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Geometric Operators in Decision Making with Minimization of Regret

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***Abstract*—**We study different types of aggregation operators and the decision making process with minimization of regret. We analyze the original work developed by Savage and the recent work developed by Yager that generalizes the MMR method creating a parameterized family of minimal regret methods by using the ordered weighted averaging (OWA) operator. We suggest a new method that uses different types of geometric operators such as the weighted geometric mean or the ordered weighted geometric operator (OWG) to generalize the MMR method obtaining a new parameterized family of minimal regret methods. The main result obtained in this method is that it allows to aggregate negative numbers in the OWG operator. Finally, we give an illustrative example.

***Keywords*—**Decision making, Regret, Aggregation operators, OWA operator, OWG operator.

I. Introduction

THE Ordered Weighted Geometric (OWG) operator was introduced by Chiclana et al. [1] and it provides a parameterized family of aggregation operators similar to the Ordered Weighted Averaging (OWA) operator introduced by Yager [2]. Since their appearance, a lot of new extensions have been developed about them. For the OWA operator, we could mention [3] – [18] and for the OWG operator [19] – [29]. Basically, the OWG operator uses in the same aggregation the OWA operator and the geometric mean.

Among the great variety of extensions developed for the OWA and the OWG operator, this work will focus on an article published recently by Yager [15] consisting in introduce the OWA aggregation in decision making with minimization of regret. The first methods for decision making with minimization of regret were introduced by Savage [30], [31] and they consisted in use the paradigm of minimization of maximal regret (MMR). These methods have been generalized by Yager in [15] with the introduction of the OWA operators in the paradigm of MMR creating a parameterized family of minimal regret methods. In this paper, we propose a method that uses the OWG operator for generalize the MMR method obtaining another parameterized family of minimal regret methods.

In order to do so, this paper is organized as follows. In Section II, we briefly comment the OWA operator, the geometric mean and the OWG operator. In Section III, we summarize the main concepts of the traditional MMR method and the generalization developed by Yager. In Section IV, we suggest a new generalization of the MMR method using the OWG operator in the aggregation step. Finally, in Section V, we give an illustrative example in order to see numerically the results obtained with the new approach.

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II. Aggregation Operators

A. OWA Operator

The OWA operator was introduced in [2] and it provides a parameterized family of aggregation operators which have been used in a wide range of applications [3] – [18]. In the following, we provide a definition of the OWA operator as introduced by Yager [2].

Definition 1. An OWA operator of dimension n is a mapping $OWA: R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is one and $w_j \in [0, 1]$, then:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \quad (1)$$

where b_j is the jth largest of the a_i.

From a more generalized perspective of the reordering step, we have to distinguish between the Descending OWA (DOWA) operator and the Ascending OWA (AOWA) operator. The weights of these operators are related by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the OWA and w^*_{n-j+1} the jth weight of the AOWA operator. Note that the AOWA operator is the dual of the DOWA operator as it is explained in [8].

The OWA operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotone, bounded and idempotent. It can also be demonstrated that the OWA operator has as special cases the maximum, the minimum and the average criteria [2].

Another issue to consider is the two measures introduced by Yager [2] for characterizing the weighting vector and the type of aggregation it performs. The first measure, the attitudinal character, is defined as:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} \right) \quad (2)$$

It can be shown that $\alpha \in [0, 1]$. The more of the weight located toward the bottom of W, the closer α to 0 and the more of the weight located near the top of W, the closer α to 1. Note that for the minimum $\alpha(W) = 0$, for the maximum $\alpha(W) = 1$, and for the average criteria $\alpha(W) = 0.5$.

The second measure introduced also in [2], is called the entropy of dispersion of W and it is used to provide a measure of the information being used. It is defined as:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \quad (3)$$

That is, if $w_j = 1/n$ for all j, then $H(W) = \ln n$, and the amount of information used is maximum. If $w_j = 1$ for some j, known as step-OWA [9], then $H(W) = 0$, and the least amount of information is used.

Note that it is also possible to study these measures with the AOWA operator. The main difference is that the reordering step used in the analysis is ascendant.

