

# Gorenstein Projective, Injective and Flat Modules Relative to Semidualizing Modules

Jianmin Xing, Rufeng Xing

**Abstract**—In this paper we study some properties of  $G_C$ -projective, injective and flat modules, where  $C$  is a semidualizing module and we discuss some connections between  $G_C$ -projective, injective and flat modules, and we consider these properties under change of rings such that completions of rings, Morita equivalences and the localizations.

**Keywords**—Semidualizing module,  $C$ -projective(injective,flat),  $G_C$ -projective (injective,flat), Commutative ring; Localizations.

## I. INTRODUCTION

UNLESS stated otherwise, throughout this paper all rings are associative with identity and all modules are unitary modules. Let  $R$  be a ring, we denote by  $R\text{-Mod}(\text{Mod-}R)$  the category of left(right)  $R$ -modules respectively. For any  $R$ -module  $M$ , we denote by  $\text{pd}_R(M)$ ,  $\text{id}_R(M)$  and  $\text{fd}_R(M)$  the projective dimension, injective dimension and flat dimension of  $M$  respectively, and denote by  $M^+ = \text{Hom}_Z(M, Q/Z)$  the characteristic module of  $M$ . For unexplained concepts and notations, we refer the reader to [2, 3, 8].

When  $R$  is two-sided Noetherian, Auslander and Bridger [1] introduced the  $G$ -dimension,  $G\text{-dim}_R(M)$  for every finitely generated  $R$ -module  $M$ . Several decades later, Enochs and Jenda [6,7] extended the ideas of Auslander and Bridger and introduced the Gorenstein projective, injective and flat dimensions. The homological properties of the Gorenstein projective dimension and some generalized versions of such a dimension have been studied by many authors, see [4,5, 6, 7, 9, 10, 11]. Foxby [9] and Golod [10] independently initiated the study of semidualizing modules (under different names). Examples include the rank 1 free module and a dualizing (canonical) module, when one exists. Golod [10] used these to define  $G_C$ -dimension, a refinement of projective dimension, for finitely generated modules. The  $G_C$ -dimension of a finitely generated  $R$ -module  $M$  is the length of the shortest resolution of  $M$  by so-called totally  $C$ -reflexive modules.

White introduced in [20] the  $G_C$ -projective modules and gave a functorial descriptions of the  $G_C$ -projective dimension of modules with respect to a semidualizing module over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [20]. Over a commutative Noetherian ring, the  $G_C$ -projective modules and the  $G_C$ -projective dimension were

called  $C$ -Gorenstein projective modules and the  $C$ -Gorenstein projective dimension in [12], respectively. Note that the non-commutative versions of almost all the results in [20] also hold true.

In this paper, based on the results mentioned above, we further investigate the properties of the  $G_C$ -projective, injective and flat modules over general rings. and investigate the relation among them. At last, we study these properties under change of rings such that completions of rings, Morita equivalences and the localizations.

This paper is organized as follows.

In Section II, we give some definitions and basic properties of  $G_C$ -projective, injective and flat modules such that semidualizing module,  $C$ -projective, injective and flat module.

In Section III, we study the relation and the properties among  $G_C$ -projective, injective and flat modules.

In Section IV, we consider the properties under change of rings. Specially, we consider the completions of rings, Morita equivalences and the localizations.

## II. PRELIMINARY NOTES

In this section we give some definitions of  $G_C$ -projective, injective flat modules, and some known results about them. At first we introduce the semidualizing module and  $C$ -projective, injective and flat modules which are defined as follows:

**Definition 1** <sup>[13]</sup> Let  $R, S$  be rings, an  $(S, R)$ -bimodule  $C = {}_S C_R$  is semidualizing if the following conditions are satisfied.

(a1)  ${}_S C$  admits a degreewise finite  $S$ -projective resolution.  
 (a2)  $C_R$  admits a degreewise finite  $R^{op}$ -projective resolution.

(b1) The homothety map  ${}_S S_S \rightarrow \text{Hom}_{R^{op}}(C, C)$  is an isomorphism.

(b2) The homothety map  ${}_R R_R \rightarrow \text{Hom}_S(C, C)$  is an isomorphism.

(c1)  $\text{Ext}_S^i(C, C) = 0$  for any  $i \geq 1$ .

(c2)  $\text{Ext}_R^i(C, C) = 0$  for any  $i \geq 1$ .

Unless otherwise stated, when  $R = S$  is commutative, all semidualizing bimodules in this paper are symmetric in the sense that the two  $R$ -actions on  $C$  agree. In this case we will use the terminology " $C$  is semidualizing over  $R$ ". Note that when  $R = S$  is commutative and noetherian, Definition 1 agrees with the established terminology; that is a finitely generated  $R$ -module  $C$  is semidualizing if the natural homothety map  $R \rightarrow \text{Hom}_R(C, C)$  is an isomorphism and

Jianmin Xing is with the College of Mathematics and Physics, Qingdao University of Science and Technology, Qingdao, 266061, P.R.China (e-mail: xingjianmin1978@163.com).

Rufeng Xing is with the College of Physical and Environmental Oceanography, Ocean University of China, Qingdao, 266100, P.R.China (e-mail: 100season@sina.com).

$\text{Ext}_R^i(C, C) = 0$  for any  $i \geq 1$ . Two examples are the free module of rank 1, and over a Cohen-Macaulay local ring, the dualizing (canonical) module, when it exists. It is easy to see that a semidualizing module is finitely generated and even finitely presented.

