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High Accuracy Eigensolutions in Elasticity for Boundary Integral Equations by Nyström Method

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Abstract—Elastic boundary eigensolution problems are converted into boundary integral equations by potential theory. The kernels of the boundary integral equations have both the logarithmic and Hilbert singularity simultaneously. We present the mechanical quadrature methods for solving eigensolutions of the boundary integral equations by dealing with two kinds of singularities at the same time. The methods possess high accuracy $O(h^3)$ and low computing complexity. The convergence and stability are proved based on Anselone's collective compact theory. Bases on the asymptotic error expansion with odd powers, we can greatly improve the accuracy of the approximation, and also derive a posteriori error estimate which can be used for constructing self-adaptive algorithms. The efficiency of the algorithms are illustrated by numerical examples.

Keywords—boundary integral equation, extrapolation algorithm, a posteriori error estimate, elasticity.

I. Introduction

THE fundamental boundary eigenproblem for the planar elasticity[1,2] is defined as follows: to find non-zero deformation $u = (u_1, u_2)^T$ in the domain Ω and on the boundary Γ satisfying:

$$\begin{cases} \sigma_{ij,j} = 0, \text{in } \Omega, \\ t_i = \lambda u_i, \text{on } \Gamma,\ k,l,i,j = 1,2, \end{cases} \tag{1}$$

where $\Omega \subset R^2$ is a bounded, simply connected domain with a smooth boundary Γ, $t = (t_1, t_2)^T$ is the traction vector on Γ, σ_{ij} is the stress tensor, and λ is the eigenvalue. Following vector computational rules, the repeated subscripts imply the summation from 1 to 2.

To obtain eigensolutions $\lambda^{(l)}$ and $u^{(l)} = (u_1^{(l)}, u_2^{(l)})^T$, we convert Eq.(1) into the following boundary integral equations[3,4,5] (BIEs) by potential theory:

$$\begin{aligned} \alpha_{ij}(y)u_j^{(l)}(y) + \int_\Gamma k_{ij}^*(y,x)u_j^{(l)}(x)ds_x \\ = \lambda^{(l)} \int_\Gamma h_{ij}^*(y,x)u_j^{(l)}(x)ds_x, \end{aligned} \tag{2}$$

where $\alpha_i(y) = \theta(y)/(2\pi)$ is related to the interior angle $\theta(y)$ of Ω at $y \in \Gamma$, especially when y is on a smooth part of the boundary Γ, $\alpha_{ij} = \delta_{ij}/2$ with the Kronecker delta δ_{ij}, and the integral kernels

$$\begin{cases} h_{ij}^* = \dfrac{1}{8\pi\mu(1-\nu)}\Big[-(3-4\nu)\delta_{ij}\ln r + r_{,i}r_{,j}\Big], \\ k_{ij}^* = \dfrac{1}{4\pi(1-\nu)r}\Big[\dfrac{\partial r}{\partial n}((1-2\nu)\delta_{ij} \\ \quad + 2r_{,i}r_{,j}) + (1-2\nu)(n_i r_{,j} - n_j r_{,i})\Big], \end{cases}$$

are Kelvin's fundamental solutions[1,6], where μ is shear modulus, ν is the Poisson ratio, $r^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2$, and $n = (n_1, n_2)^T$ is the unit outward normal on Γ.

Equations (2) are obvious singular integral equations. In particular, the second term of the left-hand side is the Hilbert singularity and the term of the right-hand side is the logarithmic singularity. Thus, the key to achieve the eigensolutions $\lambda^{(l)}$ and $u^{(l)}$ accurately is converted to approximate the logarithmic and Hilbert singularity appropriately.

Once eigensolutions are obtained, u and t on Γ can be solved from Eq.(2), and then the displacement vector in Ω can be calculated[5,7] as following:

$$\begin{aligned} u_i(y) = \int_\Gamma h_{ij}^*(y,x)t_j(x)ds_x \\ - \int_\Gamma k_{ij}^*(y,x)u_j(x)ds_x, \quad \forall y \in \Omega. \end{aligned} \tag{3}$$

