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Group of p-th roots of unity modulo n

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Abstract—Let $n \geq 3$ be an integer and p be a prime odd number. Let us consider $\mathbf{G}_p(n)$ the subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$ defined by :

$$\mathbf{G}_p(n) = \{x \in (\mathbb{Z}/n\mathbb{Z})^* \ / \ x^p = 1\}.$$

In this paper, we give an algorithm that computes a generating set of this subgroup.

Keywords—Group, p-th roots, modulo, unity.

I. Introduction

LET $n \geq 3$ be an integer, recall that $(\mathbb{Z}/n\mathbb{Z})^*$ denotes the group of units of the ring $(\mathbb{Z}/n\mathbb{Z})$. For more details on the structure of $(\mathbb{Z}/n\mathbb{Z})^*$ see [2], [3] and [4].
The group $(\mathbb{Z}/n\mathbb{Z})^*$ has several applications, the most important is cryptography, that is RSA cryptosystem (see [7]). The security of the RSA cryptosystem is based on the problem of factoring large integers and the task of finding e-th roots modulo a composite number n whose factors are not known.

Let p be a prime odd number, we notice by $\mathbf{G}_p(n)$ the part of $(\mathbb{Z}/n\mathbb{Z})^*$ formed by the elements x that verify $x^p = 1$. We can easily prove that $\mathbf{G}_p(n)$ is a subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$ which contains exactly the unity and the elements of order p.
Remember also that these elements of order p in $(\mathbb{Z}/n\mathbb{Z})^*$ exist if and only if p divides $\lambda(n)$, with λ is the Carmichael lambda function, otherwise $\mathbf{G}_p(n)$ is not reduced to $\{1\}$ if and only if p divides $\lambda(n)$.
The elements of $\mathbf{G}_p(n)$ other than 1 have the order p and so the order of $\mathbf{G}_p(n)$ is of the form p^t with t an integer. Then we obtain the following result:

Proposition :
Let $n \geq 3$ be an integer and p be a prime number, then there exists an integer t such as :

$$Card(\mathbf{G}_p(n)) = p^t$$

with $t = 0$ if and only if p does not divide $\lambda(n)$.

Our work consists to determine explicitly the integer t described in the preceding proposition and by giving at the same time with an effective manner the decomposition of $\mathbf{G}_p(n)$ in product of cyclic groups and give a generating family of this group. Finally, we give the algorithm written in Maple. The case $p = 2$ is treated in [1] and in this article, our approach is the same as it. For more details about the algorithmic number theory see [5] and [6], and for introduction to Maple see [10].

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II. P-th roots of unity modulo n

Let us consider an integer $n \geq 3$ and p a prime odd number, let $n = p^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ the decomposition of n in prime factors.
We know that the p-th roots of unity modulo n, which are nontrivial, exist if and only if p divides $\lambda(n)$, that is to say $\alpha \geq 2$ or there exists i such as p divides $p_i - 1$.
Thus, in our study, we will distinguish these following cases $\alpha = 0$, $\alpha = 1$ and $\alpha \geq 2$, but before that we are going to give some results which will be useful thereafter.

Definition 2.1: Let $n \geq 3$ be an integer and p be a prime number, we denote $\alpha_p(n)$ the number of prime factors q of n such that p divides $q - 1$.

Remark :
• $\alpha_2(n)$ is the number of prime odd factors of n.
• The function α_p is additive, that is to say if n and m are coprime numbers, then

$$\alpha_p(m.n) = \alpha_p(m) + \alpha_p(n)$$

and generally, for all the numbers not equal to 0, n and m we have:

$$\alpha_p(m.n) = \alpha_p(m) + \alpha_p(n) - \alpha_p(GCD(m, n)).$$

In the following, we consider an integer $n \geq 3$ whose the factorization is $n = p^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$, with p a prime odd number dividing $\lambda(n)$.

Proposition 2.1: Let x be a p-th root of unity modulo n. If p does not divide $p_i - 1$, then p_i divides $x - 1$.

Proof :
We have $x^p \equiv 1[n] \Longrightarrow x^p \equiv 1[p_i]$ and thus the order of x in $(\mathbb{Z}/p_i\mathbb{Z})^*$ is 1 or p, but the order of x in $(\mathbb{Z}/p_i\mathbb{Z})^*$ divides $p_i - 1$ and thus it cannot be p. Therefore $x \equiv 1[p_i]$ and then we obtain the result.■

Now, we will ameliorate the precedent result with the following lemma :

Lemma 2.1:

$$GCD(x - 1, 1 + x + x^2 + \ldots + x^{p-1}) \in \{1, p\}$$

Proof :
One can easily verify that we have:

$$(x-1)(x^{p-2} + 2x^{p-3} + 3x^{p-4} + \ldots + (p-2)x + (p-1)) - (1 + x + x^2 + \ldots + x^{p-1}) = p.■$$

Corollary 2.1: Let x be a p-th root of unity modulo n. If p does not divide $p_i - 1$ and $p \neq p_i$, then $p_i^{\alpha_i}$ divides $x-1$.

Proof :
We have $x^p \equiv 1[n] \implies x^p \equiv 1[p_i^{\alpha_i}]$ then $p_i^{\alpha_i}$ divides $x^p - 1 = (x-1)(1+x+x^2+\ldots+x^{p-1})$, or p does not divide $p_i - 1$ and thus p_i divides $x-1$ also we know that the $GCD(x-1, 1+x+x^2+\ldots+x^{p-1}) \in \{1,p\}$ and $p \neq p_i$, then $p_i^{\alpha_i}$ divides $x-1$.■

If p divides n, that is to say $\alpha \geq 1$, and x is a p-th root of unity modulo n, then p divides $x^p - 1 = (x-1)(1+x+x^2+\ldots+x^{p-1})$ and consequently p divides $x-1$ or $1+x+x^2+\ldots+x^{p-1}$ and seeing the relation given in the proof of *Lemma 2.1* we conclude that p divides both at the same time, and thus

$$PGCD(x-1, 1+x+x^2+\ldots+x^{p-1}) = p.$$

We are interested now in the case of $\alpha \geq 2$, we saw in [1] for $p=2$ that $2^{\alpha-1}$ divides $x-1$ or $x+1$, we are going to see that this result is not true for an odd prime p and more precisely we have the following result:

Proposition 2.2: Let x be a p-th root of unity modulo n $(\alpha \geq 2)$, then $p^{\alpha-1}$ divides $x-1$.

The case $\alpha = 2$ is trivial, for $\alpha \geq 3$, one needs the following lemma:

Lemma 2.2: Let p be a prime odd number and x be an integer, then we have :

$$x^p \equiv 1\,[p^3] \Longrightarrow x \equiv 1\,[p^2]$$

Proof :
It is clear that $x^p \equiv 1[p^3] \implies x \equiv 1\,[p]$, so $x = 1+kp$ $(k \in \mathbb{N})$ and consequently $x^p \equiv 1 + p^2k\,[p^3]$ (this writing is possible because $p \geq 3$) moreover p^3 divides p^2k, then p divides k and finally we obtain: $x \equiv 1\,[p^2]$.■

Remark : Notice that the precedent lemma is not true for $p=2$, for instance $3^2 \equiv 1\,[8]$ and $3 \not\equiv 1\,[4]$.

Proof of Proposition 2.2:
We have $x^p \equiv 1\,[p^\alpha]$ $(\alpha \geq 3)$ and so in particulary $x^p \equiv 1\,[p^3]$, from the precedent lemma we conclude that $x \equiv 1\,[p^2]$.
We have p^α divides $x^p - 1 = (x-1)(1+x+x^2+\ldots+x^{p-1})$ and as $PGCD(x-1, 1+x+x^2+\ldots+x^{p-1}) = p$ besides p^2 divides $x-1$, so $p^{\alpha-1}$ divides $x-1$.■

Remark :
The precedent proposition shows that $p^{\alpha-1}$ divides $x-1$, but this does not mean that the p-adic valuation of $x-1$ is $\alpha-1$ and this is proved by the following examples.

<u>An application example :</u>

- $n = 7^3 * 29 = 9947$, we have $344^7 \equiv 1\,[n]$ and $344 \equiv 1\,[7^3]$. $2402^7 \equiv 1\,[n]$ and $2402 \equiv 1\,[7^4]$.
- $n = 7^2 * 29 * 43 * 71 = 4338313$, we have $350547^7 \equiv 1\,[n]$ and $350547 \equiv 1\,[7^4]$.

Let us return to our principal aim, which is the study of the group $\mathbf{G}_p(n)$, we begin by the case $\alpha = 0$.