**Definition 2**<sup>[13]</sup> Let  $R$  be a ring,  ${}_R C_R$  be a semidualizing bimodule, a module in  $R\text{-Mod}$  is called  $C$ -projective if it has the form  $C \otimes_R P$  for some projective module  $P \in R\text{-Mod}$ . A module in  $R\text{-Mod}$  is called  $C$ -injective if it has the form  $\text{Hom}_R(C, I)$  for some injective module  $I \in R\text{-Mod}$ . A module in  $R\text{-Mod}$  is called  $C$ -flat if it has the form  $C \otimes_R F$  for some flat module  $F \in R\text{-Mod}$ . Set

$$\begin{aligned}\mathcal{P}_C(R) &= \{C \otimes_R P \mid P \text{ is projective}\}, \\ \mathcal{I}_C(R) &= \{\text{Hom}_R(C, I) \mid I \text{ is injective}\} \text{ and} \\ \mathcal{F}_C(R) &= \{C \otimes_R F \mid F \text{ is flat}\}\end{aligned}$$

Let  $M \in R\text{-Mod}$ . We denote  $\text{Add}_R M$  (resp.  $\text{Prod}_R M$ ) the subclass of  $R\text{-Mod}$  consisting of all modules isomorphic to direct summands of direct sums (resp. direct products) of copies of  $M$ . The following result was proved in [15, Proposition 2.4.].

**Lemma 1**<sup>[15]</sup>

(1)  $\mathcal{P}_C(R) = \text{Add}_R C$ ;  
(2)  $\mathcal{I}_C(R) = \text{Prod}_R C^+$ , where  $C^+ = \text{Hom}_R(C, E)$  with  ${}_R E$  a injective cogenerator for  $\text{Mod-}R$ .

The following notions were introduced by Holm in [12] and White in [20] for commutative rings. The non-commutative versions of them were given in [15]. Now we give the commutative versions of them.

**Definition 3**<sup>[15]</sup>

(1) A complete  $\mathcal{PP}_C$ -resolution is a  $\text{Hom}_R(-, \text{Add}_R C)$  exact exact sequence:

$$\mathcal{X} = \cdots \rightarrow P_0 \rightarrow C \otimes_R P^0 \rightarrow C \otimes_R P^1 \rightarrow \cdots \quad (1)$$

in  $R\text{-Mod}$  with all  $P^i$  and  $P_i$  projective. A module  $M \in R\text{-Mod}$  is called  $G_C$ -projective if there exists a complete  $\mathcal{PP}_C$ -resolution as in (1) with  $M = \text{coker}(P_1 \rightarrow P_0)$ . Set

$\mathcal{GP}_C(R) =$  the class of  $G_C$ -projective modules in  $R\text{-Mod}$ .

(2) A complete  $\mathcal{I}_C \mathcal{I}$ -resolution is a  $\text{Hom}_R(\text{Prod } C^+, -)$  exact exact sequence:

$$\mathcal{Y} = \cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \quad (2)$$

in  $\text{Mod-}R$  with all  $I^i$  injective and  $I_i \in \text{Prod}_R C^+$ . A module  $M \in \text{Mod-}R$  is called  $G_C$ -injective if there exists a complete  $\mathcal{I}_C \mathcal{I}$ -resolution as in (2) with  $M = \text{Im}(I_0 \rightarrow I^0)$ . Set

$\mathcal{GI}_C(R) =$  the class of  $G_C$ -injective modules in  $\text{Mod-}R$ .

(3) A complete  $\mathcal{FF}_C$ -resolution is a  $\mathcal{I}_C(R) \otimes -$  exact exact sequence:

$$\mathcal{X} = \cdots \rightarrow F_0 \rightarrow C \otimes_R F^0 \rightarrow C \otimes_R F^1 \rightarrow \cdots \quad (3)$$

in  $R\text{-Mod}$  with all  $F^i$  and  $F_i$  flat. A module  $M \in R\text{-Mod}$  is called  $G_C$ -flat if there exists a complete  $\mathcal{FF}_C$ -resolution as in (3) with  $M = \text{coker}(F_1 \rightarrow P_0)$ . Set

$\mathcal{GF}_C(R) =$  the class of  $G_C$ -flat modules in  $R\text{-Mod}$ .

It is trivial that in case  ${}_R C_R = {}_R R_R$ , the  $G_C$ -projective (injective and flat) modules are just the usual Gorenstein projective (injective and flat) modules, respectively.

Using the definitions, we immediately get the following results.

**Proposition 1**

(1) If  $(M_i)_{i \in I}$  is a family of  $G_C$ -projective modules, then  $\bigoplus M_i$  is  $G_C$ -projective.

(2) If  $(F_i)_{i \in I}$  is a family of  $G_C$ -flat modules, then  $\bigoplus F_i$  is  $G_C$ -flat.

### III. THE $G_C$ PROPERTY

In this section we always assume that  $R$  is a commutative ring without special instruction and  $C$  is a semidualizing  $R$ -bimodule, then we study the properties and relationship among  $G_C$ -projective, injective and flat modules.

**Lemma 2** Let  $R$  be a commutative left coherent ring. Then

(1)  $M$  is an  $C$ -flat left  $R$ -module if and only if  $M^+$  is an  $C$ -injective right  $R$ -module.

(2)  $M$  is an  $C$ -injective  $R$ -module if and only if  $M^+$  is an  $C$ -flat right  $R$ -module.