A third measure that could be used for the analysis of the weighting vector W is what Yager called the balance operator [12]. It is useful to analyse the balance between favouring the arguments with high values or the arguments with low values. It can be defined as follows.

$$BAL(W) = \sum_{j=1}^{n} \left(\frac{n+1-2j}{n-1} \right) w_j \quad (4)$$

It can be shown that $BAL(W) \in [-1, 1]$. Note that for the maximum we get $BAL(W) = 1$, for the minimum, $BAL(W) = -1$ and for the average criteria, $BAL(W) = 0$. Also note that for the median and the olympic average, $BAL(W) = 0$. For the Arrow-Hurwicz aggregation, assuming that the usual aggregation of this method is $\lambda \text{Max}\{a_i\} + (1 - \lambda)\text{Min}\{a_i\}$, $BAL(W) = 2\lambda - 1$. As it can be shown, for an optimistic situation, where $\lambda > 0.5$, the balance is positive and for a pessimistic situation, where $\lambda < 0.5$, the balance is negative.

If we analyse the balance in the AOWA operator, we can use a similar formulation.

$$BAL(W) = \sum_{j=1}^{n} \left(\frac{2j-1-n}{n-1} \right) w_j \quad (5)$$

In this case, we also get that $BAL(W) \in [-1, 1]$. We also obtain the same results about the special cases such as the maximum with $BAL(W) = 1$, the minimum with $BAL(W) = -1$, the average criteria, the median and the olympic average with $BAL(W) = 0$, and the Arrow-Hurwicz aggregation with $BAL(W) = 2\lambda - 1$.

B. Geometric Mean

The geometric mean is a traditional aggregation operator which has been used for different applications such as in [32], [33]. It can be defined as follows:

Definition 2. A geometric mean operator of dimension n is a mapping $GM: R^{+n} \to R^{+}$, such that:

$$GM(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} (a_i)^{\frac{1}{n}} \quad (6)$$

where R^{+} is the set of positive real numbers. The geometric mean is commutative, monotonic, bounded and idempotent.

If we consider that the arguments of the geometric mean are not equally important, then, we can use the weighted geometric mean in the aggregation. The weighted geometric mean is a generalization of the geometric mean as it can include it as a special case of the formulation. It can be defined as follows.

Definition 3. A weighted geometric mean is a mapping $WGM: R^n \to R$ that has an associated weighting vector W of dimension n such that the sum of the weights is one and $w_i \in [0, 1]$, then:

$$WGM(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} a_i^{w_i} \quad (7)$$

Note that the weighted geometric mean becomes the geometric mean when $w_i = 1/n$ for all i.

C. OWG Operator

The OWG operator was introduced in [1] and it provides a family of aggregation operators similar to the OWA operator as it includes the maximum and the minimum among its special cases. It consists in combine the OWA operator with the geometric mean. In the following, we provide a definition of the OWG operator as introduced by Xu and Da [27].

Definition 4. An OWG operator of dimension n is a mapping $OWG: R^{+n} \to R^{+}$ that has an associated weighting vector W of dimension n such that the sum of the weights is one and $w_j \in [0, 1]$, then:

$$OWG(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} b_j^{w_j} \quad (8)$$

where b_j is the jth largest of the a_i, and R^{+} is the set of positive real numbers.

From a more generalized perspective of the reordering step in the OWG operator, we have to distinguish between the Descending OWG (DOWG) operator and the Ascending OWG (AOWG) operator [27]. The weights of these operators are related by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the OWG and w^*_{n-j+1} the jth weight of the AOWG operator. Note that the AOWG operator is the dual of the DOWG operator.

As it is seen in [1], the OWG operator is commutative, monotonic, bounded and idempotent.

By choosing a different manifestation of the weighting vector, we are able to obtain different types of aggregation operators [1]. For example, we get the maximum when $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, the minimum when $w_n = 1$ and $w_j = 0$ for all $j \neq n$, and the geometric mean when $w_j = 1/n$ for all j. Other examples of aggregations with OWG operators can be seen in [27].