<u>Case 1 : $\alpha = 0$</u>

Let n be an integer whose decomposition into prime factors is $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ with $p_i \neq p$ for all i. Let x be a p-th root of unity modulo n, we have shown in the above results that if p does not divide $p_i - 1$, then $p_i^{\alpha_i}$ divides $x-1$. The condition p divides $\lambda(n)$ implies that it exists at least an integer i such that p divides $p_i - 1$, let σ be a permutation of the set $\{1,2,..,m\}$ such that $n = p_{\sigma(1)}^{\alpha_{\sigma(1)}} p_{\sigma(2)}^{\alpha_{\sigma(2)}} \ldots p_{\sigma(d)}^{\alpha_{\sigma(d)}} p_{\sigma(d+1)}^{\alpha_{\sigma(d+1)}} \ldots p_{\sigma(m)}^{\alpha_{\sigma(m)}}$ and p divides only $p_{\sigma(1)}^{\alpha_{\sigma(1)}}, p_{\sigma(2)}^{\alpha_{\sigma(2)}} \ldots$ and $p_{\sigma(d)}^{\alpha_{\sigma(d)}}$, then $p_{\sigma(d+1)}^{\alpha_{\sigma(d+1)}} \ldots p_{\sigma(m)}^{\alpha_{\sigma(m)}}$ divides $x-1$.
We start our study by the following theorem:

Theorem 2.1: Let n be an integer whose decomposition in prime factors is $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ with $p_i \neq p$ for all i and p divides only $p_1 - 1$, then $\mathbf{G}_p(n)$ is a cyclic subgroup of $(\mathbb{Z}/n\mathbb{Z})^*$ of order p.

Proof :
Let x be a p-th root of unity modulo n, we have $p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ divides $x-1$, then x is a solution of one of the following systems :

$$\begin{cases} x - 1 = p_2^{\alpha_2} \ldots p_m^{\alpha_m} K \\ 1 + x + x^2 + \ldots + x^{p-1} = p_1^{\alpha_1} K' \end{cases}$$

$$\begin{cases} x - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m} K \\ 1 + x + x^2 + \ldots + x^{p-1} = K' \end{cases}$$

Clearly, 1 is the unique solution of the second system. Now, we will show that the first system have exactly $p-1$ solutions, which follows immediately from the two following lemmas.

Lemma 2.3: The systems

$$\begin{cases} x - 1 = p_2^{\alpha_2} \ldots p_m^{\alpha_m} K \\ 1 + x + x^2 + \ldots + x^{p-1} = p_1^{\alpha_1} K' \end{cases} \quad (\star)$$

$$\begin{cases} x - 1 = p_2^{\alpha_2} \ldots p_m^{\alpha_m} K \\ 1 + x + x^2 + \ldots + x^{p-1} = p_1 K' \end{cases} \quad (\star\star)$$

have the same number of solutions respectively modulo n and $n/p_1^{\alpha_1 - 1}$.

Proof :
It is clear that any solution of $(\star)$ is a solution of $(\star\star)$. Reciprocally let x be a solution of $(\star\star)$, then $x^p \equiv 1\,[p_1 p_2^{\alpha_2} \ldots p_m^{\alpha_m}]$

that is to say $x^p = 1 + p_1 p_2^{\alpha_2} \dots p_m^{\alpha_m} K_1$ and therefore

$$\begin{aligned} x^{pp_1^{\alpha_1-1}} &= (1 + p_1 p_2^{\alpha_2} \dots p_m^{\alpha_m} K_1)^{p_1^{\alpha_1-1}} \\ &= 1 + \sum_{i=1}^{p_1^{\alpha_1-1}-1} \mathbf{C}^{\,i}_{p_1^{\alpha_1-1}} (p_1 p_2^{\alpha_2} \dots p_m^{\alpha_m} K_1)^i + \\ & \quad (p_1 p_2^{\alpha_2} \dots p_m^{\alpha_m} K_1)^{p_1^{\alpha_1-1}} \end{aligned}$$

It is easily verified that all $\mathbf{C}^{\,i}_{p_1^{\alpha_1-1}}$ are divisible by $p_1^{\alpha_1-1}$ and $p_1^{\alpha_1-1} \geq \alpha_1$, then $x^{pp_1^{\alpha_1-1}} \equiv 1\,[n]$. From the other hand

$$\begin{aligned} x^{p_1^{\alpha_1-1}} &= (1 + p_2^{\alpha_2} \dots p_m^{\alpha_m} K)^{p_1^{\alpha_1-1}} \\ &= 1 + \sum_{i=1}^{p_1^{\alpha_1-1}-1} \mathbf{C}^{\,i}_{p_1^{\alpha_1-1}} (p_2^{\alpha_2} \dots p_m^{\alpha_m} K)^i + \\ & \quad (p_2^{\alpha_2} \dots p_m^{\alpha_m} K)^{p_1^{\alpha_1-1}} \end{aligned}$$

and as the $\mathbf{C}^{\,i}_{p_1^{\alpha_1-1}}$ are divisible by p_1 and K is not divisible by p_1, then $x^{p_1^{\alpha_1-1}} - 1$ is divisible by all the p_i except p_1 and consequently $x^{p_1^{\alpha_1-1}}$ is a solution of $(\star)$.
Let x and y be two solutions of $(\star\star)$ such as $x^{p_1^{\alpha_1-1}} = y^{p_1^{\alpha_1-1}}\,[n]$ and thus $x^{p_1^{\alpha_1-1}} = y^{p_1^{\alpha_1-1}}\,[p_1]$, hence $x \equiv y\,[p_1]$, on the other hand it is clear that $x \equiv y\,[p_2^{\alpha_2} \dots p_m^{\alpha_m}]$ and consequently $x \equiv y\,[p_1 p_2^{\alpha_2} \dots p_m^{\alpha_m}]$. We therefore conclude that the number of solutions of $(\star)$ is greater than or equal to that of $(\star\star)$. Thus the systems $(\star)$ and $(\star\star)$ have the same number of solutions modulo n and $n/p_1^{\alpha_1-1}$ respectively.■

Lemma 2.4: The following system

$$\begin{cases} x - 1 = p_2^{\alpha_2} \dots p_m^{\alpha_m} K \\ 1 + x + x^2 + \dots + x^{p-1} = p_1 K' \end{cases} \qquad (\star\star)$$

has $p - 1$ solutions modulo $n/p_1^{\alpha_1-1}$.

Proof :
We know that $\mathbb{Z}/p_1\mathbb{Z}$ is the field of decomposition of the polynomial $X^{p_1} - X$, and more precisely we have :

$$X^{p_1} - X = \prod_{i=0}^{p_1-1} (X - i)$$

and therefore

$$X^{p_1-1} - 1 = \prod_{i=1}^{p_1-1} (X - i)$$

and as p divides $p_1 - 1$ then the polynomial $X^p - 1$ divides $X^{p_1-1} - 1$ and therefore the polynomial $X^p - 1$ is also a product of factors of degree 1, that is to say

$$X^p - 1 = \prod_{i=1}^{p} (X - \gamma_i)$$

and as 1 is a root of $X^p - 1$ then we take $\gamma_1 = 1$ and finally we obtain

$$1 + X + X^2 + \dots X^{p-1} = \prod_{i=2}^{p} (X - \gamma_i)$$

and consequently the system $(\star\star)$ is equivalent to the following systems:

$$\begin{cases} x - 1 = p_2^{\alpha_2} \dots p_m^{\alpha_m} K_2 \\ x - \gamma_2 = p_1 K_2' \end{cases} \quad \begin{cases} x - 1 = p_2^{\alpha_2} \dots p_m^{\alpha_m} K_3 \\ x - \gamma_3 = p_1 K_3' \end{cases}$$

$$\dots \begin{cases} x - 1 = p_2^{\alpha_2} \dots p_m^{\alpha_m} K_p \\ x - \gamma_p = p_1 K_p' \end{cases}$$

It is clear that each of these systems has only one solution modulo $p_1 p_2^{\alpha_2} \dots p_m^{\alpha_m}$. Also the solutions of these systems are 2 by 2 distinct. Indeed if we denote x_i the solution of the following system

$$\begin{cases} x - 1 = p_2^{\alpha_2} \dots p_m^{\alpha_m} K_i \\ x - \gamma_i = p_1 K_i' \end{cases}$$

then $x_i \equiv \gamma_i\,[p_1]$. Since the γ_i are distinct modulo p_1, then the x_i are also distinct. We conclude that $(\star\star)$ have $p - 1$ solutions modulo $n/p_1^{\alpha_1-1}$.■

Remark :
The proof of the previous theorem gives an algorithm for calculating the solutions of $(\star)$, and this is done in two steps:
<u>Step 1</u>
We resolve $(\star\star)$, the most difficult point in this step is to determinate the γ_i. We must give the factorization of the polynomial $1 + X + X^2 + \dots + X^{p-1}$ in the field $\mathbb{Z}/p_1\mathbb{Z}[X]$ and for this we can use Berlekamp's algorithm [8] or Cantor-Zassenhaus algorithm [9]. Then we decompose $(\star\star)$ in small systems that are resolved easily with Euclidian's algorithm.
<u>Step 2</u>
To find the solutions of $(\star)$, it is sufficient to see that they are also solutions of $(\star\star)$ set to the power $p_1^{\alpha_1-1}$ modulo n.