**Proof** By virtue of the conclusion in Lemma 4.1 of [19].

Now we give the relation of  $G_C$ -flat modules and  $G_C$ -injective modules. The first conclusion is the Lemma 5.2 in [19].

**Proposition 2**<sup>[19]</sup> Let  $C$  be a semidualizing  $R$ -module. If  $M$  is an  $R$ -module, then  $M$  is in  $\mathcal{GF}_C(R)$  if and only if its characteristic module  $M^+$  is in  $\mathcal{GI}_C(R)$ .

**Proposition 3** Let  $R$  be a commutative artinian ring, if  $M$  is a  $G_C$ -injective left  $R$ -module, then  $M^+$  is a  $G_C$ -flat right  $R$ -module.

**Proof** There exists a complete  $\mathcal{I}_C \mathcal{I}$ -exact sequence

$$\cdots \rightarrow \text{Hom}_R(C, I_1) \rightarrow \text{Hom}_R(C, I_0) \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

with  $I_i, I^i$  injective for  $i \geq 0$  and

$$M = \text{coker}(\text{Hom}_R(C, I_1) \rightarrow \text{Hom}_R(C, I_0))$$

. Let  $J$  be any injective left  $R$ -module. Then  $J = \bigoplus_{\Lambda} J_{\alpha}$ , where  $J_{\alpha}$  is an injective envelope of some simple left  $R$ -module for any  $\alpha \in \Lambda$  by [14, Theorem 6.6.4], and hence  $\text{Tor}_1^R(M^+, C \otimes J) = \bigoplus_{\Lambda} \text{Tor}_1^R(M^+, C \otimes J_{\alpha}) = \bigoplus_{\Lambda} \text{Ext}_1^R(C \otimes J_{\alpha}, M) = 0$  by [8, Theorem 3.2.13] for all  $i \geq 1$ . Therefore  $M^+$  is a  $G_C$ -flat right  $R$ -module.

**Proposition 4** Let  $R$  be a commutative ring and  $Q$  a projective  $R$ -module. If  $M$  is an  $G_C$ -projective left  $R$ -module, then  $M \otimes_R Q$  is a  $G_C$ -projective left  $R$ -module.

**Proof** There is an exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow C \otimes_R P^0 \rightarrow C \otimes_R P^1 \rightarrow \cdots$$

with  $P_i, P^i$  projective and  $M = \text{coker}(P_1 \rightarrow P_0)$ . Then the sequence

$$\cdots \rightarrow P_1 \otimes Q \rightarrow P_0 \otimes Q \rightarrow C \otimes_R P^0 \otimes Q \rightarrow C \otimes_R P^1 \otimes Q \rightarrow \cdots$$

is exact with  $P_i \otimes Q, P^i \otimes Q$  projective by [18, Ch. 2, 1 Theorem 3]. Let  $Q'$  be any projective left  $R$ -module. Then  $\text{Ext}_R^1(M \otimes_R Q, C \otimes Q') = \text{Hom}_R(Q, \text{Ext}_R^1(M, C \otimes Q')) =$

0 by [17, p. 258, 9.20] for all  $i \geq 1$ . Hence  $M \otimes_R Q$  is a  $G_C$ -projective  $R$ -module.

**Proposition 5** Let  $R$  be a commutative ring and  $F$  a flat left  $R$ -module. If  $M$  is an  $G_C$ -flat left  $R$ -module, then  $M \otimes_R F$  is a  $G_C$ -flat left  $R$ -module.

**Proof** There is an exact sequence

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow C \otimes_R F^0 \rightarrow C \otimes_R F^1 \rightarrow \cdots$$

with  $F_i, F^i$  flat and  $M = \text{coker}(F_1 \rightarrow F_0)$ . Then the sequence

$$\cdots \rightarrow F_1 \otimes F \rightarrow F_0 \otimes F \rightarrow C \otimes_R F^0 \otimes F \rightarrow C \otimes_R F^1 \otimes F \rightarrow \cdots$$

is exact with  $F_i \otimes F, F^i \otimes F$  flat by [18, Ch. 2, 1 Theorem 3]. Let  $I$  be any injective  $R$ -module and  $\mathcal{F}$  be a flat resolution of  $\text{Hom}(C, I)$ . Then

$$\begin{aligned} & \text{Tor}_1^R(M \otimes_R F, \text{Hom}(C, I)) \\ &= H_i((M \otimes_R F) \otimes \mathcal{F}) \simeq H_i(M \otimes_R (F \otimes \mathcal{F})) \\ &\simeq \text{Tor}_1^R(M, F \otimes_R \text{Hom}(C, I)) = 0 \end{aligned}$$

by [17, p. 258, 9.20] for all  $i \geq 1$ , since  $F \otimes_R \text{Hom}(C, I) \simeq \text{Hom}(C, F \otimes_R I)$  is a  $C$ -injective module by [8, Theorem 3.2.16] and [13, Lemma 1.12]. Hence  $M \otimes_R F$  is a  $G_C$ -flat  $R$ -module.

**Theorem 1** If  $R$  is commutative right coherent and  $C$  is faithfully semidualizing  $R$ -bimodule, then the class  $\mathcal{GF}_C(R)$  of  $G_C$ -flat  $R$ -modules is projectively resolving and closed under direct summands.

Furthermore, if  $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots$  is a sequence of  $G_C$ -flat  $R$ -modules, then the direct limit  $\varinjlim M_n$  is again  $G_C$ -flat.

