. All results were computed using Matlab 7.0.

Example1 Let us first consider the nonlinear Fredholm integral equation

$$y(x) = x^2 - \frac{8}{15}x - \frac{7}{6} + \int_0^1 (x+t)[y(t)]^2 dt$$

with the exact solution $y(x) = x^2 - 1$. Table 1 shows the numerical results for Example 1 with $N = 2, M = 10$.

TABLE I

Numerical results for Example 1

x	Exact solution	Approximated solution	Absolution error
0.1	-9.900000000000e-001	-9.900000000007e-001	7.158718062783e-013
0.2	-9.600000000000e-001	-9.600000000007e-001	7.920331057675e-013
0.3	-9.100000000000e-001	-9.100000000008e-001	8.679723606519e-013
0.4	-8.400000000000e-001	-8.400000000009e-001	9.439116155363e-013
0.5	-7.500000000000e-001	-7.500000000010e-001	1.019850870420e-012
0.6	-6.400000000000e-001	-6.400000000010e-001	1.096012169909e-012
0.7	-5.100000000000e-001	-5.100000000011e-001	1.171951424794e-012
0.8	-3.599999999999e-001	-3.600000000012e-001	1.247890679678e-012
0.9	-1.900000000000e-001	-1.900000000013e-001	1.323829934563e-012

Example2 As the second example consider the following integral equation

$$y(x) = x^2 - e^x - \frac{(1+2e^3)x}{9} + \int_0^1 xt[y(t)]^3 dt[13]$$

with the exact solution $y(x) = e^x$. The comparison among the hybrid solution, Haar wavelets solution and the analytic solution for $x \in [0, 1]$ is shown in Table 2 for $N = 2, M = 10$, which confirms that the hybrid function method in section 4 gives almost the same solution as the analytic method.

TABLE II
COMPARISON RESULTS FOR EXAMPLE 2

x	Analytic solution	Hybrid function solution	Haar wavelets solution k=32
0.1	1.10517091807565	1.10521832988095	1.107217811
0.2	1.22140275816017	1.22149758177075	1.218102916
0.3	1.34985880757600	1.35000104299025	1.341165462
0.4	1.49182469764127	1.49201434483251	1.474918603
0.5	1.64872127070013	1.64895832944478	1.667402633
0.6	1.82211880039051	1.82240326946041	1.833861053
0.7	2.01375270747048	2.01408458179746	2.016679830
0.8	2.22554092849247	2.22592019104068	2.217456630
0.9	2.45960311115695	2.46002971282342	2.437978177

Example3

$$y(x) = f(x) + \int_0^1 xt[y(t)]^2 dt[13]$$

where

$$f(x) = f_1(x) - \left(\frac{9}{128} - \frac{9}{32e} + \frac{7}{16e^2} + \frac{1}{16e^4}\right),$$

and

$$f_1(x) = \begin{cases} e^{2x-2}, & 0 \le x \le \frac{1}{2}; \\ -x^2 + \frac{1}{e} + \frac{1}{4}, & \frac{1}{2} \le x \le 1. \end{cases}$$

The exact solution is $y(x) = f_1(x)$. Table 3 illustrates the numerical results of Example3 with $N = 2, M = 10$.

TABLE III
COMPARISON RESULTS FOR EXAMPLE 3

x	Present method	Exact solution	Method in [13],k=32
0.1	0.16529887499956	0.16529888822159	0.1625177090
0.2	0.20189649154656	0.20189651799466	0.1921647474
0.3	0.24659692403710	0.24659696394161	0.2290236855
0.4	0.30119415470771	0.30119421191220	0.2744773506
0.5	0.36787933407020	0.36787944117144	0.3240321944
0.6	0.25787944563142	0.25787944117144	0.2134699868
0.7	0.12787944145781	0.12787944117144	0.8532965420
0.8	-0.02212055879412	-0.02212055882856	-0.0603813505
0.9	-0.19212055862314	-0.19212055882856	-0.2236779332

VI. CONCLUSION

We have solved the nonlinear Fredholm integral equations of second kind by using hybrid of block-pulse functions and Chebyshev polynomials. The properties of hybrid of block-pulse functions and Chebyshev polynomials are used to reduce the equation to the solution of nonlinear algebraic equations. Illustrative examples are given to demonstrate the validity and applicability of the proposed method. The advantages of hybrid functions are that the values of N and M are adjustable as well as being able to yield more accurate numerical solutions than Haar wavelets functions [13], for the solutions of integral equations. Also hybrid functions have good advantage in dealing with piecewise continuous functions, as are shown. The method can be extended and applied to the system of nonlinear integral equations, linear and nonlinear integro-differential equations, but some modifications are required.

ACKNOWLEDGMENT

The authors are grateful to the Huaihai Institute of Technology which provided support through Grant No.Z2011151.

REFERENCES

[1] F.Bloom, *Asymptotic bounds for solutions to a system of damped integro-differential equations of electromagnetic theory*, J. Math. Anal. Appl. 73(2) (1980), 524-542.
[2] K. Holmaker, *Global asymptotic stability for stationary solution of system of integro-differential equations describing the formation of liver zones*, SIAM J. Math. Anal. 24(1) (1993), 116-128.
[3] J. Frankel, *A Galerkin solution to regularized Cauchy singular integro-differential equations*, Quart. Appl. Math. 52 (2) (1995), 145-258.
[4] K. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge, England: Cambridge University Press, 1997.
[5] L.M. Delves, *A fast method for the solution of Fredholm integral equations*, IMA.J. Appl. Math. 20 (2) (1977), 173-182.
[6] L.M. Delves, J.L. Mohamed, *Computational Methods for Integral Equations*, Cambridge, England: Cambridge University Press, 1988.
[7] L.M. Delves, J. Walsh, *Numerical Solution of Integral Equations*, Oxford, England: Oxford Univ. Press, 1974.
[8] K. Atkinson , H. Weimin, *Theoretical Numerical Analysis, A Functional Analysis Framework*, 3rd ed. New York, USA: Springer, 2009.
[9] C. Yang, J. Hou ang B. Qin *Numerical solution of Riccati differential equations by using hybrid functions and tau method*, International Journal of mathematical and Comutational Sciences 6(2012) 205-208.
[10] M. Razzaghi, H.R. Marzban, *Direct method for variational problems via hybrid of Block-pulse and Chebyshev functions*, Math. Probl. Eng. 6 (2000), 85-97.
[11] C. Canuto, M.Y. Hussaini, A.Quarteroni, T. A. Zang, *Spectral Methods on Fluid Dynamics*, Annu. Rev. Fluid. Mech. 19(1)(1987)339-367.
[12] M.T. Kajani, A.H. Vencheh, *Solving second kind integral equations with Hybrid Chebyshev and Block-Pulse functions*, App.Math.Comput. 163(2005), 71-77.
[13] E. Babolian, A. Shahsavaran, *Numerical solution of nonlinear Fredholm integral equations of the second kind using Haar wavelets*, J. Comput. Appl. Math. 25(2009),87-95.

Jianhua Hou She received her bachelor in computer science in 1998 from the Shanxi Normal University, Shanxi, China. Her master degree was obtained in 2004 from the Chengdu University of Technology, Sichuan, China. Since 2005 she is lecturer in Department of Science, Huaihai Institute of Technology, Jiangsu, China. Her main research interests are in parallel numerical algorithms.