Note also that the set of solutions of $(\star)$ forms with 1 a cyclic group of order p, then any solution of $(\star)$ generates this group. Thus in practice it is sufficient to determine a solution of $(\star)$ to find the others.

<u>A sample calculation :</u>

We want to determine the elements of order 7 modulo n with $n = 10609215 = 29^4 * 5 * 3$. The first step consists to give the factorization of $1 + X + X^2 + \dots + X^6$ in the field $\mathbb{Z}/29\mathbb{Z}[X]$, by using Berlekamp's algorithm, we obtain :

$$\begin{aligned} & 1 + X + X^2 + \dots + X^6 \\ = \; & (X+4)(X+5)(X+6)(X+9)(X+13)(X+22). \end{aligned}$$

Let's consider the following system

$$\begin{cases} x - 1 = 15K \\ x + 4 = 29K' \end{cases}$$

which gives $29K' - 15K = 5$, and by the euclidian algorithm we obtain $K' = -5$ and $K = -10$.

Therefore $x = -149 = 286$ modulo $435 = 29 * 5 * 3$. Thereby $286^{29^3} \mod n = 1006441$ is an element of order 7 modulo n and consequently the elements of $\mathbf{G}_7(n)$ are

$$\mathbf{G}_7(n) = \{1006441, 1006441^2, \ldots, 1006441^7\}$$

that is to say

$$\mathbf{G}_7(n) = \{1006441, 8684356, 6860611, 4797001, 5450251, 9979951, 1\}$$

Now, we give an algorithm in $MAPLE$ which allows us for any fixed integer n and a prime odd number p, as described in the last theorem, to give a generator of the cyclic group $\mathbf{G}_p(n)$.

```
Gene_p := proc(n,p)   local LB, LD, P, gen, i, LFact;
LD := [ ]; LB := [ ];
LFact := ifactors(n)[2];
for i from 1 to nops(LFact) do
if (LFact[i][1] - 1 mod p = 0) then
LD := [op(LD), LFact[i]];
end :
end :
P := convert(Berlekamp(x^p - 1, x) mod LD[1][1], list);
if(P[1] - x + 1 mod LD[1][1] <> 0) then
LB := Bezout(LD[1][1], n/(LD[1][1]^LD[1][2]), P[1] - x + 1);
gen := ((LD[1][1] * LB[1] - (P[1] - x) mod n))&^
(LD[1][1]^(LD[1][2] - 1)) mod n;
else
LB := Bezout(LD[1][1], n/(LD[1][1]^LD[1][2]), P[2] - x + 1);
gen := (LD[1][1] * LB[1] - (P[2] - x) mod n)&^
(LD[1][1]^(LD[1][2] - 1)) mod n;
end :
eval(gen);
end :
end :
```

Algorithm 2.1

Remark :
The Berlekamp's procedure used in this algorithm is predefined in $MAPLE$.

In the remainder of this paragraph, considering an integer n whose decomposition in prime factors is $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ and p a prime odd number such that $p_i \neq p$ for all i. For a fixed permutation we can write $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_d^{\alpha_d} p_{d+1}^{\alpha_{d+1}} \ldots p_m^{\alpha_m}$ with p divides $p_i - 1$ for all $i \in \{1, .., d\}$. We have seen that if x is a p-th root of unity modulo n, then $p_{d+1}^{\alpha_{d+1}} \ldots p_m^{\alpha_m}$ divides $x - 1$. Thus $p_{d+1}^{\alpha_{d+1}} \ldots p_m^{\alpha_m}$ don't have a significant role in our study, for the rest we set $p_{d+1}^{\alpha_{d+1}} \ldots p_m^{\alpha_m} = A$.

Definition 2.2: Let x a p-th root of unity modulo n, we say that x is initial if all the p_i, $i \in \{1, .., d\}$ divides $x - 1$ except for only one p_i. We say that this p-th root is associated to p_i, and we write :

$$x - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK.$$

with K is an integer not divisible par p_i.

We denote by $\mathbf{G}_p^{p_i}(n)$ the set formed by the unity and the initial p-th roots of unity associated to p_i, and we have the following theorem :

Theorem 2.2: $\mathbf{G}_p^{p_i}(n)$ is a cyclic subgroup of $\mathbf{G}_p(n)$ with cardinality p.

Proof :
The initial p-th roots of unity associated to p_i are the solutions of the system :

$$\begin{cases} x - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK \\ 1 + x + x^2 + .. + x^{p-1} = p_i^{\alpha_i} K' \end{cases} \qquad (\star)$$

We saw in the foregoing that this system have $p - 1$ solutions modulo n and then $Card(\mathbf{G}_p^{p_i}(n)) = p$. Let's prove now that $\mathbf{G}_p^{p_i}(n)$ is a subgroup. Let x and y be two solutions of $(\star)$, we have

$$x - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK \ and$$

$$y - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK'$$

and therefore

$$\begin{aligned} x.y &= 1 + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} A(K \\ &+ K' + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AKK') \end{aligned}$$

Note that $x.y$ is a p-th root of unity and therefore at this stage we have two case. If p_i divides $(K + K' + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AKK')$, then $p_i^{\alpha_i}$ divides $x.y - 1$ and we obtain $x.y = 1$. If p_i does not divide $(K + K' + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AKK')$, then $x.y$ is an initial to p-th root of unity associated to p_i. It is clear that if x is a p-th root of unity, then its inverse $x^{-1} = x^{p-1}$ is an element of $\mathbf{G}_p^{p_i}(n)$. Whereof $\mathbf{G}_p^{p_i}(n)$ is a cyclic subgroup of $\mathbf{G}_p(n)$ because its cardinality is a prime number p.■

Proposition 2.3: Let x and y be two initial p-th roots of unity associated to p_i and p_j with $i \neq j$, then $x.y$ is a p-th root of unity satisfying

$$x.y - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots \overset{\vee}{p_j^{\alpha_j}} \ldots p_d^{\alpha_d} AK$$

with K is an integer which is not divisible by p_i and p_j.

Proof :
We have

$$x - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK_1 \ and$$

$$y - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_j^{\alpha_j}} \ldots p_d^{\alpha_d} AK_2$$

and therefore

$$x.y = 1 + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots \overset{\vee}{p_j^{\alpha_j}} \ldots p_d^{\alpha_d} A(p_j^{\alpha_j} K_1 + p_i^{\alpha_i} K_2)$$

and as p_i does not divide K_1 also p_j does not divide K_2, then $(p_i^{\alpha_j} K_1 + p_i^{\alpha_i} K_2)$ is not divisible by both p_i and p_j.■

Definition 2.3: Let x be a p-th root of unity modulo n, we say that it is final if all the p_i, $i \in \{1, .., d\}$ does not divide $x - 1$, that is to say $x - 1 = AK$, with K an integer not divisible by any p_i, $i \in \{1, .., d\}$.

Remark :
The existence of final p-th roots of unity modulo n is ensured by the preceding proposition, in fact if for all $i \in \{1, .., d\}$ we take x_i an initial p-th root of unity associated to p_i, then $\prod_{i=1}^{d} x_i$ is a final p-th root of unity modulo n.

Definition 2.4: Let x and y be two p-th roots of unity modulo n, we say that y is a final conjugate of x if $x.y - 1$ is not divisible by any of the p_i, $i \in \{1, .., d\}$, that is to say $x.y$ is a final p-th root of unity modulo n.

Proposition 2.4: Any p-th root of unity modulo n have a final conjugate.