**Proof** Using the dual of Theorem 2.8 in [20] and together with the Proposition 2 above, we see that  $\mathcal{GF}_C(R)$  is projectively resolving. Now, comparing Proposition 2.5 with Proposition 1.4 in [11], we get that  $\mathcal{GF}_C(R)$  is closed under direct summands.

Concerning the last statement, we know that if  $R$  is coherent and  $C$  is faithfully semidualizing, then the class  $\mathcal{F}_C(R)$  is preenveloping on the category of  $R$ -modules by [13, Proposition 5.10.]. So we pick for each  $n$  a co-proper right  $\mathcal{F}_C$ -resolution  $F_n$  of  $M_n$ , as illustrated in the next diagram.

$$\begin{array}{ccccccc} F_0 & : & 0 & \rightarrow & M_0 & \rightarrow & C \otimes F_0^0 \rightarrow C \otimes F_0^1 \rightarrow \cdots \\ & & \vdots & & \downarrow & & \downarrow \\ F_1 & : & 0 & \rightarrow & M_1 & \rightarrow & C \otimes F_1^0 \rightarrow C \otimes F_1^1 \rightarrow \cdots \\ & & \vdots & & \downarrow & & \downarrow \\ & & \vdots & & \vdots & & \vdots \end{array}$$

By Proposition 1.8 in [11], each map  $M_n \rightarrow M_{n+1}$  can be lifted to a chain map  $F_n \rightarrow F_{n+1}$  of complexes. Since we are dealing with sequences (and not arbitrary direct systems), each column above is again a direct system. Thus it makes sense to apply the exact functor  $\varinjlim$  to the upon exact sequences, and doing so, we obtain an exact complex,

$$F = \varinjlim F_n = 0 \rightarrow \varinjlim M_n \rightarrow C \otimes \varinjlim F_n^0 \rightarrow \cdots$$

where each module  $C \otimes F^k = C \otimes \varinjlim F_n^k, k = 0, 1, 2, \dots$  is  $C$ -flat. When  $I$  is injective right  $R$ -module, then  $\text{Hom}_R(C, I) \otimes_R F_n$  is exact since

$$C \otimes F = C \otimes \text{Hom}_Z(I, Q/Z) \simeq \text{Hom}_Z(\text{Hom}_R(C, I), Q/Z)$$

is a  $C$ -flat (left)  $R$ -module, while the first isomorphism comes from that  $R$  is coherent and the second isomorphism holds by [13, Lemma 1.14], we get exactness of  $\text{Hom}_R(F_n, C \otimes F) = \text{Hom}_R(F_n, \text{Hom}_Z(\text{Hom}_R(C, I), Q/Z)) = \text{Hom}_Z(\text{Hom}_R(C, I) \otimes_R F_n, Q/Z)$  and hence of  $\text{Hom}_R(C, I) \otimes_R F_n$ , since  $Q/Z$  is a faithfully injective  $Z$ -module. Since  $\varinjlim$  commutes with the homology functor, we also get exactness of  $\text{Hom}_R(C, I) \otimes_R F = \varinjlim (\text{Hom}_R(C, I) \otimes_R F_n)$ . Thus we have constructed the "right half",  $F$ , of a complete  $\mathcal{F}_C$ -resolution for  $\varinjlim M_n$ .

Since  $M_n$  is  $G_C$ -flat, we also have

$$\text{Tor}_i^R(\text{Hom}_R(C, I), \varinjlim M) \simeq \varinjlim \text{Tor}_i^R(\text{Hom}_R(C, I), M) = 0$$

for  $i > 0$ , and all injective right modules  $I$ . Thus  $\varinjlim M_n$  is  $G_C$ -flat.

**Proposition 6** Let  $R$  be commutative artinian and  $C$  is semidualizing  $R$ -module. Then the class  $\mathcal{GF}_C(R)$  of all  $G_C$ -flat right  $R$ -modules is closed under arbitrary direct products.

**Proof** Let  $M = \prod_{i \in I} M_i$ , and  $M_i \in \mathcal{GF}_C(R)$  for all  $i \geq 1$ . There exists an exact sequence

$$0 \rightarrow M_i \rightarrow C \otimes F_i^0 \rightarrow C \otimes F_i^1 \rightarrow C \otimes F_i^2 \rightarrow \cdots$$

for  $i \geq 1$  and  $F_i^j$  flat for  $j \geq 0$ . Then

$$0 \rightarrow \prod_{i \in I} M_i \rightarrow \prod_{i \in I} C \otimes F_i^0 \rightarrow \prod_{i \in I} C \otimes F_i^1 \rightarrow \cdots$$

is exact, where  $\prod_{i \in I} F_i^j$  is a flat right  $R$ -modules and  $\prod_{i \in I} C \otimes F_i^j \simeq C \otimes \prod_{i \in I} F_i^j$ , since  $C$  is finitely presented and  $R$  is left artinian. Let  $E$  be any injective left  $R$ -module. Then  $E = \bigoplus_{\alpha} E_{\alpha}$ , where  $E_{\alpha}$  is an injective envelope of some simple left  $R$ -module for any  $\alpha \in \Lambda$  by [18, Theorem 6.6.4]. Thus

$$\begin{aligned} & \text{Tor}_n^R(\prod_{i \in I} M_i, \text{Hom}_R(C, E)) \\ &= \bigoplus_{\alpha} \text{Tor}_n^R(\prod_{i \in I} M_i, \text{Hom}_R(C, E_{\alpha})) \\ &= \bigoplus_{\alpha} \prod_{i \in I} \text{Tor}_n^R(M_i, \text{Hom}_R(C, E_{\alpha})) = 0 \end{aligned}$$

by [8, Theorem 3.2.26] for all  $n \geq 1$ . Therefore  $M$  is an  $G_C$ -flat right  $R$ -module.