A considerable part of articles have researched on planar elasticity. Parton and Perlin[8] introduced the eigenvalue λ into the boundary conditions of the elasticity and obtained some analytical solutions. Hadjesfandiari and Dargush[1,9,10] gave the general theory of fundamental boundary eigensolution for elasticity and potential problem. They also gave the finite element method to solve the planar elasticity and achieved the error estimate of the approximate solution. Cohen et al.[11] presented perfectly matched layers for modeling unbounded domains to construct a mixed formulation of a spectral finite element approximation in the linear elasticity system. Pavarino[12] introduced preconditioned mixed spectral finite element methods for the indefinite elasticity systems which showed that the convergence rate was independent of the penalty parameter. Talbot and Crampton[13] approached to 2D vibrational problems by a pseudo-spectral since the governing partial differential equations were translated into matrix eigenvalue problems which can be solved by collocation method. Chen et al.[3] used collocation method to research the degenerate scale problem in plane elasticity, and they got the degenerate scale (or the eigenvalue) to avoid the special geometry size resulting in a non-unique solution. Cai

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et al.[14,15] gave first-order system least-squares methods for the solution of linear elastic problems about two dimensions and three dimensions.

To solve the boundary integral equations (BIEs), mechanical quadrature methods are constructed by using a new type of numerical quadrature method. It preserves the advantage of requiring less computational cost than projection methods since each element in the discretization matrixes of integral equations is evaluated directly by the mechanical quadrature methods. An extrapolation algorithms (EAs) based on asymptotic expansion of errors are pretty effective parallel algorithms, which possess high accuracy degrees, good stability and almost optimal computational complexity. The EAs have been applied to many problems, such as the numerical integrations, finite difference methods and finite element methods[16,17,18].

We firstly use the Sidi's quadrature rules[19,20] to calculate weakly singular integrals and Hilbert singular integrals in equations (2) simultaneously. Secondly, by Anselone's collective compact and asymptotically compact theory[16,21], we prove the $O(h^3)$ convergence rate. Finally, basing on the asymptotic expansion of error with an odd power, we establish extrapolation algorithms (EAs). After h^3-extrapolation, we get the $O(h^5)$ convergence rate. Numerical examples support our algorithms and show that the MQMs are fit for practice in this paper.

II. Integral operators

We firstly define some boundary integral operators on Γ as follows:

$$\begin{cases} (K_{ij}w)(y) = \displaystyle\int_\Gamma k^*_{ij}(y,x)w(x)ds_x \ y\in\Gamma, i,j=1,2, \\ (H_{ij}w)(y) = \displaystyle\int_\Gamma h^*_{ij}(y,x)w(x)ds_x \ y\in\Gamma, i,j=1,2. \end{cases} \tag{4}$$

Thus Eq.(2) can be converted into the following operator equations:

$$\begin{pmatrix} \frac{1}{2}I_0+K_{11} & K_{12} \\ K_{21} & \frac{1}{2}I_0+K_{22} \end{pmatrix}\begin{pmatrix} u_1^{(l)} \\ u_2^{(l)} \end{pmatrix} = \lambda^{(l)}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} u_1^{(l)} \\ u_2^{(l)} \end{pmatrix}, \tag{5}$$

where I_0 is an identity operator.

Assume that Γ can be described by a regular parameter mapping $x(s) = (x_1(s),\ x_2(s)) : [0,2\pi] \to \Gamma$, satisfying $|x'(s)|^2 = |x_1'(s)|^2 + |x_2'(s)|^2 > 0$, and $x_i(s) \in C^{2m+1}[0,2\pi], i=1,2$.

Define the integral operator on $C^{2m+1}[0,2\pi]$:

$$\begin{aligned}
(A_0\omega)(t) &= \textstyle\int_0^{2\pi} a_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= \bar{c}_0\textstyle\int_0^{2\pi} \ln|2e^{-1/2}sin(\frac{t-\tau}{2})|\omega(\tau)|x'(\tau)|d\tau, \\
(B_0\omega)(t) &= \textstyle\int_0^{2\pi} b_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= \bar{c}_0\textstyle\int_0^{2\pi} \ln|\frac{x(t)-x(\tau)}{2e^{-1/2}sin((t-\tau)/2)}|\omega(\tau)|x'(\tau)|d\tau, \\
(B_{ij}\omega)(t) &= \textstyle\int_0^{2\pi} b_{ij}(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_1\textstyle\int_0^{2\pi} \frac{(x_i(t)-x_i(\tau))(x_j(t)-x_j(\tau))}{|x(t)-x(\tau)|^2}\omega(\tau)|x'(\tau)|d\tau, \\
(C_0\omega)(t) &= \textstyle\int_0^{2\pi} c_0(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_2\textstyle\int_0^{2\pi} \{(n_1r._2-n_2r._1)/r\}\omega(\tau)|x'(\tau)|d\tau, \\
(M_{ii}\omega)(t) &= \textstyle\int_0^{2\pi} m_{ii}(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_3\textstyle\int_0^{2\pi} \{[\frac{\partial r}{\partial n}[(1-2v)+2r._ir._i]/r\}\omega(\tau)|x'(\tau)|d\tau, \\
(M_{ij}\omega)(t) &= \textstyle\int_0^{2\pi} m_{ij}(t,\tau)\omega(\tau)|x'(\tau)|d\tau \\
&= c_3\textstyle\int_0^{2\pi} \{\frac{\partial r}{\partial n}(2r._ir._j)/r\}\omega(\tau)|x'(\tau)|d\tau \quad i\neq j,
\end{aligned}$$

where $\bar{c}_0 = -(3-4\nu)/[8\pi\mu(1-\nu)]$, $c_1 = 1/[8\pi\mu(1-\nu)]$, $c_2 = -(1-2\nu)/[4\pi(1-\nu)]$, $c_3 = -1/[4\pi(1-\nu)]$. As $t\to s$, depending on the properties of the kernels and using Taylor expansion, we know that A_0 is the logarithmic weak singular operator, and C_0 is the Hilbert singularity operator since

$$\frac{n_ir._j - n_jr._i}{r} = (-1)^i\frac{1+O(t-s)}{(t-s)+O(t-s)} \quad i\neq j. \tag{6}$$

And B_0, B_{ij}, M_{ij} are smooth operators.

Then Eq.(5) is equivalent to

$$\begin{cases} (\dfrac{1}{2}I+C+M)u^{(l)} = \lambda^{(l)}(A+B)u^{(l)}, \\ \|u^{(l)}\|^2_{0,\Gamma} = \displaystyle\int_0^{2\pi}|u^{(l)}(s)|^2|x'(s)|ds = 1, \end{cases} \tag{7}$$

where

$$I = \begin{pmatrix} I_0 & 0 \\ 0 & I_0 \end{pmatrix}, \quad A = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix},$$

$$B = \begin{pmatrix} B_0+B_{11} & B_{12} \\ B_{21} & B_0+B_{22} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & C_0 \\ -C_0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

III. Mechanical quadrature methods

Let $h = \pi/n$, $(n\in N)$ be the mesh width and $t_q = qh$, $(q = 0,1,\cdots,2n-1)$ be the nodes. Since B_0, B_{ij}, M_{ij} are smooth integral operators with the period 2π and $\omega(t) \in C^{2m+1}[0,2\pi]$, we can obtain the highly accurate Nyström's approximation by the midpoint or the trapezoidal rule[17,19]. For example, the Nyström's approximation operator B_0^h of B_0 can be defined as:

$$(B_0^h\omega)(t) = h\sum_{j=0}^{2n-1} b_0(t,\tau_j)\omega(\tau_j), \tag{8}$$

and the error is

$$(B_0\omega)(t) - (B_0^h\omega)(t) = O(h^{2m}). \tag{9}$$

The Nyström's approximation B_{ij}^h of B_{ij} and M_{ij}^h of M_{ij} can be defined similarly.

For the logarithmic weak singular operator A_0, the continuous approximation of its kernel $a_n(t,\tau)$ is defined as:

$$a_n(t,\tau) = \begin{cases} a_0(t,\tau), & \texttt{for } |t-\tau|\geq h, \\ \bar{c}_0h\ln|e^{-1/2}h/(2\pi)|, & \texttt{for } |t-\tau|<h. \end{cases} \tag{10}$$

By Sidi's quadrature rules[19], its Nyström's approximation operator A_0^h can be defined as:

$$(A_0^h\omega)(t) = h\sum_{j=0}^{2n-1} a_n(t,\tau_j)\omega(\tau_j)|x'(\tau_j)|, \tag{11}$$

which has the following error estimate:

$$\begin{aligned}(A_0\omega)(t) &- (A_0^h\omega)(t)\\ &= 2\sum_{\mu=1}^{m-1}\frac{\varsigma'(-2\mu)}{(2\mu)!}\omega^{(2\mu)}(t)h^{2\mu+1} + O(h^{2m}),\end{aligned} \tag{12}$$

where $\varsigma'(t)$ is the derivative of Riemann zeta function.