Proof :
If $x = 1$ or x is a final p-th root of unity modulo n, then we have the result. When $d = 1$, then a final p-th root of unity modulo n is also an initial p-th root of unity associated to p_1 and thus all the p-th roots of unity distinct from 1 are final.
Now, we suppose that $d \geq 2$ and $x - 1$ is divisible by a nonempty subset of p_i of cardinality $t < d$ and we can assume that, for a fixed permutation, this p_i are $p_1, p_2, \dots$ are p_t and thus

$$x - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t} AK$$

with K is an integer which is not divisible by any of the p_i, $i \in \{t+1, .., d\}$. For all $i \in \{1, .., t\}$ let x_i be an initial p-th root of unity associated to p_i and therefore

$$x_i = 1 + p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \dots p_d^{\alpha_d} AK_i$$

with K_i not divisible by p_i, and thus

$$\prod_{i=1}^{t} x_i = \prod_{i=1}^{t} (1 + p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \dots p_d^{\alpha_d} AK_i)$$

$$= 1 + p_{t+1}^{\alpha_{t+1}} \dots p_d^{\alpha_d} A \sum_{i=1}^{t} p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_t^{\alpha_t} K_i + K'n$$

but $\sum_{i=1}^{t} p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_t^{\alpha_t} K_i$ is not divisible by any of the p_i, $i \in \{1, .., t\}$ therefore $y = \prod_{i=1}^{t} x_i$ is a p-th root of unity satisfies $y = 1 + p_{t+1}^{\alpha_{t+1}} \dots p_d^{\alpha_d} AM$ with M an integer which is not divisible by p_i, $i \in \{1, .., t\}$. So

$$x.y = 1 + A(p_{t+1}^{\alpha_{t+1}} \dots p_d^{\alpha_d} AM + p_1^{\alpha_1} \dots p_t^{\alpha_t} AK)$$

It is clear that $(p_{t+1}^{\alpha_{t+1}} \dots p_d^{\alpha_d} AM + p_1^{\alpha_1} \dots p_t^{\alpha_t} AK)$ is not divisible by any of the p_i, $i \in \{1, .., d\}$, and hence the result.■

Theorem 2.3: Let x be a final p-th root of unity modulo n, then it exists d integers $K_1, K_2, \dots, K_d$ such as:

$$x = 1 + \sum_{i=1}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_d^{\alpha_d} AK_i$$

and

$$(1 + p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_d^{\alpha_d} AK_i)^p = 1\,[n] \quad \forall\ 1 \leq i \leq d.$$

Proof :
Since $p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_d^{\alpha_d}}$ and $p_d^{\alpha_d}$ are coprime then it exists two integers $\widetilde{K}'_d$ and $\widetilde{K}_d$ such as

$$1 = p_d^{\alpha_d} \widetilde{K}'_d + p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_d^{\alpha_d}}\ \widetilde{K}_d\ (\star)$$

and therefore

$$x - 1 = p_d^{\alpha_d} AK'_d + p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_d^{\alpha_d}}\ AK_d$$

with $K'_d = ((x-1)/A)\widetilde{K}'_d$ and $K_d = ((x-1)/A)\widetilde{K}_d$.
We have :

$$\begin{aligned}
(x - p_d^{\alpha_d} AK'_d)^p &= (x - (x-1)p_d^{\alpha_d} \widetilde{K}'_d)^p \\
&= (a(1 - p_d^{\alpha_d} \widetilde{K}'_d) + p_d^{\alpha_d} \widetilde{K}'_d)^p \\
&= (x p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_d^{\alpha_d}}\ \widetilde{K}_d + p_d^{\alpha_d} \widetilde{K}'_d)^p \\
&= (p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_d^{\alpha_d}}\ \widetilde{K}_d)^p + (p_d^{\alpha_d} \widetilde{K}'_d)^p \quad [p_1^{\alpha_1} \\
&\quad p_2^{\alpha_2} \dots p_d^{\alpha_d}] \\
&= 1\,[p_1^{\alpha_1} p_2^{\alpha_2} \dots p_d^{\alpha_d}] \quad \text{from } (\star)
\end{aligned}$$

On the other hand

$$\begin{aligned}
x - (x-1)p_d^{\alpha_d} \widetilde{K}'_d &= 1 + (x-1)(1 - p_d^{\alpha_d} \widetilde{K}'_d) \\
&= 1\ [A]
\end{aligned}$$

Thus $(x - (x-1)p_d^{\alpha_d} \widetilde{K}'_d)^p = 1[n]$ and consequently $(1 + p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_d^{\alpha_d}}\ AK_d)^p = 1[n]$.
Suppose that it exists some integers $K_t, K_2, \dots, K_d$ and K'_t such as :

$$x = 1 + \sum_{i=t}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_d^{\alpha_d} AK_i + p_t^{\alpha_t} \dots p_d^{\alpha_d} AK'_t$$

and

$$(1 + p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_i^{\alpha_i}} \dots p_d^{\alpha_d} AK_i)^p = 1\,[n] \quad \forall\ t \leq i \leq d$$

Let $\widetilde{K}_{t-1}$ and $\widetilde{K}'_{t-1}$ be two integers such as

$$1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots \overset{\vee}{p_{t-1}^{\alpha_{t-1}}}\ \widetilde{K}_{t-1} + p_{t-1}^{\alpha_{t-1}} \widetilde{K}'_{t-1}\ (\star\star)$$

and therefore

$$p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \dots p_d^{\alpha_d} AK'_t = p_1^{\alpha_1} \dots \overset{\vee}{p_{t-1}^{\alpha_{t-1}}} \dots p_d^{\alpha_d} AK'_t \widetilde{K}_{t-1} +$$

$$p_{t-1}^{\alpha_{t-1}} \dots p_d^{\alpha_d} AK'_t \widetilde{K}'_{t-1}.$$

We have

$$\begin{aligned}
& (p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1 - p_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK'_t \widetilde{K}'_{t-1})^p \\
= & ((p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1)(1 - p_{t-1}^{\alpha_{t-1}} \widetilde{K}'_{t-1}) + \\
& p_{t-1}^{\alpha_{t-1}} \widetilde{K}'_{t-1})^p \\
= & ((p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1) p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_{t-1}^{\alpha_{t-1}} \widetilde{K}_{t-1} + \\
& p_{t-1}^{\alpha_{t-1}} \widetilde{K}'_{t-1})^p \\
= & (p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1)^p (p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_{t-1}^{\alpha_{t-1}} \widetilde{K}_{t-1})^p + \\
& (p_{t-1}^{\alpha_{t-1}} \widetilde{K}'_{t-1})^p \ [p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{t-1}^{\alpha_{t-1}}]
\end{aligned}$$

however

$$\begin{aligned}
& (p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1)^p \\
= & (x - \sum_{i=t}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i)^p \\
= & x^p \ [p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{t-1}^{\alpha_{t-1}} A] \\
= & 1 \ [p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{t-1}^{\alpha_{t-1}} A]
\end{aligned}$$

and consequently

$$\begin{aligned}
& (p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1 - p_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK'_t \widetilde{K}'_{t-1})^p \\
= & (p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_{t-1}^{\alpha_{t-1}} \widetilde{K}_{t-1})^p + \\
& (p_{t-1}^{\alpha_{t-1}} \widetilde{K}'_{t-1})^p \ \ [p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{t-1}^{\alpha_{t-1}}] \\
= & 1 \ \ [p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{t-1}^{\alpha_{t-1}}] \quad \text{from } (\star\star)
\end{aligned}$$

also it is clear that

$$(p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1 - p_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK'_t \widetilde{K}'_{t-1})^p = 1 \ [p_d^{\alpha_d} \ldots p_t^{\alpha_t} A]$$

and so

$$(p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK'_t + 1 - p_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK'_t \widetilde{K}'_{t-1})^p = 1 \ [n]$$

That means

$$(1 + p_1^{\alpha_1} \ldots \check{p}_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK'_t \widetilde{K}_{t-1})^p = 1 \ [n].$$

We set $K_{t-1} = K'_t \widetilde{K}_{t-1}$ and $K'_{t-1} = K'_t \widetilde{K}'_{t-1}$, we obtain so

$$\begin{aligned}
x = & 1 + \sum_{i=t}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i + \\
& p_1^{\alpha_1} \ldots \check{p}_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK_{t-1} + p_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK'_{t-1} \\
= & 1 + \sum_{i=t-1}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i + \\
& p_{t-1}^{\alpha_{t-1}} \ldots p_d^{\alpha_d} AK'_{t-1}
\end{aligned}$$

with

$$(1 + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i)^p = 1 \, [n] \quad \forall \ t-1 \le i \le d$$

Thus by induction, we obtain

$$\begin{aligned}
x = & 1 + \sum_{i=1}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i + p_1^{\alpha_1} \ldots p_d^{\alpha_d} AK'_1 \\
= & 1 + \sum_{i=1}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i \quad [n]
\end{aligned}$$

with $(1 + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i)^p = 1 \, [n]$, $\forall \ 1 \le i \le d$.■

Corollary 2.2: Any final p-th root of unity modulo n is a product of d initial p-th roots associated respectively to p_1,$p_2 \ldots$ and p_d.