#### IV. CHANGE OF RINGS

Let  $(R, m)$  be a commutative local noetherian ring with residue field  $k$  and let  $E(k)$  be the injective envelope of  $k$ .  $\hat{R}$  and  $\hat{M}$  will denote the  $m$ -adic completion of a ring  $R$  and an  $R$ -module  $M$ , and  $M^v$  will denote the Matlis dual  $\text{Hom}_R(M, E(k))$ .

**Lemma 3**<sup>[13]</sup> Let  $Q \rightarrow R$  be a flat ring homomorphism between commutative rings. If  $E$  is semidualizing over  $Q$ , then  $E \otimes_Q R$  is semidualizing over  $R$ .

**Corollary 1** (1) Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $C$  is a semidualizing  $R$ -module, then  $C[x]$  is a semidualizing  $R[x]$ -module and  $S^{-1}C$  is a semidualizing  $S^{-1}R$ -module.

$$\cdots \rightarrow \mathrm{Hom}(\hat{C}, \bar{I}_1) \longrightarrow \mathrm{Hom}(\hat{C}, \bar{I}_0) \rightarrow \mathrm{Hom}_R(\hat{R}, M) \rightarrow 0$$

**Lemma 4**<sup>[17]</sup> Let  $A$  be an commutative noetherian with subset  $S$ , and  $A, B$  be  $R$ -modules with  $A$  finitely generated.

There are isomorphisms, for all  $n \geq 0$ ,

$$\text{Ext}_{S^{-1}R}^n(S^{-1}A, S^{-1}B) \cong S^{-1}\text{Ext}_R^n(A, B)$$

**Lemma 5**<sup>[21]</sup> Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $S^{-1}R$  is a projective  $R$ -module, then  $\bar{A}$  is a projective  $R$ -module if and only if  $\bar{A}$  is a projective  $S^{-1}R$ -module for any  $\bar{A} \in S^{-1}R\text{-Mod}$ .

For convenience, nextly we note the  $G_{S^{-1}C}$ -projective(injective,flat)  $S^{-1}R$ -module by the  $G_C$ -projective(injective,flat)  $S^{-1}R$ -module.

**Proposition 10** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ , then

(1) If  $S^{-1}R$  is a projective  $R$ -module and  $A$  is a finitely generated  $G_C$ -projective  $R$ -module, then  $S^{-1}A$  is an  $G_C$ -projective  $S^{-1}R$ -module;

(2) If  $S^{-1}R$  is a faithfully flat  $R$ -module and  $\bar{B}$  is a finitely generated  $G_C$ -projective  $R$ -module if and only if  $\bar{B}$  is an finitely generated  $G_C$ -projective  $S^{-1}R$ -module for any  $\bar{B} \in S^{-1}R\text{-mod}$ .

**Proof** (1) There exists a complete  $\mathcal{PP}_C$ -exact sequence  $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow C \otimes P^0 \rightarrow C \otimes P^1 \rightarrow \cdots$  with  $P_i, P^i$  projective for  $i \geq 0$  and  $M = \text{coker}(P_1 \rightarrow P_0)$ . Then there exists a complete  $\mathcal{PP}_C$  exact sequence  $\cdots \rightarrow S^{-1}P_1 \rightarrow S^{-1}P_0 \rightarrow S^{-1}C \otimes S^{-1}P^0 \rightarrow S^{-1}C \otimes S^{-1}P^1 \rightarrow \cdots$  in  $S^{-1}R\text{-Mod}$ , with  $S^{-1}A = \text{coker}(S^{-1}P_1 \rightarrow S^{-1}P_0)$  and  $S^{-1}P_i, S^{-1}P^i$  are projective  $S^{-1}R$ -module for  $i \geq 0$ . Let  $\bar{Q}$  be any projective  $S^{-1}R$ -module, then  $\bar{Q}$  is projective  $R$ -module by Lemma 5. Since  $A$  is an  $G_C$ -projective, we have  $\text{Ext}_R^i(A, C \otimes \bar{Q}) = 0$  for all  $i \geq 1$ . So  $\text{Ext}_{S^{-1}R}^i(S^{-1}A, S^{-1}C \otimes \bar{Q}) \simeq \text{Ext}_{S^{-1}R}^i(S^{-1}A, S^{-1}C \otimes S^{-1}\bar{Q}) \simeq S^{-1}\text{Ext}_R^i(A, C \otimes \bar{Q}) = 0$  by Lemma 4 for all  $i \geq 1$ . Hence  $S^{-1}A$  is an  $G_C$ -projective  $S^{-1}R$ -module.

(2) ( $\Rightarrow$ ) By (1), Since  $B \simeq S^{-1}B$  by [16.Prop.5.17].

