International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:7, No:11, 2013

Geometrically Non-Linear Free Vibration Analysis of Functionally Graded Rectangular Plates

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Abstract**—**In the present study, the problem of geometrically non-linear free vibrations of functionally graded rectangular plates (FGRP) is studied. The theoretical model, previously developed and based on Hamilton's principle, is adapted here to determine the fundamental non-linear mode shape of these plates. Frequency parameters, displacements and stress are given for various power-law distributions of the volume fractions of the constituents and various aspect ratios. Good agreement with previous published results is obtained in the case of linear and non-linear analyses.

Keywords**—**Non-linear vibration- functionally graded materials-rectangular plates.

I. Introduction

FUNCTIONALLY graded materials (FGM) are a new generation of composites presented first in 1990's by a group of Japanese scientists [1], [2]. The accomplishment of functionally graded material is the realization of contemporary and distinct functions that cannot be achieved by the traditional composite materials. These are advanced composite materials with a microscopically inhomogeneous anatomy and are usually made from mixtures of ceramic and metal using powder metallurgy techniques. Continuous changes in their microstructure distinguish FGM from other traditional composite materials. The material property of the FGM can be tailored to obtain the specific demand in different engineering applications in order to exploit the advantage of the properties of individual constituent. This is possible, because the material composition changes gradually in a preferred direction.

The advantage of using this material is that it eliminates the interface problem due to smooth and continuous change of material properties from one surface to other [1], [2].

Large amplitude free flexural vibration (LAFFV) behavior of a plate arises in many engineering applications, particularly in aircraft structure substantially deviated panels.

In order to solve plate problems, the choice of the plate theory and the type of solution method are necessary. Through the study of comprehensive survey of literature, it is found that many researchers have been carried out on free vibration of the FG plates. Free vibration of simply supported and clamped rectangular thin FG plates was presented by Abrate [4] based on the classical plate theory. In another work, Abrate [5] also investigated free vibration, buckling and static deflections of different shapes of FG, such as square, circular and skew plates with several boundary conditions on the basis of the classical plate theory, first order shear deformation theory and third order shear deformation theory. Zhao et al. [6] have presented a free vibration analysis for FG square and skew plates with different boundary conditions using the element-free k_p-Ritz method on the basis of the first-order shear deformation theory (FSDT).

The objective of this work is the calculation of the non-linear frequencies associated to the fundamental non-linear mode shape of fully clamped FGM rectangular plates. A theoretical model, based on Hamilton's principle, was developed in a general form, which reduces the large vibration amplitude problem to a set of non-linear algebraic equations solved numerically for each value of the amplitude of vibration.

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II. Functionally Graded Materials

In this study, the properties of the plate are assumed to vary through the thickness of the plate with a power-law distribution of the volume fractions of the two materials in between the two surfaces. The top surface (z = h/2) of the plate is assumed ceramic-rich whereas the bottom surface (z = -h/2) is metal-rich. Poisson's ratio ν is assumed to be constant, equal to 0.3 throughout the analysis. In such a way, an arbitrary material property P (Young's modulus E, and mass density ρ) of the functionally graded plate is assumed to vary through the thickness of the plate, as a function of the volume fraction and properties of the constituent material as

$$P = P_t V_c + P_b V_m \tag{1}$$

In which the subscripts m and c refer to the metallic and ceramic constituents, respectively. P_t and P_b properties of the top and bottom surfaces of the plate. V_C and V_m are the ceramic and metal volume fractions are related by

$$V_C + V_m = 1 \tag{2}$$

For a plate with a uniform thickness h and a reference plane at its middle plane the volume fraction can be written as

$$V_j(z) = \left(\frac{2z+h}{2h}\right)^n \tag{3}$$

where n is the power-law exponent, $0 \leq n \leq \infty$. For a functionally graded solid with two constituent materials, the property variation P can be expressed as