Proof :
From the precedent theorem, it exists some integers $K_1, K_2, \ldots, K_d$ such as:

$$x = 1 + \sum_{i=1}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i$$

and

$$(1 + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i)^p = 1 \, [n] \quad \forall \ 1 \le i \le d$$

If we set $x_i = 1 + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i$, then x_i is a p-th root of unity modulo n also from the construction of K_i in the preceding proof, K_i is not divisible by p_i. Thus x_i is an initial p-th root associated to p_i. On the other hand we have

$$\begin{aligned}
\prod_{i=1}^{d} x_i = & \prod_{i=1}^{d} (1 + p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i) \\
= & 1 + \sum_{i=1}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \check{p}_i^{\alpha_i} \ldots p_d^{\alpha_d} AK_i \ [n] = x.■
\end{aligned}$$

Corollary 2.3: Every p-th root of unity modulo n is a product of initial p-th roots.

Proof :
Let x be a p-th root of unity modulo n, if this root is final, then the result is immediate, otherwise there is $x_1, x_2, \ldots$ and x_t such as $x.\prod_{i=1}^{t} x_i$ is final p-th root of unity modulo n and from the precedent corollary there exists $y_1, y_2, \ldots$ and y_d initial p-th roots of unity modulo n associated respectively to p_1,$p_2 \ldots$ and p_d such as $x.\prod_{i=1}^{t} x_i = \prod_{i=1}^{d} y_i$ and thus $x = \prod_{i=1}^{t} x_i^{-1}.\prod_{i=1}^{d} y_i$ and as the set of initial p-th roots of unity modulo n associated to p_i form with 1 a group, then x can be written like following $x = \prod_{i=1}^{d} z_i$ with z_i is either 1 or an initial p-th root associated to p_i.■

Corollary 2.4: $\mathbf{G}_p(n)$ is generated by the initial $p-$th roots of unity modulo n.

Remark :
As for each p_i the set of initial p-th roots of unity modulo n associated to p_i form with 1 a cyclic group then

$$\mathbf{G}_p(n) = < x_1, x_2, \ldots, x_d >$$

with x_i an initial p-th root of unity modulo n associated to p_i.

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Theorem 2.4: The map

$$\begin{array}{rcl}\varphi : \mathbf{G}_p^{p_1}(n) \times \mathbf{G}_p^{p_2}(n) \ldots \times \mathbf{G}_p^{p_d}(n) & \longrightarrow & \mathbf{G}_p(n) \\ (x_1, x_2, \ldots, x_d) & \longmapsto & x_1.x_2, \ldots x_d\end{array}$$

is an isomorphism of groups.

Proof :
We have shown that φ is a surjective morphism of groups, remains to prove that it is injective.
We have $\varphi(x_1, x_2, \ldots, x_d) = 1 \iff x_1.x_2, \ldots x_d = 1$, assume that there exists an integer i such that $x_i \neq 1$, then we can easily verify that $x_1.x_2, \ldots x_d - 1$ is also not divisible by p_i but this is absurd, thus $x_i = 1$ for all i and hence φ is injective.■

From the previous theorem it is clear that $Card(\mathbf{G}_p(n)) = p^d$, where d is a number of distinct prime factors q of n such that p divides $q-1$, that is to say $d = \alpha_p(n)$ and we obtain the following result :

Corollary 2.5:

$$Card(\mathbf{G}_p(n)) = p^{\alpha_p(n)}.$$

Remark :
From the previous theorem we have

$$\mathbf{G}_p(n) = \{ \prod_{(i_1,i_2,..,i_d)\in I^d} x_1^{i_1} x_2^{i_2} \ldots x_d^{i_d} \quad , \text{with } I = \{1, 2, .., p\}\}$$

with x_i is a generator of the cyclic group $\mathbf{G}_p^{p_i}(n)$.

We give now an algorithm written in Maple that allows us from an integer n and an odd prime p, as described in this foregoing, to give a generating set of $\mathbf{G}_p(n)$.

```
Gene_p := proc(n, p)   local LB, LD, i, LFact, GEN, P;
LD := [ ]; LB := [ ]; GEN := [ ];
LFact := ifactors(n)[2];
for i from 1 to nops(LFact) do
if (LFact[i][1] - 1 mod p = 0) then
LD := [op(LD), LFact[i]];
end :
end :
for i from 1 to nops(LD) do
P := convert(Berlekamp(x^p - 1, x) mod LD[i][1], list);
if(P[1] - x + 1 mod LD[i][1] <> 0) then
LB := Bezout(LD[i][1], n/(LD[i][1]^LD[i][2]), P[1] - x + 1);
GEN := [op(GEN), ((LD[i][1] * LB[1] - (P[1] - x) mod n))&^(LD[i][1]^(LD[i][2] - 1)) mod n];
else
LB := Bezout(LD[i][1], n/(LD[i][1]^LD[i][2]), P[2] - x + 1);
GEN := [op(GEN), (LD[i][1]*LB[1]-(P[2]-x) mod n)&^ (LD[i][1]^(LD[i][2] - 1)) mod n];
end :
end :
if(GEN = [ ]) then
GEN := [1];
end :
eval(GEN);
end :
```

Algorithm 2.2

A sample application :

Let $n = 53 * 79 * 131 * 17 * 19$ and $p = 13$, to find a generating set of the group formed by the p-th roots of unity modulo n, it suffices to use the previous algorithm with the command line $Gene\_p(n, 13)$. The displayed result is $[50140906, 174921943, 71677254]$, which represents the list of generators of this group.

Remark :
In the case when this algorithm return [1], then this means that $G_p(n) = \{1\}$.

Case 2 : $\alpha = 1$

Let n be an integer whose decomposition into prime factors is $n = p\, p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}$ with $p_i \neq p$ for all i and let x be a p-th root of unity modulo n, the above results show that if p does not divide $p_i - 1$ then $p_i^{\alpha_i}$ divides $x - 1$, on the other hand we have $x^p = 1[n]$ implies that p divides $(x-1)(1 + x + .. + x^{p-1})$ and from the lemma 2.1 we obtain p divides $x - 1$ and $1 + x + .. + x^{p-1}$.
Also provided p divides $\lambda(n)$ implies that there exists at least one integer i such that p divides $p_i - 1$. For a fixed permutation we can write $n = p\, p_1^{\alpha_1} \ldots p_d^{\alpha_d} \ldots p_m^{\alpha_m}$ with p divides $p_i - 1$ for all $i \in \{1, .., d\}$ and does not divide $p_i - 1$ for every $i \in \{d+1, .., m\}$. Assume for the following $p_{d+1}^{\alpha_{d+1}} \ldots p_m^{\alpha_m} = A$.
We define in the same manner the initial p-th roots of unity modulo n by replacing A with pA. The initial p-th roots of unity modulo n associated to p_i, $i \in \{1, .., d\}$ are the solutions of the system :

$$\begin{cases} x - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} pAK \\ 1 + x + x^2 + .. + x^{p-1} = p_i^{\alpha_i} K' \end{cases}$$

We show in the same manner that this system has exactly $p - 1$ roots modulo n. Thus for all $i \in \{1, .., d\}$ there are $p - 1$ initial p-th roots associated to p_i. We also show that the initial p-th roots of unity modulo n associated to p_i form with 1 a cyclic subgroup of $\mathbf{G}_p(n)$ of cardinality p and it is denoted as $\mathbf{G}_p^{p_i}(n)$.
We define in the same way a final p-th root of unity and its conjugate by replacing A by pA and we obtain the following theorem :

Theorem 2.5: Let x be a final p-th root of unity modulo n, then there exists integers $K_1, K_2, \ldots, K_d$ such that :

$$x = 1 + \sum_{i=1}^{d} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} pAK_i$$

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and

$$(1+p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}pAK_i)^p = 1\,[n] \quad \forall\ 1\le i\le d.$$

Indeed to prove this result we can just proceed as above and replacing A by pA.
We deduce that any final p-th root of unity modulo n is the product of d initial p-th roots associated respectively to p_1,p_2,...and p_d. Hence every p-th root of unity is the product of initial p-th roots, and we can show that $\mathbf{G}_p(n)$ is generated by the initial p-th roots of unity and more precisely if we denote x_i an initial p-th root of unity associated to p_i, then

$$\mathbf{G}_p(n) =< x_1, x_2, \dots, x_d > .$$

Also we have the following results :

Theorem 2.6: The map

$$\begin{array}{rcl}\varphi : \mathbf{G}_p^{p_1}(n)\times \mathbf{G}_p^{p_2}(n)\dots\times \mathbf{G}_p^{p_d}(n) & \longrightarrow & \mathbf{G}_p(n)\\ (x_1,x_2,\dots,x_d) & \longmapsto & x_1.x_2,\dots x_d\end{array}$$

is an isomorphism of groups.