( $\Leftarrow$ ) There exists a complete  $\mathcal{PP}_C$ -exact sequence  $\cdots \rightarrow \bar{P}_1 \rightarrow \bar{P}_0 \rightarrow S^{-1}C \otimes \bar{P}^0 \rightarrow S^{-1}C \otimes \bar{P}^1 \rightarrow \cdots$  in  $S^{-1}R\text{-Mod}$  with  $\bar{P}_i, \bar{P}^i$  projective for  $i \geq 0$  and  $\bar{B} = \text{coker}(\bar{P}_1 \rightarrow \bar{P}_0)$  then  $\bar{P}_i, \bar{P}^i$  are projective  $R$ -module by Lemma 5. Let  $Q$  be any projective  $R$ -module, then  $S^{-1}Q$  is a projective  $S^{-1}R$ -module. So  $S^{-1}\text{Ext}_R^i(\bar{B}, C \otimes Q) \simeq \text{Ext}_{S^{-1}R}^i(\bar{B}, S^{-1}C \otimes S^{-1}Q) = 0$ , then  $\text{Ext}_R^i(\bar{B}, C \otimes Q) = 0$  since  $S^{-1}R$  is faithfully flat  $R$ -module. Therefore  $\bar{B}$  is an  $G_C$ -projective  $R$ -module.

**Lemma 6** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$  and  $S^{-1}R$  be a finitely generated projective  $R$ -module, If  $I$  is a injective  $R$ -module, then  $\text{Hom}_R(S^{-1}R, \text{Hom}(C, I))$  is a  $C$ -injective  $S^{-1}R$ -module.

**Proof** Since  $S^{-1}R$  be a finitely generated projective  $R$ -module, then

$$\begin{aligned} & \text{Hom}_R(S^{-1}R, \text{Hom}(C, I)) \\ & \simeq S^{-1}\text{Hom}_R(S^{-1}R, \text{Hom}_R(C, I)) \\ & \simeq \text{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}\text{Hom}_R(C, I)) \\ & \simeq \text{Hom}_{S^{-1}R}(S^{-1}R, \text{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I)) \\ & \simeq \text{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I). \end{aligned}$$

Since  $S^{-1}I$  is injective  $S^{-1}R$ -modules, hence  $\text{Hom}_R(S^{-1}R, \text{Hom}(C, I))$  is a  $C$ -injective  $S^{-1}R$ -module.

**Proposition 11** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $S^{-1}R$  is a finitely generated projective  $R$ -module and  $C$  is a finitely generated semidualizing  $R$ -module, then

(1) If  $A$  is an  $G_C$ -injective  $R$ -module, then  $\text{Hom}_R(S^{-1}R, A)$  is an  $G_C$ -injective  $S^{-1}R$ -module;

(2) For any  $B \in R\text{-Mod}$ ,  $\text{Hom}_R(S^{-1}R, B)$  is an  $G_C$ -injective  $R$ -module if and only if  $\text{Hom}_R(S^{-1}R, B)$  is an  $G_C$ -injective  $S^{-1}R$ -module.

**Proof**(1) There exists a complete  $\mathcal{IC}\mathcal{I}$ -exact sequence  $\cdots \rightarrow \text{Hom}(C, I_1) \rightarrow \text{Hom}(C, I_0) \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$  with  $I_i, I^i$  injective for  $i \geq 0$  and

$$A = \text{coker}(\text{Hom}(C, I_1) \rightarrow \text{Hom}(C, I_0)).$$

Then there exists a complete  $\mathcal{IC}\mathcal{I}$ -exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Hom}_R(S^{-1}R, \text{Hom}(C, I_1)) & \rightarrow \text{Hom}_R(S^{-1}R, \text{Hom}(C, I_0)) \\ & \rightarrow \text{Hom}_R(S^{-1}R, I^0) \rightarrow \text{Hom}_R(S^{-1}R, I^1) \rightarrow \cdots \end{aligned}$$

in  $S^{-1}R\text{-Mod}$ , with

$$\begin{aligned} \text{Hom}_R(S^{-1}R, A) &= \text{coker}(\text{Hom}_R(S^{-1}R, \text{Hom}(C, I_1)) \\ & \rightarrow \text{Hom}_R(S^{-1}R, \text{Hom}(C, I_0))) \end{aligned}$$

and  $\text{Hom}_R(S^{-1}R, I_i)$  is an injective  $S^{-1}R$ -module for  $i \geq 0$  by [8.Theorem 3.2.9] and  $\text{Hom}_R(S^{-1}R, \text{Hom}(C, I_i))$  is an  $C$ -injective  $S^{-1}R$ -module by Lemma 6. Let  $\bar{I}$  be any injective  $S^{-1}R$ -module, then  $\bar{I}$  is an injective  $R$ -module by [4. Lemma 1.2]. So

$$\begin{aligned} & \text{Ext}_{S^{-1}R}^i(\text{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}), \text{Hom}_R(S^{-1}R, A)) \\ & \simeq \text{Ext}_R^i(\text{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}), A) \\ & \simeq \text{Ext}_R^i(\text{Hom}_R(C, \text{Hom}_{S^{-1}R}(S^{-1}R, \bar{I})), A) = 0 \end{aligned}$$

by [14.P.258.9.21] for all  $i \geq 1$ , and Hence  $\text{Hom}_R(S^{-1}R, A)$  is an  $G_C$ -injective  $S^{-1}R$ -module.

(2)( $\Rightarrow$ ) is obvious.