$$P(z) = (P_C + P_m)V_C(z) + P_m \tag{4}$$

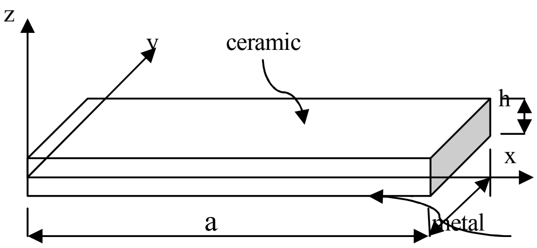

Fig. 1 Geometry of a FGM rectangular plate

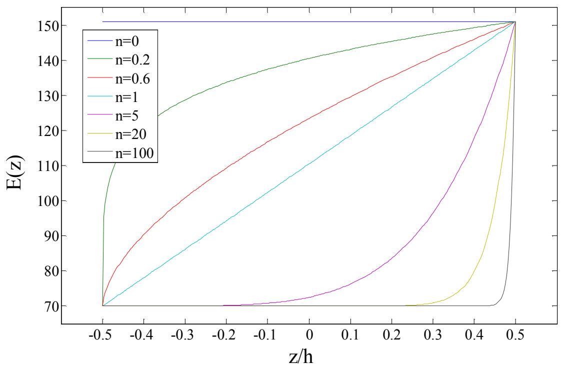

Fig. 2 Variation of Young's modulus through the dimensionless thickness

In what follows the metal and ceramic are respectively Aluminum and Zirconia (ZrO2), there material properties are given in numerical results.

Fig. 2 shows the Variation of Young's modulus $E(z)$ through the dimensionless thickness z/h with different values of volume fraction index n. If $n = 0$ then the plate reduces to a pure ceramic plate. As the volume fraction index n increases, the ceramic volume fraction decreases. It is seen that in a same z/h, by increasing n the Young's modulus decreases.

III. General Mathematical Model

Consider a fully clamped thin rectangular plate as shown in Fig. 1. The expression for the total strain energy V_t and the kinetic energy T are given by:

$$V_t = V_a + V_b + V_c \tag{5}$$

with:

$$V_a = \frac{A_{11}}{8}\int_S \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^2 ds \tag{6}$$

$$V_b = \frac{D_{11}}{2}\int_S \left[\begin{array}{l}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2v\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \\ + 2(1+v)\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\end{array}\right] ds \tag{7}$$

$$V_c = -\frac{B_{11}}{2}\int_S \left[\begin{array}{l}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \\ v\left[\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] \\ + 2(1-v)\frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{array}\right] ds \tag{8}$$

and

$$T = \frac{I_0}{2}\int_S \left(\frac{\partial w}{\partial t}\right)^2 ds \tag{9}$$

In which w is the deflection function and S the plate area. The terms involving the in plane displacement u and v and their derivatives have been omitted in this study.

A_{11}, B_{11} and D_{11} are respectively in-plane, bending-stretching coupling and bending stiffness and are given by:

$$(A_{11}, B_{11}, C_{11}) = \int_{-h/2}^{h/2} (1, z, z^2)\frac{E(z)}{1-v^2} dz \tag{10}$$

and the unit area density of the plate is:

$$I_0 = \int_{-h/2}^{h/2} \rho(z) dz \tag{11}$$

Then, Hamilton's principle is formulated as follow:

$$\partial \int_0^{\omega/\pi} (V_t - T) dt = 0 \tag{12}$$

The Von Karman's non linear strain-displacement relations for a rectangular plate are:

$$\varepsilon_x^m = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \tag{13}$$

$$\varepsilon_x^m = \frac{\partial u}{\partial y} + \frac{1}{2}\left(\frac{\partial u}{\partial y}\right)^2 \tag{14}$$

$$\gamma_x^m = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \tag{15}$$

$$\varepsilon_x^f = -z\frac{\partial^2 w}{\partial x^2} \tag{16}$$

$$\varepsilon_{yx}^f = -z\frac{\partial^2 w}{\partial y^2} \tag{17}$$

$$\gamma_{xy}^f = -z\frac{\partial^2 w}{\partial x \partial y} \tag{18}$$

Assuming that the time and space functions are separable and that harmonic motion takes place, the transverse displacement function w is expanded as a series of n basic functions:

$$w = a_i w_i(x, y)\sin \omega t \qquad i = 1,.........,n \tag{19}$$

Replacing the transverse displacement function w by the basic functions (19), the expressions for the flexural strain energy V_b, the axial strain energy V_a, the coupling strain energy V_c, and the kinetic energy T can be written respectively as:

$$V_a = \frac{1}{8} a_i a_j a_k a_l b_{ijkl} \sin^4 \omega t \tag{20}$$

$$V_b = \frac{1}{8} a_i a_j K_{ij} \sin^4 \omega t \tag{21}$$

$$V_C = \frac{1}{2} a_i a_j a_k C_{ijk} \sin^4 \omega t \tag{22}$$

$$T = \frac{1}{2} \omega^2 a_i a_j m_{ij} \cos^2 \omega t \tag{23}$$

Discrediting the total strain energy V_t and the kinetic energy T, replacing in (12), and integrating the time functions, the following set of non-linear algebraic equations is written in a non-dimensional form as:

$$3a_i a_j a_k b^{*s}_{ijkr} + 2a_i k^{*s}_{ir} + 2a_i a_j c^{*s}_{ijr} - 2a_i m^*_{ir} = 0 \qquad r = 1,........,n \tag{24}$$

where k^*_{ij} , b^*_{ijkl}, c^*_{ijk} and m^*_{ij} are non-dimensional generalised parameters given by:

$$k^{*s}_{ij} = \left(k^*_{ij} + k^*_{ji}\right)/2 \tag{25}$$

$$c^{*s}_{ijk} = \left(c^*_{ijk} + c^*_{kij} + c^*_{ikj}\right)/3 \tag{26}$$

$$b^{*s}_{ijkl} = b^*_{ijkl} \tag{27}$$

where W^*: non-dimensional transverse displacement , $x^* = {}^{x}/_{a}$ and $y^* = {}^{y}/_{b}$ are non-dimensional co-ordinates and λ=b/a is the aspect ratio.

$$m^*_{ij} = \int_{s^*} w^*_i w^*_j dx^* dy^* \tag{28}$$

$$k^*_{ij} = \int_{s^*} \begin{pmatrix} \lambda^4 \frac{\partial^2 w^*_i}{\partial x^{*2}} \frac{\partial^2 w^*_j}{\partial x^{*2}} + \frac{\partial^2 w^*_i}{\partial y^{*2}} \frac{\partial^2 w^*_j}{\partial y^{*2}} \\ +2(1+v)\lambda^2 \frac{\partial^2 w^*_i}{\partial x \partial y} \frac{\partial^2 w^*_j}{\partial x \partial y} \\ +2v\lambda^2 \frac{\partial^2 w^*_i}{\partial x^{*2}} \frac{\partial^2 w^*_j}{\partial y^{*2}} \end{pmatrix} dx^* dy^* \tag{29}$$

$$c^*_{ijk} = -\frac{4hB_{11}}{\pi D_{11}} \int_{S^*} \begin{bmatrix} \lambda^4 \frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial x^*} \frac{\partial^2 w^*_k}{\partial x^{*2}} \\ + \frac{\partial w^*_i}{\partial y^*} \frac{\partial w^*_j}{\partial y^*} \frac{\partial^2 w^*_k}{\partial y^{*2}} \\ +2(1-v) \frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial y^*} \frac{\partial^2 w^*_k}{\partial x \partial y} \\ 2v\lambda^2 \begin{pmatrix} \frac{\partial w^*_i}{\partial y^*} \frac{\partial w^*_j}{\partial y^*} \frac{\partial^2 w^*_k}{\partial x^{*2}} \\ + \frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial x^*} \frac{\partial^2 w^*_k}{\partial y^{*2}} \end{pmatrix} \end{bmatrix} dx^* dy^* \tag{30}$$