Corollary 2.6:

$$Card(\mathbf{G}_p(n)) = p^{\alpha_p(n)}.$$

Remark :
From the previous theorem we can easily show that

$$\mathbf{G}_p(n) = \{\prod_{(i_1,i_2,..,i_d)\in I^d} x_1^{i_1}x_2^{i_2}\dots x_d^{i_d} \quad \text{, with } I=\{1,2,..,p\}\}$$

with x_i is a generator of the cyclic group $\mathbf{G}_p^{p_i}(n)$.

Finally, note that *Algorithm 2.2* remains valid in this case.

Case 3 : $\alpha\ge 2$

Let n be an integer whose decomposition into prime factors is $n=p^\alpha p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m}$ with $p_i\ne p$ for all i and $\alpha\ge 2$. The fact that $\alpha\ge 2$ ensures that $\mathbf{G}_p(n)$ is not reduced to $\{1\}$.
Suppose that for every i, p does not divide p_i-1 and let x be a p-th root of unity modulo n, then $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m}$ divides $x-1$ and by *Proposition 2.2* it follows that $p^{\alpha-1}$ divides $x-1$. So x is a solution of the system

$$\left\{\begin{array}{l} x-1 = p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m}K\\ \\ 1+x+x^2+..+x^{p-1} = K'\end{array}\right.$$

But this system has p solutions modulo n which are $1, 1+n/p, 1+2n/p, ..$ and $1+(p-1)n/p$. Then we obtain the following result:

Proposition 2.5: Let $n = p^\alpha p_1^{\alpha_1}p_2^{\alpha_2}\dots p_m^{\alpha_m}$ with $\alpha\ge 2$ and p does not divide p_i-1 for all i, then

$$\mathbf{G}_p(n) = \{1+kn/p;\quad 0\le k\le p-1\}$$

Remark:
It is clear that $\mathbf{G}_p(n)$ is a cyclic group of order p.

We will now exclude this case from our study, that is, there exists at least i such that p divides p_i-1. For a fixed permutation we can write $n=p^\alpha p_1^{\alpha_1}\dots p_d^{\alpha_d}\dots p_m^{\alpha_m}$ with p divides p_i-1 for all $i\in\{1,..,d\}$ and does not divide p_i-1 for all $i\in\{d+1,..,m\}$ and assume for the rest of this paper $p_{d+1}^{\alpha_{d+1}}\dots p_m^{\alpha_m}=A$.

Definition 2.5: Let x be a p-th root of unity modulo n, x is said of class zero if $x-1=p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK$ with K an integer.

It is clear that there are p p-th roots of unity of class zero which are $\{1+kn/p;\quad 0\le k\le p-1\}$ and one can easily verify that they form a cyclic group of order p denoted $\mathbf{G}_p^0(n)$.

Definition 2.6: Let x be a p-th root of unity modulo n, it said initial root if every p_i, $i\in\{1,..,d\}$ divides $x-1$ except for only one p_i. We said that this root is associated to p_i. And we write :

$$x-1 = p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}AK.$$

with K an integer that is not divided by p_i.

Theorem 2.7: There exists p^2-p initial p-th roots of unity associated to p_i for all $1\le i\le d$.

Proof :
We may assume $i=1$, the initial p-th roots associated to p_1 are the solutions of the system :

$$\left\{\begin{array}{l} x-1 = p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK\\ \\ 1+x+x^2+..+x^{p-1} = p_1^{\alpha_1}K'\end{array}\right. \quad (\star)$$

and we conclude with the following lemmas.■

Lemma 2.5: The following systems have the same number of solutions respectively modulo n and $n/p_1^{\alpha_1-1}$.

$$\left\{\begin{array}{l} x-1 = p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK\\ \\ 1+x+x^2+\dots+x^{p-1} = p_1^{\alpha_1}K'\end{array}\right. \quad (\star)$$

$$\left\{\begin{array}{l} x-1 = p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK\\ \\ 1+x+x^2+\dots+x^{p-1} = p_1K'\end{array}\right. \quad (\star\star)$$

Proof :
It is clear that any solution of $(\star)$ is a solution of $(\star\star)$. Reciprocally let x be a solution of $(\star\star)$, then $x^p \equiv 1\,[p^\alpha p_1p_2^{\alpha_2}\dots p_d^{\alpha_d}A]$ that is to say $x^p = 1+p^\alpha p_1p_2^{\alpha_2}\dots p_d^{\alpha_d}AK_1$ and therefore

$$\begin{array}{rcl} x^{pp_1^{\alpha_1-1}} & = & (1+p^\alpha p_1p_2^{\alpha_2}\dots p_d^{\alpha_d}AK_1)^{p_1^{\alpha_1-1}}\\ & = & 1+\displaystyle\sum_{i=1}^{p_1^{\alpha_1-1}-1}\mathbf{C}^i_{p_1^{\alpha_1-1}}(p_1p_2^{\alpha_2}\dots p_d^{\alpha_d}AK_1)^i\\ & + & (p^\alpha p_1p_2^{\alpha_2}\dots p_d^{\alpha_d}AK_1)^{p_1^{\alpha_1-1}}\end{array}$$

It is easily verified that all $\mathbf{C}^{\,i}_{p_1^{\alpha_1-1}}$ are divisible by $p_1^{\alpha_1-1}$ and $p_1^{\alpha_1-1} \geq \alpha_1$, then $x^{pp_1^{\alpha_1-1}} \equiv 1\,[n]$. On the other hand

$$\begin{aligned} x^{p_1^{\alpha_1-1}} &= (1+p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK)^{p_1^{\alpha_1-1}} \\ &= 1+\sum_{i=1}^{p_1^{\alpha_1-1}-1} \mathbf{C}^{\,i}_{p_1^{\alpha_1-1}}(p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK)^i \\ &+ (p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK)^{p_1^{\alpha_1-1}} \end{aligned}$$

And as $\mathbf{C}^{\,i}_{p_1^{\alpha_1-1}}$ are divisible by p_1 and K is not divisible by p_1, then $x^{p_1^{\alpha_1-1}}-1$ is divisible by all p_i except p_1. Consequently $x^{p_1^{\alpha_1-1}}$ is a solution of $(\star)$.

Let x and y be two solutions of $(\star\star)$ such that $x^{p_1^{\alpha_1-1}} = y^{p_1^{\alpha_1-1}}\,[n]$ thus $x^{p_1^{\alpha_1-1}} = y^{p_1^{\alpha_1-1}}\,[p_1]$. Hence $x \equiv y\,[p_1]$, on the other hand it is clear that $x \equiv y\,[p_2^{\alpha_2}\dots p_d^{\alpha_d}A]$ therefore $x \equiv y\,[p_1p_2^{\alpha_2}\dots p_d^{\alpha_d}A]$. We conclude then that the systems $(\star)$ and $(\star\star)$ have the same number of solutions respectively modulo n and $n/p_1^{\alpha_1-1}$.■

Lemma 2.6: The following system have p^2-p solutions modulo $n/p_1^{\alpha_1-1}$.

$$\begin{cases} x-1 = p^{\alpha-1}p_2^{\alpha_2}\dots p_m^{\alpha_m}K \\ 1+x+x^2+\dots+x^{p-1} = p_1K' \end{cases} \quad (\star\star)$$

Proof :
We know that

$$X^p-1=\prod_{i=1}^{p}(X-\gamma_i)$$

and as 1 is a root of X^p-1 then we take $\gamma_1=1$. Finally, we obtain

$$1+X+X^2+\dots X^{p-1}=\prod_{i=2}^{p}(X-\gamma_i)$$

and consequently $(\star\star)$ is equivalent to the following systems :

$$\begin{cases} x-1=p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK_2 \\ x-\gamma_2=p_1K'_2 \end{cases}$$

$$\vdots$$

$$\begin{cases} x-1=p^{\alpha-1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK_p \\ x-\gamma_p=p_1K'_p \end{cases}$$

It is clear that for each one of these systems have p solutions modulo $n/p_1^{\alpha_1-1}$. Since, the solutions of these systems are distinct, we conclude that $(\star\star)$ have $p(p-1)$ solutions modulo $n/p_1^{\alpha_1-1}$.■

Proposition 2.6: The set formed by the initial p-th roots of unity modulo n associated to p_i and by the elements of $\mathbf{G}_p^0(n)$ is a subgroup of $\mathbf{G}_p(n)$ denoted $\mathbf{G}_p^{p_i}(n)$ and we have $Card(\mathbf{G}_p^{p_i}(n))=p^2$.