Because C_0 is a Hilbert singular operator, its Nyström's approximation operator C_0^h can be defined by Sidi's quadrature rules[19]:

$$\begin{aligned}(C_0^h\omega)(t_i) =& 2c_2 a_1(t_i,t_i)\\ & h\sum_{j=0}^{2n-1}\cot((t_j-t_i)/2)\omega(t_j)|x'(t_j)|\varepsilon_{ij},\end{aligned} \tag{13}$$

where

$$a_1(t,s) = \frac{1}{(t-s)+O(t-s)}\frac{\tan((t-s)/2)}{1/2},$$

and

$$\varepsilon_{ij} = \begin{cases} 1, \text{ if } |i-j| \text{ is odd number} \\ 0, \text{ if } |i-j| \text{ is even number}\end{cases}. \tag{14}$$

The Nyström's approximation has the following error bounds:[19]

$$(C_0\omega)(t_i) - (C_0^h\omega)(t_i) = O(h^{2m}). \tag{15}$$

Thus we obtain the numerical approximate equations of Eq.(7),

$$\begin{cases} (\frac{1}{2}I + C^h + M^h)u_h^{(l)} = \lambda_h^{(l)}(A^h+B^h)u_h^{(l)}, \\ h\sum_{i=1}^{2}\sum_{j=0}^{2n-1}(u_{ih}^{(l)}(t_j))^2|x'(t_j)| = 1, \end{cases} \tag{16}$$

where A^h, B^h, C^h and M^h are discrete matrices of order $4n$ corresponding to the operators A ,B, C and M, respectively. $\lambda_h^{(l)}$ and $u_h^{(l)}$ are the approximate solution of eigenvalue $\lambda^{(l)}$ and eigenvector $u^{(l)}$, respectively.

From Eq.(1), $u^{(l)}$ is a trivial solution as $\lambda^{(l)} = 0$, and if $\lambda^{(l)} \neq 0$, we have $\lambda_h^{(l)} \neq 0$. Let $\gamma_h^{(l)} = 1/\lambda_h^{(l)}$ and also suppose that the eigenvalues of $(\frac{1}{2}I + C)^{-1}M$ and $(\frac{1}{2}I + C^h)^{-1}M^h$ do not include -1, then the Eqs.(7) and (16) can be rewritten as follows: find $\gamma^{(l)}$ and $u^{(l)} \in V^{(0)}$ satisfying

$$\begin{aligned}&\gamma^{(l)}u^{(l)} = Lu^{(l)},\\ &\quad\text{with } \|u^{(l)}\|_{0,\Gamma}^2 = \int_0^{2\pi}|u^{(l)}(s)|^2|x'(s)|ds = 1\end{aligned} \tag{17}$$

and find $\gamma_h^{(l)}$ and $u_h^{(l)}$ satisfying

$$\begin{aligned}&\gamma_h^{(l)}u_h^{(l)} = L^h u_h^{(l)},\\ &\quad\text{with } h\sum_{i=1}^{2}\sum_{j=0}^{2n-1}(u_{ih}^{(l)}(t_j))^2|x'(t_j)| = 1,\end{aligned} \tag{18}$$

where $L^h = [I + (\frac{1}{2}I + C^h)^{-1}M^h]^{-1}(\frac{1}{2}I + C^h)^{-1}(A^h+B^h)$, and $L = [I + (\frac{1}{2}I + C)^{-1}M]^{-1}(\frac{1}{2}I + C)^{-1}(A+B)$, and the space $V^{(m)} = C^{(m)}[0,2\pi]\times C^{(m)}[0,2\pi]$, $m = 0,1,2,\ldots.$

According to Anselone's collective compact and asymptotically compact theory[16,21], we know that B_0^h is collectively compact convergent to B_0, B_{ij}^h is collectively compact convergent to B_{ij} and the approximate operator $\{A_0^h\}$ is asymptotically compact convergent to A_0 as $n \to \infty$ so we obtain the Theorem 1.