( $\Leftarrow$ ) There is an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_1) & \rightarrow \text{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_0) \\ & \rightarrow \bar{I}^0 \rightarrow \bar{I}^1 \rightarrow \cdots \end{aligned}$$

in  $S^{-1}R\text{-Mod}$  with  $\bar{I}_i$  and  $\bar{I}^i$  injective for all  $i \geq 0$  and  $\text{Hom}_R(S^{-1}R, B) = \ker(\bar{I}^0 \rightarrow \bar{I}^1)$ . Then  $\bar{I}_i$  and  $\bar{I}^i$  are injective  $R$ -module. But we have

$$\begin{aligned} \text{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_i) & \simeq \text{Hom}_R(C, \text{Hom}_{S^{-1}R}(S^{-1}R, \bar{I}_i)) \\ & \simeq \text{Hom}_R(C, \bar{I}_i) \end{aligned}$$

Hence  $\text{Hom}_{S^{-1}R}(S^{-1}C, \bar{I}_i)$  is a  $C$ -injective  $R$ -module. Let  $I$  be any injective  $R$ -module, then  $S^{-1}I$  is an injective  $S^{-1}R$ -module. So

$$\begin{aligned} & \text{Ext}_R^i(\text{Hom}_R(C, I), \text{Hom}_R(S^{-1}R, B)) \\ & \simeq \text{Ext}_R^i(\text{Hom}_R(C, I), \text{Hom}_{S^{-1}R}(S^{-1}R, \text{Hom}_R(S^{-1}R, B))) \\ & \simeq \text{Ext}_{S^{-1}R}^i(S^{-1}\text{Hom}_R(C, I), \text{Hom}_R(S^{-1}R, B)) \\ & \simeq \text{Ext}_{S^{-1}R}^i(\text{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I), \text{Hom}_R(S^{-1}R, B)) = 0 \end{aligned}$$

by [17.P.107. Theorem 3.84] and [14.P.258.9.21] for all  $i \geq 1$ . Hence  $\text{Hom}_R(S^{-1}R, B)$  is an  $G_C$ -injective  $R$ -module.

**Proposition 12** Let  $R$  be a commutative ring and  $S$  a multiplicatively closed set of  $R$ . If  $C$  be a semidualizing  $R$ -module and  $S^{-1}R$  is a finitely generated projective  $R$ -module, then

(1) If  $A$  is a  $G_C$ -flat  $R$ -module, then  $S^{-1}A$  is a  $G_C$ -flat  $R$ -module;

(2) If  $A$  is an  $G_C$ -flat  $R$ -module, then  $S^{-1}A$  is an  $G_C$ -flat  $S^{-1}R$ -module;

(3) For any  $\bar{B} \in S^{-1}R\text{-Mod}$ ,  $\bar{B}$  is an  $G_C$ -flat  $R$ -module if and only if  $\bar{B}$  is an  $G_C$ -flat  $S^{-1}R$ -module.

**Proof** (1) There is an exact sequence  $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow C \otimes_R F^0 \rightarrow C \otimes_R F^1 \rightarrow \cdots$  in  $R\text{-Mod}$  where  $F_i$  and  $F^i$  are flat for  $i \geq 0$  and  $A = \text{coker}(F_1 \rightarrow F_0)$ . Then  $\cdots \rightarrow S^{-1}F_1 \rightarrow S^{-1}F_0 \rightarrow S^{-1}C \otimes_R S^{-1}F^0 \rightarrow S^{-1}C \otimes_R S^{-1}F^1 \rightarrow \cdots$  is exact and  $S^{-1}F_i$  and  $S^{-1}F^i$  are flat  $S^{-1}R$ -module for all  $i \geq 0$ . Hence  $S^{-1}F_i$  and  $S^{-1}F^i$  are flat  $R$ -module for all  $i \geq 0$ . Since  $S^{-1}R$  is a finitely generated projective  $R$ -module,  $S^{-1}R \oplus P \simeq R^{(n)}$ , where  $P$  is projective. So  $(S^{-1}C \otimes_R S^{-1}F^i) \oplus (P \otimes C \otimes_R S^{-1}F^i) \simeq (S^{-1}R \otimes C \otimes_R S^{-1}F^i) \oplus (P \otimes C \otimes_R S^{-1}F^i) \simeq R^{(n)} \otimes C \otimes_R S^{-1}F^i \simeq C \otimes_R (S^{-1}F^i)^{(n)}$ , hence  $S^{-1}C \otimes_R S^{-1}F^i$  is  $C$ -flat  $R$ -module for all  $i \geq 0$  by [13, Proposition 5.5.]. Let  $I$  be any injective  $R$ -module. Then  $\text{Tor}_i^R(\text{Hom}_R(C, I), S^{-1}A) \simeq \text{Tor}_i^R(S^{-1}\text{Hom}_R(C, I), A) = 0$  by [16, Prop. 5.17], since  $S^{-1}\text{Hom}_R(C, I) \simeq \text{Hom}_{S^{-1}R}(S^{-1}C, S^{-1}I) \simeq \text{Hom}_R(C, S^{-1}I)$  by [13, Lemma 1.2] and so  $S^{-1}\text{Hom}_R(C, I)$  is  $C$ -injective  $R$ -module. Hence  $S^{-1}A$  is an  $G_C$ -flat  $R$ -module.