Other types of aggregations that could be obtained with the OWG operator are the weighted geometric median and the E-Z OWG weights. For the weighted geometric median, we will use a similar approach than the one used by Yager in [10] for the weighted OWA median. The difference with the median is that in this case, we consider the weights associated with the arguments. Then, instead of looking for the argument with the *(n/2)*th ordered position, we will look for the ordered position where the sum of the weights is 0.5. That is, we will select the argument $OWG(a_1,\ldots, a_n) = b_k$ where b_k is the kth largest argument of the a_i such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k-1$ is less than 0.5. Note that when $j = i$, for all i and j, where j is the jth argument of b_j and i is the ith argument of a_i, it is found the weighted geometric median for the weighted geometric mean.

For the E-Z OWG weights based on the E-Z OWA weights [14], we could distinguish between two classes. In the first class, which has an optimistic point of view, we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$. In the second class, which has a pessimistic point of view, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n.

If we use the same methodology in the AOWG operators, we can also obtain different types of aggregation operators by using a different manifestation in the weighting vector. The weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DOWG (or OWG) operator and w^*_{n+1-j} the jth weight of the AOWG operator.

Note that in this case it is also possible to analyse different measures about the weighting vector such as the attitudinal character, the entropy of dispersion and the balance operator. For the attitudinal character, we could use the formulation explained in [16] when it uses the particular case of OWG operators. For the entropy of dispersion and for the balance operator, as we are strictly interested in the weighting vector, we could use the same formulation as it has been explained in Section 2.1.

As we can see, the OWG operator cannot aggregate negative numbers in the aggregation because the results become inconsistent. If we analyse the results, we can observe that depending on the number of arguments with negative values, the result will be positive or negative. If the sum the number of arguments with negative values is even, then, the final result will be positive. If the sum is odd, then, the final result will be negative. As we can see, this situation is completely inconsistent with the aggregation where we should expect similar results independently that the number of arguments is even or odd. In the following Section, we are going to suggest a methodology that is able to deal with negative numbers when using the OWG operator.

III. Decision Making using Minimization of Maximal Regret

The use of minimization of maximal regret in decision making was suggested by Savage in [30], [31]. It can be summarized as follows.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{S_1, \ldots, S_n\}$. c_{ij} is the payoff to the decision maker if he selects alternative A_i and the state of nature is S_j. The matrix R whose components are the r_{ij}, is the regret matrix. The objective of the problem is to select the alternative which best satisfies the payoff to the decision maker. In order to do that, we should follow the following steps:

Step 1: Calculate the payoff matrix.
Step 2: Calculate $C_j = \text{Max}\{c_{ij}\}$ for each S_j.
Step 3: Calculate $r_{ij} = C_j - c_{ij}$; for each pair A_i and S_j.
Step 4: Calculate $R_i = \text{Max}\{r_{ij}\}$ for each A_i.
Step 5: Select A_{i*} such that $R_{i*} = \text{Min}\{R_i\}$.

As we can see, once established the regret matrix, this method uses a pessimistic criteria. Using a similar methodology, we could use other criteria instead of the pessimistic one. For example, we could use the average criteria, the Hurwicz criteria, the weighted mean, the OWA operator or the OWG operator. As the OWA operator generalizes a wide range of aggregation operators such as the average, the Hurwicz criteria and the weighted mean, we are going to consider this case when taking decisions with the MMR method.

This generalization was suggested by Yager in [15]. He proposed to use the OWA operator in the regret matrix. Then, all the other criteria could be included in this aggregation as particular cases of using an established attitudinal character such as the maximum, the minimum, the average and the weighted average. Yager called this generalization the Min-W-Regret (MWR) procedure. In order to distinguish between the use of the average, the weighted average and the OWA operator in the regret matrix, we prefer to call the case with OWA operators as the Min-OWA-Regret procedure. It can be summarized as follows:

Step 1: Calculate the payoff matrix.

Step 2: Calculate $C_j = \text{Max}\{c_{ij}\}$ for each S_j.

Step 3: Calculate $r_{ij} = C_j - c_{ij}$; for each pair A_i and S_j.