$$b^*_{ijkl} = \frac{h^2 A_{11}}{4D_{11}} \int_{s} \begin{bmatrix} \left(\lambda^2 \frac{\partial w^*_i}{\partial x^*} \frac{\partial w^*_j}{\partial x^*} + \frac{\partial w^*_i}{\partial y^*} \frac{\partial w^*_i}{\partial y^*} \right) \\ \left(\lambda^2 \frac{\partial w^*_k}{\partial x^*} \frac{\partial w^*_l}{\partial x^*} + \frac{\partial w^*_k}{\partial y^*} \frac{\partial w^*_l}{\partial y^*} \right) \end{bmatrix} dx^* dy^* \tag{31}$$

IV. Numerical Results and Discussions

In the problem considered here, the top surface of the plate is Zirconia (ZrO2): (E_c=151 GPa, υ_c=0.3, ρ_c=3000 Kg/m^3) whereas the bottom surface of the plate is Aluminum Zirconia (ZrO2): (E_m=70 GPa, υ_m=0.3, ρ_m=2707 Kg/m^3).

TABLE I
FREQUENCY RATIO Ω^*_{NL}/Ω^*_L OF A CLAMPED SQUARE ISOTROPIC PLATE

W^*_{Max}	ω^*_{nl}/ω^*_l	
	Present Work	[3]
0.2	1.0072	1.0105
0.4	1.0376	1.0411
0.6	1.0856	1.0894
0.8	1.1478	1.1521
1	1.2207	1.2260
1.2	1.3017	1.3084
1.4	1.3887	1.3970
1.5	1.4339	1.4431

In Table I, the first nonlinear frequency ratios ω^*_{nl}/ω^*_l calculated in the present work at various vibration amplitudes, is compared with the results published in [3]. It is noted that the results are in very good accordance for different values of the dimensionless amplitude, since the maximum deviation doesn't exceed 0.63% for central amplitude W^*_{max}=1.5.

TABLE II
FUNDAMENTAL NONLINEAR FREQUENCY Ω_{NL} OF SQUARE FGM PLATE WITH CCCC BOUNDARY CONDITION

n	W^{*}_{Max}	ω_{nl} Present Work	ω_{nl} [7]
0	0.2	50.477	50.220
	0.4	51.263	50.677
	0.6	52.525	51.448
	0.8	54.212	52.456
	1	56.273	53.700
	1.2	58.673	55.102
0.5	0.2	45.910	45.196
	0.4	46.630	45.669
	0.6	47.799	46.378
	0.8	49.351	47.322
	1	51.2595	48.503
	1.2	53.4665	49.763
2	0.2	42.956	41.701
	0.4	43.570	42.015
	0.6	44.573	42.644
	0.8	45.925	43.586
	1	47.587	44.528
	1.2	49.523	45.785
1000	0.2	36.242	36.117
	0.4	36.806	36.424
	0.6	37.701	36.966
	0.8	38.918	37.66
	1	40.395	38.521
	1.2	42.115	39.534

In Table II, the fundamental nonlinear Frequency ω_{nl} of square FGM for all clamped calculated in the present work at various vibration amplitudes, is compared with the results published in [7]. It is noted that the results are in very good accordance for different values of the dimensionless amplitude, since the maximum deviation doesn't exceed 8% for central amplitude W^{*}_{max}=1.2.