Proof :
Let x and y be two elements of $\mathbf{G}_p^{p_i}(n)$, there are three cases to distinguish :

• If x and y are in $\mathbf{G}_p^0(n)$, then in this case xy belongs $\mathbf{G}_p^0(n)$ since the latter is a group and hence xy is in $\mathbf{G}_p^{p_i}(n)$.

• If x and y are respectively in $\mathbf{G}_p^{p_i}(n)\setminus\mathbf{G}_p^0(n)$ and $\mathbf{G}_p^0(n)$, then we have $x-1=p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}AK$ and $y-1=p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots p_d^{\alpha_d}AK'$ with K an integer not divisible by p_i thus

$$xy=1+p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}A(K+p_i^{\alpha_i}K')$$

The term $K+p_i^{\alpha_i}K'$ is not divided by p_i and therefore xy is a p-th root of unity associated to p_i. Hence xy is in $\mathbf{G}_p^{p_i}(n)$.

• If x and y are in $\mathbf{G}_p^{p_i}(n)\setminus\mathbf{G}_p^0(n)$, then : $x-1=p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}AK$ and $y-1=p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}AK'$ with K and K' are two integers not divided by p_i therefore

$$\begin{aligned} xy &= 1+p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}A(K+K' \\ &+ p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}AKK') \end{aligned}$$

If the term $K+K'+p^{\alpha-1}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}AKK'$ is divided by p_i then xy belongs to $\mathbf{G}_p^0(n)\subset\mathbf{G}_p^{p_i}(n)$, otherwise xy is a p-th root associated to p_i and consequently xy is in $\mathbf{G}_p^{p_i}(n)$.

Thus $\mathbf{G}_p^{p_i}(n)$ is stable for the product and as the inverse of the element x is x^{p-1}, then $\mathbf{G}_p^{p_i}(n)$ is stable by the inverse operation which proves that $\mathbf{G}_p^{p_i}(n)$ is a subgroup of $\mathbf{G}_p(n)$. Finally, we can see that $\mathbf{G}_p^0(n)$ does not contain an initial p-th root associated to p_i which allows us to conclude that $Card(\mathbf{G}_p^{p_i}(n))=(p^2-p)+p=p^2$. ■

Definition 2.7: Let x be a p-th root, we said that x is of the first class if p^α divides $x-1$, otherwise it said to be of the second class.

Proposition 2.7: There are $p-1$ initial p-th roots of unity associated to p_i which are of the first class.

Proof :
The initial p-th roots associated to p_i which are of first class are solutions of the system :

$$\begin{cases} x-1=p^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\dots \overset{\vee}{p_i^{\alpha_i}}\dots p_d^{\alpha_d}AK \\ x+1=p_i^{\alpha_i}K' \end{cases}$$

And from the previous we know that this system has $p-1$ solutions modulo n.■

Let denote by $\overset{+}{\mathbf{G}}{}_p^{p_i}(n)$ the set formed by 1 and the initial p-th roots of unity associated to p_i that are of the first class and we can easily verify that $\overset{+}{\mathbf{G}}{}_p^{p_i}(n)$ is a cyclic subgroup of $\mathbf{G}_p(n)$ of cardinality p and we have the following result :

Proposition 2.8: The map

$$\begin{aligned} \varphi:\overset{+}{\mathbf{G}}{}_p^{p_i}(n)\times\mathbf{G}_p^0(n) &\longrightarrow \mathbf{G}_p^{p_i}(n) \\ (x,y) &\longmapsto x.y \end{aligned}$$

is an isomorphism of groups.

Proof :
It is clear that φ is surjective morphism of groups. For the injectivity, let us consider two elements x and y of $\overset{+}{\mathbf{G}}_p^{p_1}(n)$ and $\mathbf{G}_p^0(n)$ respectively such that $x.y = 1$, we have : $x - 1 = p^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK$ and $y - 1 = p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_d^{\alpha_d} AK'$, therefore

$$xy = 1 + p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} A(K + p_i^{\alpha_i} K').$$

As $x.y = 1$, then the term $K + p_i^{\alpha_i} K'$ is divided by $p_i^{\alpha_i}$ therefore $p_i^{\alpha_i}$ divides K, hence $x = y = 1$.■

Definition 2.8: Let x be a p-th root of unity modulo n, we said x is final if all the p_i, $i \in \{1, .., d\}$ does not divide $x - 1$, which means $x - 1 = p^{\alpha-1} AK$, with K an integer not divisible by p_i, $i \in \{1, .., d\}$.

Proposition 2.9: Any final p-th root of unity modulo n can be written in a single manner as product of a final p-th root of the first class by a class zero's p-th root.

Proof :
Let x be a final p-th root of unity modulo n and let's consider an integer y of the form $y = 1 + p^\alpha AK$ and z a class zero's p-th root. We have :

$$\begin{aligned} x = yz &\iff x = (1 + p^\alpha AK)(1 + p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_d^{\alpha_d} AK') \\ &\iff x - 1 = p^\alpha AK + p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_d^{\alpha_d} AK' \\ &\iff \frac{x-1}{p^{\alpha-1} A} = pK + p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_d^{\alpha_d} K' \end{aligned}$$

This equation has solutions K and K', also $(1 + p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_d^{\alpha_d} AK')^p = 1$, therefore $(1 + p^\alpha AK)^p = 1$ and as $x - 1$ is divisible by none of the p_i which implies that K is divisible by none of the p_i, this proves that $(1 + p^\alpha AK)$ is a final p-th root of the first class. Also it is clear that if we take K and K' as other solutions, then $1 + p^\alpha AK$ and $1 + p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_d^{\alpha_d} AK'$ are the same modulo n.■

Remark :
If for all $i \in \{1, .., d\}$ we take x_i an initial p-th root of the first class associated to p_i, then $\prod_{i=1}^{d} x_i$ is a final root of the first class. The following theorem shows that any final root of the first class is a product of this form.

Theorem 2.8: Any final p-th root of the first class is product of d initial p-th roots of the first class associated respectively to $p_1, p_2, ..$ and p_d.

Proof :
Let x be a final p-th root of the first class, we know that there exist $K_1, K_2, ..$ and K_d such that

$$x = 1 + \sum_{i=1}^{d} p^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK_i$$

and

$$(1 + p^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK_i)^p = 1\,[n] \quad \forall\ 1 \leq i \leq d.$$

If we set $x_i = 1 + p^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_d^{\alpha_d} AK_i$, then x_i is an initial p-th root of the first class associated to p_i and we can easily verify that $x = \prod_{i=1}^{d} x_i$.■

Definition 2.9: Let x and y be two p-th roots of unity modulo n, we say y is a final conjugate root of x if $x.y - 1$ is divisible by none of the p_i, $i \in \{1, .., d\}$, that means $x.y$ is a final p-th root modulo n.

Proposition 2.10: Any p-th root of unity modulo n have a final conjugate.

Proof :
Let x be a p-th root of unity modulo n, if $x \in \mathbf{G}_p^0(n)$ or x is a final p-th root then we have the expected result. When $d = 1$, a final p-th root is an initial p-th root associated to p_1 and therefore any root that not belongs to $\mathbf{G}_p^0(n)$ are finals. Assume that $d \geq 2$ and $x - 1$ is divisible by a nonempty subfamily of p_i of cardinality $t < d$ and for a permutation, we can assume them $p_1, p_2, \ldots$ and p_t. Thus

$$x - 1 = p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_t^{\alpha_t} AK$$

with K an integer not divisible by p_i, $i \in \{t+1, .., d\}$. For all $i \in \{1, .., t\}$, let x_i be an initial p-th associated to p_i therefore

$$x_i = 1 + p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK_i$$

with K_i not divided by p_i, whereof

$$\begin{aligned} \prod_{i=1}^{t} x_i &= \prod_{i=1}^{t} (1 + p^{\alpha-1} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_t^{\alpha_t} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AK_i) \\ &= 1 + p^{\alpha-1} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} A \sum_{i=1}^{t} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_t^{\alpha_t} K_i + K'n \end{aligned}$$

but $\sum_{i=1}^{t} p_1^{\alpha_1} p_2^{\alpha_2} \ldots \overset{\vee}{p_i^{\alpha_i}} \ldots p_t^{\alpha_t} K_i$ is divisible by none of the p_i, $i \in \{1, .., t\}$. Consequently $y = \prod_{i=1}^{t} x_i$ is a root which verify $y = 1 + p^{\alpha-1} p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AM$ with M an integer that not divided by p_i, $i \in \{1, .., t\}$. Thereby

$$x.y = 1 + p^{\alpha-1} A(p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AM + p_1^{\alpha_1} \ldots p_t^{\alpha_t} AK)$$

It is clear that $(p_{t+1}^{\alpha_{t+1}} \ldots p_d^{\alpha_d} AM + p_1^{\alpha_1} \ldots p_t^{\alpha_t} AK)$ is divisible by none of the p_i, $i \in \{1, .., d\}$, hence the result.■

Corollary 2.7: Every p-th root of unity is a product of a first class initial p-th roots by a class zero's p-th root.