Theorem 1.[16,17] *The approximate operator sequence $\{L^h\}$ is the asymptotically compact sequence and convergence to L in $V^{(0)}$, i.e.*

$$L^h \overset{a.c}{\to} L, \tag{19}$$

where $\overset{a.c}{\to}$ means the asymptotically compact convergence.

Following the error estimates in Eqs.(12) and (15) by the mechanical quadrature methods, we also have the conclusion:

Theorem 2. *Under the hypotheses of Theorem 1, there exist constants d_1, d_2 and vector functions $w_{i1}, w_{i2} \in V^{(5)}, i = 1,\cdots,\chi_1$, independent of h, such that*

$$\lambda_h^{(l)} - \lambda^{(l)} = d_1h^3 + d_2h^5 + O(h^7), \tag{20}$$

$$u_{(i)h}^{(l)} - u_{(i)}^{(l)} = w_{i1}h^3 + w_{i2}h^5 + O(h^7). \tag{21}$$

IV. Extrapolation algorithms

Let $(\lambda_h^{(l)}, u_h^{(l)})$ and $(\lambda_{h/2}^{(l)}, u_{h/2}^{(l)})$ be the solutions of Eq.(18) according to mesh widths h and $h/2$, respectively. From Eqs.(20) and (21), the h^3-Richardson extrapolations of the eigenvalue

$$\tilde{\lambda}_h^{(l)} = (8\lambda_{h/2}^{(l)} - \lambda_h^{(l)})/7, \tag{22}$$

and the eigenvector

$$\begin{aligned}\tilde{u}_{(i)h}^{(l)}(s_j) = (8u_{(i)h/2}^{(l)}(s_j) - u_{(i)h}^{(l)}(s_j))/7,\\ s_j = jh,\ j = 0,\cdots,2n-1.\end{aligned} \tag{23}$$

They have the error estimates $|\tilde{\lambda}_h^{(l)} - \lambda^{(l)}| = O(h^5)$ and $||\tilde{u}_{(i)h}^{(l)}(s_j) - u_{(i)}^{(l)}(s_j)|| = O(h^5)$, respectively. The extrapolations algorithm is very effective to improve the error accuracy, although it is not very complicated.

From the error estimates Eqs.(22) and (23), we can derive the following posteriori error estimates

$$\begin{aligned}|\lambda_{h/2}^{(l)} - \lambda^{(l)}| &\leq |8/7\lambda_{h/2}^{(l)} - 1/7\lambda_h^{(l)} - \lambda_{h/2}^{(l)}| + O(h^5)\\ &\leq 1/7|\lambda_{h/2}^{(l)} - \lambda_h^{(l)}| + O(h^5)\end{aligned}$$

and

$$\begin{aligned}&||u_{(i)h/2}^{(l)}(s_j) - u_{(i)}^{(l)}(s_j)||\\ &\leq ||8/7u_{(i)h/2}^{(l)}(s_j) - 1/7u_{(i)h}^{(l)}(s_j) - u_{(i)h/2}^{(l)}(s_j)|| + O(h^5)\\ &\leq 1/7||u_{(i)h/2}^{(l)}(s_j) - u_{(i)h}^{(l)}(s_j)|| + O(h^5).\end{aligned} \tag{24}$$

Note that this equation can be used to construct self-adaptive algorithms.

V. Numerical examples

Let $e_l^h = |\lambda_h^{(l)} - \lambda^{(l)}|$, $r_l^h = e_l^h / e_l^{h/2}$, and $\tilde{e}_l^h$ be the error after EAs, and p_l^h be the posteriori error estimate.

Example 1: Consider a circular isotropic elastic body with radius a in the plane strain deformation. Parton and Perlin[8] presented some analytic solutions about eigenvalues for this problem as follows:

$$\lambda_l = \frac{2\,\mu\, l}{a},\ l = 1, 2, \cdots . \tag{25}$$

Setting circular plane strain deformation problem with $a = 1$ and the material properties $\mu = 1.0$ and $\nu = 0.3$, we compute the boundary numerical eigensolutions by Eq. (18) and list the results in Table 1.