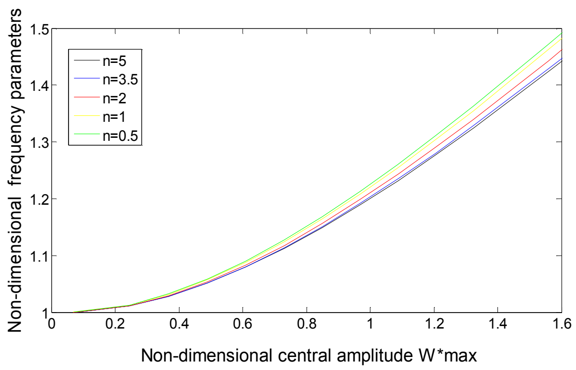

Fig. 2 Comparison of the change frequency of the first mode for n=0.5, 1, 2, 3.5 and 5 for a square FGM plate

Fig. 2 shows the effect of volume fraction index n on the frequency ratio (ω^{*}_{nl} / ω^{*}_{l}) of a CCCC square Aluminum and Zirconia oxide (ZrO2) FGM plate. It is observed that the non-dimensional non- linear frequency ratio decreases with the increase in the volume fraction index n. This decrease in frequency ratio is on expected lines, because increase in the volume fraction index n means that a plate has a smaller ceramic component. It can be additionally viewed that the frequency ratio enhances with the rise in amplitude ratio.

TABLE III
NOMENCLATURE

S	Quantity	U Units SI
a, b	Length, width of plate	m
h	Plate thickness,	m
E	Young's modulus	N/ m^2
K^{*}_{ij}, m	Terms for non-dimensional rigidity tensor and mass respectively	
b^{*}_{ijkl}	non-dimensional non-linearity tensor	
C^{*}_{ijK}	non-dimensional coupling tensor	
S, S^{*}	The dimensional and non-dimensional surfaces	m
T	Kinetic energy	Nm
u, v	In-plane displacements at (x,y)	m/s
$w(x,y$	Transverse displacement at (x,y)	m/s
V_a, V_b, l	Axial bending and coupling energy	Nm
$w^{*}(x,y$	Non-dimensional transverse displacement at point (x,y)	
w^{*}_{max}	Maximum of the non-dimensional transverse displacement	
x, y, z	Position co-ordinates	m
λ	Non dimensional parameter (aspect ratio)	
ν	Poisson ratio	
ρ	Mass per unit volume	kg /m^3
ω^{*}	Non dimensional frequency parameters	

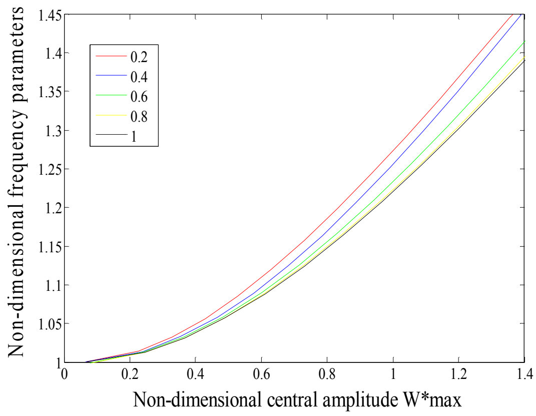

Fig. 3 Comparison of the change frequency of the first mode for λ=0.2, 0.4, 0.6, 0.8 and 1for (n=1)

Fig. 3 shows the effect of aspect ratio index λ on the frequency ratio (ω^{*}_{nl} /ω^{*}_{l}) of a CCCC rectangular FGM plates square, for volume fraction index (n=1). It is observed from the Fig. 3 that for a given central amplitude, the non-dimensional non- linear frequency ratio increases when the aspect ratio decreases.

V. CONCLUSION

The present study deals with the problem of geometrically nonlinear free vibrations of FG fully clamped rectangular plate. The theoretical model previously developed and based on Hamilton's principle and spectral analysis, is adapted here to determination of the non-linear natural frequencies associated to the fundamental non-linear mode shape of these plates. Frequency parameters for various central amplitudes are given for various volume index and various aspect ratios. A very good agreement is obtained in the case of isotropic plates with the published results.

A good agreement is obtained In the case of FGM square plate plates with the published results in [7], however increasing discrepancy when the amplitude grows, which is mainly due to the fact that the axial displacements have been neglected in the expression of the axial strain energy.

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