Proof :
Let x be a p-th root modulo n, if x is final then we can write it as a product of a final p-th root of unity of the first class by a class zero's p-th root and from the previous results this final p-th root of the first class is product of d initial p-th roots of the first class associated respectively to $p_1, p_2, ..$ and p_d, hence the result. Now let us assume that x is not a final p-th root so there exists x_1,x_2, .. and x_t initial p-th roots such that $x_1x_2..x_t$ is a final conjugate of x, then $xx_1x_2..x_t$ is a final p-th root, and we have :

$$xx_1x_2..x_t = y_1y_2..y_dy_0$$

with y_i is an initial p-th root of the first class associated to p_i and y_0 is a class zero's p-th root.
From *Proposition 2.8* any initial p-th root associated to p_i can be written uniquely as a product of an initial first class p-th root associated to p_i by class zero's p-th root. Thereby $x_i = \overset{+}{x}_i\ z_i$, with $\overset{+}{x}_i \in \overset{+}{\mathbf{G}}{}_p^{p_1}(n)$ and $z_i \in \mathbf{G}_p^0(n)$. So

$$x = y_1y_2..y_d(\overset{+}{x}_1\overset{+}{x}_2\ ...\ \overset{+}{x}_t)^{-1}(z_1z_2...z_t)^{-1}y_0$$

and as $\overset{+}{\mathbf{G}}{}_p^{p_1}(n)$ and $\mathbf{G}_p^0(n)$ are groups, then we obtain the result.■

Remark :
The previous result shows that $\mathbf{G}_p(n)$ is generated by the initial p-th roots of the first class and the class zero's p-th roots and as $\mathbf{G}_p^0(n)$ and $\overset{+}{\mathbf{G}}{}_p^{p_1}(n)$ are cyclic groups, then

$$\mathbf{G}_p(n) =< x_1, x_2, \dots, x_d, x_0 >$$

with x_i is an initial p-th root of the first class associated to p_i and x_0 is a p-th root of the class zero distinct from 1. More generally, we have the following result :

Theorem 2.9: The map

$$\varphi \quad : \quad \overset{+}{\mathbf{G}}{}_p^{p_1}(n) \times \overset{+}{\mathbf{G}}{}_p^{p_2}(n) \dots \times \overset{+}{\mathbf{G}}{}_p^{p_m}(n) \times \mathbf{G}_p^0(n) \longrightarrow \mathbf{G}_p(n)$$
$$(x_1, x_2, \dots, x_m, y) \longmapsto x_1.x_2, \dots x_m.y$$

is an isomorphism of groups.

Proof :
It is clear that φ is a surjective morphism of groups and we show that it is injective as in the analogous previous results.■

Corollary 2.8:

$$Card(\mathbf{G}_p(n)) = p^{\alpha_p(n)+1}.$$

Remark :
From the previous theorem we have

$$\mathbf{G}_p(n) = \{ \prod_{(i_1,i_2,..,i_d,i)\in I^{d+1}} x_1^{i_1}x_2^{i_2}\dots x_d^{i_d}x_0^{i}\}$$

with $I = \{1, 2, .., p\}$, x_i is one generator of the cyclic group $\mathbf{G}_p^{p_i}(n)$ for $i \neq 0$ and x_0 is a p-th root of the first class different from 1.

We now give an algorithm in $MAPLE$ that allows us to find a generating set of $\mathbf{G}_p(n)$. For the computing of x_0 it suffices to take $x_0 = 1 + n/p$ and for the others x_i, we proceed as above.

```
Gene_p := proc(n, p)   local LB, LD, i, LFact, GEN, P;
LD := [ ]; LB := [ ]; GEN := [ ];
GEN := [op(GEN), 1 + n/p];
LFact := ifactors(n)[2];
for i from 1 to nops(LFact) do
if (LFact[i][1] − 1 mod p = 0) then
LD := [op(LD), LFact[i]];
end :
end :
for i from 1 to nops(LD) do
P := convert(Berlekamp(x^p − 1, x) mod LD[i][1], list);
if(P[1] − x + 1 mod LD[i][1] <> 0) then
LB := Bezout(LD[i][1], n/(LD[i][1]^LD[i][2]), P[1] − x +
1);
GEN := [op(GEN), ((LD[i][1] ∗ LB[1] − (P[1] −
x) mod n))&^(LD[i][1]^(LD[i][2] − 1)) mod n];
else
LB := Bezout(LD[i][1], n/(LD[i][1]^LD[i][2]), P[2] − x +
1);
GEN := [op(GEN), (LD[i][1] ∗ LB[1] − (P[2] −
x) mod n)&^(LD[i][1]^(LD[i][2] − 1)) mod n];
end :
end :
if(GEN = [ ]) then
GEN := [1];
end;
eval(GEN);
end :
```

Algorithm 2.3

III. Conclusion

For the cardinality of $\mathbf{G}_p(n)$, we can summarize it in the following theorem :

Theorem 3.1: Let $n \geq 3$ be an integer and p be a prime odd number which does not divide n, then :
- $Card(\mathbf{G}_p(n)) = p^{\alpha_p(n)}$
- $Card(\mathbf{G}_p(pn)) = p^{\alpha_p(n)}$
- $Card(\mathbf{G}_p(p^\alpha n)) = p^{\alpha_p(n)+1}$ with $\alpha \geq 2$

We will now give an algorithm which help us to find, from a fixed integer n, a generating set of $\mathbf{G}_p(n)$.

```
Gene_p := proc(n, p)   local LB, LD, i, LFact, GEN, P;
LD := [ ]; LB := [ ]; GEN := [ ];
if (n mod p^2 = 0) then
GEN := [op(GEN), 1 + n/p];
LFact := ifactors(n)[2];
for i from 1 to nops(LFact) do
if (LFact[i][1] − 1 mod p = 0) then
```

$LD := [op(LD), LFact[i]];$
$end :$
$end :$
$for\ i\ from\ 1\ to\ nops(LD)\ do$
$P := convert(Berlekamp(x\hat{}p-1, x)\ mod\ LD[i][1], list);$
$if(P[1]-x+1\ mod\ LD[i][1] <> 0)\ then$
$LB := Bezout(LD[i][1], n/(LD[i][1]\hat{}LD[i][2]), P[1]-x+1);$
$GEN := [op(GEN), ((LD[i][1]*LB[1]-(P[1]-x)\,mod\,n))\&\hat{}(LD[i][1]\hat{}(LD[i][2]-1))\,mod\,n];$
$else$
$LB := Bezout(LD[i][1], n/(LD[i][1]\hat{}LD[i][2]), P[2]-x+1);$
$GEN := [op(GEN), (LD[i][1]*LB[1]-(P[2]-x)\,mod\,n)\&\hat{}(LD[i][1]\hat{}(LD[i][2]-1))\,mod\,n];$
$end :$
$end :$
$else$
$LFact := ifactors(n)[2];$
$for\ i\ from\ 1\ to\ nops(LFact)\ do$
$if\ (LFact[i][1]-1\ mod\ p=0)\ then$
$LD := [op(LD), LFact[i]];$
$end :$
$end :$
$for\ i\ from\ 1\ to\ nops(LD)\ do$
$P := convert(Berlekamp(x\hat{}p-1, x)\ mod\ LD[i][1], list);$
$if(P[1]-x+1\ mod\ LD[i][1] <> 0)\ then$
$LB := Bezout(LD[i][1], n/(LD[i][1]\hat{}LD[i][2]), P[1]-x+1);$
$GEN := [op(GEN), ((LD[i][1]*LB[1]-(P[1]-x)\,mod\,n))\&\hat{}(LD[i][1]\hat{}(LD[i][2]-1))\,mod\,n];$
$else$
$LB := Bezout(LD[i][1], n/(LD[i][1]\hat{}LD[i][2]), P[2]-x+1);$
$GEN := [op(GEN), (LD[i][1]*LB[1]-(P[2]-x)\,mod\,n)\&\hat{}(LD[i][1]\hat{}(LD[i][2]-1))\,mod\,n];$
$end :$
$end :$
$end :$
$if(GEN = [\,])\ then$
$GEN := [1];$
$end;$
$eval(GEN);$
$end :$

Algorithm 2.4

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