TABLE I
THE ERRORS $e_l^h, \tilde{e}_l^h$ AND A POSTERIORI ERROR ESTIMATE p_l^h, WHERE $l = 1, ..., 4$.

n	24	48	96	192	384
e_1^h	4.53E-4	5.65E-5	7.06E-6	8.82E-7	1.10E-7
r_1^h		8.02	8.00	8.00	8.00
$\tilde{e}_1^h$		1.30E-7	4.58E-9	1.5E-10	4.9E-12
p_1^h		5.66E-5	7.06E-6	8.82E-7	1.10E-7
e_2^h	7.29E-3	9.06E-4	1.13E-4	1.41E-5	1.76E-6
r_2^h		8.05	8.02	8.00	8.00
$\tilde{e}_2^h$		6.29E-6	2.61E-7	9.16E-9	3.0E-10
p_2^h		9.06E-4	1.13E-4	1.41E-5	1.76E-6
e_3^h	3.71E-2	4.60E-3	5.72E-4	7.15E-5	8.93E-6
r_3^h		8.05	8.02	8.00	8.00
$\tilde{e}_3^h$		4.79E-5	2.60E-6	9.86E-8	3.35E-9
p_3^h		4.65E-3	5.75E-4	7.16E-5	8.93E-6
e_4^h	1.51E-4	1.88E-5	2.35E-6	2.94E-7	3.67E-8
r_4^h		8.00	8.00	8.00	8.00
$\tilde{e}_4^h$		5.12E-9	1.7E-10	5.7E-12	1.8E-13
p_4^h		1.88E-5	2.35E-6	2.94E-7	3.67E-8

From Table 1, we can numerically see $r_l^h \approx 2^3$ and $\tilde{e}_l^h / \tilde{e}_l^{h/2} \approx 2^5$, which agrees with Theorem 2 very well and shows that the convergent rate of λ_h is $O(h^5)$ after using the EAs. Moreover, since $\tilde{e}_l^h / \tilde{e}_l^{h/2} \approx 2^5$, we can apply the EAs again to obtain higher accuracy orders of approximations.

Example 2: Consider an elliptical elastic body $x^2/a^2 + y^2/b^2 \le 1$ with $a = 3$, $b = 2$ in plane strain deformation, where the material properties $\mu = 1.0$ and $v = 0.3$. we compute the boundary numerical eigensolutions by Eq.(18) and list the approximate eigenvalues as $\lambda_1 = 0.17292249292766\ldots$, $\lambda_2 = 0.81236868516390\ldots$, $\lambda_3 = 1.2185529275728\ldots$. Here we use the same notation as the previous example.

From Table 2, we can numerically see that the convergent rate of λ_h is $O(h^3)$, and is $O(h^5)$ after the EAs, which coincides with Theorem 2. It also verifies the accuracy of the approximate eigenvalues.

TABLE II
THE ERRORS $e_l^h, \tilde{e}_l^h$ AND A POSTERIORI ERROR ESTIMATE p_l^h, WHERE $l = 1, 2, 3$.

n	24	48	96	192	384
e_1^h	3.35E-5	4.17E-6	5.21E-7	6.51E-8	8.14E-9
r_1^h		8.03	8.01	8.00	8.00
$\tilde{e}_1^h$		1.68E-8	4.7E-10	1.4E-11	4.1E-13
p_1^h		4.19E-6	5.21E-7	6.51E-8	8.14E-9
e_2^h	1.03E-2	1.27E-3	1.58E-4	1.97E-5	2.47E-6
r_2^h		8.13	8.04	8.01	8.00
$\tilde{e}_2^h$		2.30E-5	8.24E-7	2.76E-8	8.7E-10
p_2^h		1.29E-3	1.59E-4	1.98E-5	2.47E-6
e_3^h	5.35E-2	6.47E-3	8.02E-4	9.99E-5	1.25E-5
r_3^h		8.27	8.08	8.02	8.01
$\tilde{e}_3^h$		2.51E-4	8.62E-6	2.91E-7	9.62E-9
p_3^h		6.72E-3	8.10E-4	1.00E-4	1.25E-5

VI. Conclusion

Generally, there are two main advantages of the MQMs:

(1) Evaluating each element of discretization matrices is very simple and straightforward, which does not require any singular integrals;

(2) The algorithm has a high accuracy order $O(h^3)$ and an asymptotic expansion of the errors with odd powers. Harnessing the Richardson extrapolation algorithms, a higher accuracy order $O(h^5)$ can be obtained.

(3)The extrapolations algorithm is very effective to improve the error accuracy, although it is not very complicated.

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