(2) There is an exact sequence  $\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow C \otimes_R F^0 \rightarrow C \otimes_R F^1 \rightarrow \cdots$  in  $R\text{-Mod}$  where  $F_i$  and  $F^i$  are flat for  $i \geq 0$  and  $A = \text{coker}(F_1 \rightarrow F_0)$ . Then  $\cdots \rightarrow S^{-1}F_1 \rightarrow S^{-1}F_0 \rightarrow S^{-1}C \otimes_R S^{-1}F^0 \rightarrow S^{-1}C \otimes_R S^{-1}F^1 \rightarrow \cdots$  is exact and  $S^{-1}F_i$  and  $S^{-1}F^i$  are flat  $S^{-1}R$ -module for all  $i \geq 0$ . Let  $\bar{I}$  be any injective  $S^{-1}R$ -module. Then  $\bar{I}$  be any injective  $R$ -module by [3, Lemma 1.2]. So

$$\begin{aligned} & \text{Tor}_i^{S^{-1}R}(\text{Hom}(S^{-1}C, \bar{I}), S^{-1}A) \\ & \simeq \text{Tor}_i^{S^{-1}R}(S^{-1}\text{Hom}(C, \bar{I}), S^{-1}A) \\ & \simeq S^{-1}\text{Tor}_i^R(\text{Hom}(C, \bar{I}), A) = 0 \end{aligned}$$

for all  $i \geq 1$ . Hence  $S^{-1}A$  is an  $G_C$ -flat  $S^{-1}R$ -module.

(3)  $(\Rightarrow)$  by (2)

$\Leftarrow$  There is an exact sequence  $\cdots \rightarrow \bar{F}_1 \rightarrow \bar{F}_0 \rightarrow S^{-1}C \otimes_R \bar{F}^0 \rightarrow S^{-1}C \otimes_R \bar{F}^1 \rightarrow \cdots$  in  $S^{-1}R\text{-Mod}$  where  $\bar{F}_i$  and  $\bar{F}^i$  are flat for  $i \geq 0$  and  $\bar{B} = \text{coker}(\bar{F}_1 \rightarrow \bar{F}_0)$ . Then  $\bar{F}_i$  and  $\bar{F}^i$  are flat  $R$ -module and  $S^{-1}C \otimes_R \bar{F}^1$  is a  $C$ -flat  $R$ -module by the proof of (1). Let  $I$  be any injective  $R$ -module. Then

$$\begin{aligned} & S^{-1}\text{Tor}_i^R(\text{Hom}(C, I), \bar{B}) \\ & \simeq \text{Tor}_i^{S^{-1}R}(S^{-1}\text{Hom}(C, I), \bar{B}) \\ & \simeq \text{Tor}_i^{S^{-1}R}(\text{Hom}(S^{-1}C, S^{-1}I), \bar{B}) = 0 \end{aligned}$$

So  $\bar{B}$  is an  $G_C$ -flat  $R$ -module.

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#### REFERENCES

- [1] M. Auslander and M. Bridger, *Stable Module Theory*, Amer. Math. Soc, Providence, Rhode Island, (1969).
- [2] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, 2nd edition, in: Graduate Texts Mathematics, vol.13, Springer-Verlag, New York, 1992.
- [3] H. Bass, Injective dimension in Noetherian rings, *Trans. Amer. Math. Soc.* 102(1962)18-29.
- [4] L.W. Christensen, *Gorenstein dimensions*, Lecture Notes in Math., Springer-Verlag, Berlin, 2000.
- [5] L.W. Christensen, H.B. Foxby, H. Holm, Beyond Totally Reflexive Modules and Back. A Survey on Gorenstein Dimensions, to appear in *Commutative Algebra: Noetherian and non-Noetherian perspectives*, Springer-Verlag. Available from arXiv:0812.3807v2.
- [6] E.E. Enochs, O.M.G. Jenda, Gorenstein injective and projective modules, *Math. Z.* 220(1995)611-633.
- [7] E.E. Enochs, O.M.G. Jenda, On Gorenstein injective modules, *Comm. Algebra*, 21 (1993)3489-3501.
- [8] E.E. Enochs, O.M.G. Jenda, *Relative homological algebra*, Walter de Gruyter, Berlin-New York, 2000.
- [9] Foxby, Gorenstein modules and related modules, *Math. Scand.* 31(1972)267-284.
- [10] E. S. Golod, G-dimension and generalized perfect ideals, *Trudy Mat. Inst. Steklov.* 165(1984) 62-66.
- [11] H. Holm, Gorenstein homological dimensions, *J. Pure Appl. Algebra* 189 (2004)167-193.
- [12] H. Holm and P. Jorgensen, Semidualizing modules and related Gorenstein homological dimensions, *J. Pure Appl. Algebra*, 205(2006)423-445.
- [13] H. Holm and D. White, Foxby equivalence over associative rings, *J. Math. Kyoto Univ.* 47(2007)781-808.
- [14] F. Kasch, *Modules and Rings*, London Math. Soc. Monogr., 17, Academic Press, London, 1981.
- [15] Z. Liu, Z. Huang, A. Xu, Gorenstein Projective Dimension Relative to a Semidualizing Bimodule. <http://math.nju.edu.cn/~huangzy/paper>.
- [16] M.S. Osborne, *Basic Homological Algebra*, Grad. Texts in Math., Springer-Verlag, New York, Berlin, 2003.
- [17] J.J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.
- [18] W.T. Tong, *An Introduction to Homological Algebra*, Higher Education, 1996 (in Chinese).
- [19] Sean Sather-Wagstaff, Tirdad Sharif, Diana White, AB-Contexts and Stability for Gorenstein Flat Modules with Respect to Semidualizing Modules, *Algebra Represent Theor*, 14(2011)403-428.
- [20] D. White, Gorenstein projective dimension with respect to a semidualizing module, *J. Comm. Algebra* 2 (2010)111-137.
- [21] X. Yang and Z. Liu, Strongly Gorenstein projective, injective and flat modules, *J. Algebra*, 320(2008)2659-2674.