Step 4: Calculate $R_i = OWA(r_{i1}, \ldots, r_{in})$ using (1), for each A_i.

Step 5: Select A_{i*} such that $R_{i*} = \text{Min}\{R_i\}$.

As we can see, by choosing a different manifestation in the weighting vector of step 4, we can obtain different criteria such as the original work developed by Savage [30], [31]:

1) When $w_1 = 1$ and $w_j = 0$, $\forall\ j \neq 1$; we get the traditional Min-Max regret method. Thus, the original work developed by Savage is a particular case of this generalization.
2) When $w_n = 1$ and $w_j = 0$, $\forall\ j \neq n$; we associate with each alternative the minimal regret.
3) When $w_j = 1/n$, $\forall\ j$; we are aggregating the regret matrix with the average criteria.

Other families of aggregation operators could be obtained by using different manifestations of the weighting vector. For example, when $w_k = 1$ and $w_j = 0$ for all $j \neq k$ we are using the step-OWA [9] in the regret matrix. Note that if $k = 1$, the step-OWA is transformed in the maximum and if $k = n$, the step-OWA becomes the minimum.

When $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$, we are using the window-OWA [9] in the regret matrix. Note that k and m must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, the window-OWA is transformed in the maximum, if $m = 1$ and $k = n$, the window-OWA becomes the minimum and if $m = n$ and $k = 1$, the window-OWA is transformed in the average criteria.

If $w_1 = w_n = 0$ and for all others $w_j = 1/(n - 2)$, we are using the olympic average [13] in the regret matrix. Note that if $n = 3$ or $n = 4$, the olympic average is transformed in the OWA median [10] and if $m = n - 2$ and $k = 2$, the window-OWA is transformed in the olympic average.

Another type of aggregation that could be used in the regret matrix is the E-Z OWA weights. In this case, we should distinguish between two classes. In the first class, we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$, and in the second class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n.

We note that the median and the weighted median can also be used in the regret matrix. For the median, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and if n is even we assign for example $w_{n/2} = w_{(n/2)+1} = 0.5$. For the weighted median, we follow a different procedure than [10]. We select the kth largest argument of the r_i such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

Another interesting family is the S-OWA operator [9], [11]. We can divide it in three types: the orlike, the andlike and the generalized S-OWA operator. The orlike S-OWA operator is obtained when $w_1 = (1/n)(1 - \alpha) + \alpha$, and $w_j = (1/n)(1 - \alpha)$ for $j = 2$ to n with $\alpha \in [0, 1]$. Note that if $\alpha = 0$, we obtain the average and if $\alpha = 1$, we obtain the maximum. The andlike S-OWA operator is obtained when $w_n = (1/n)(1 - \beta) + \beta$ and $w_j = (1/n)(1 - \beta)$ for $j = 1$ to $n - 1$ with $\beta \in [0, 1]$. In this type of S-OWA, if $\beta = 0$ we obtain the average and if $\beta = 1$, the minimum. Finally, the generalized S-OWA operator is obtained when $w_1 = (1/n)(1 - (\alpha + \beta) + \alpha$, $w_n = (1/n)(1 - (\alpha + \beta) + \beta$, and $w_j = (1/n)(1 - (\alpha + \beta)$ for $j = 2$ to $n - 1$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. In this case, if $\alpha = 0$, the generalized S-OWA operator is transformed in the andlike S-OWA operator and if $\beta = 0$, in the orlike S-OWA operator. Also note that if $\alpha + \beta = 1$, the generalized S-OWA operator is transformed in the Hurwicz criteria.

Another type of OWA operator is the centered-OWA weights. It has been recently suggested by Yager [17] and it says that an OWA operator is a centered aggregation if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$, then $w_i < w_j$ and when $i > j \geq (n + 1)/2$, then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a relaxation of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. These cases are known as softly decaying centered OWA operator. A particular case of this situation is the average because all its weights are equal. Another special case of centered-OWA appears when the third condition is not accomplished. This type is known as non-inclusive centered-OWA operator. A particular case of this situation is the OWA-median.

As we can see, the generalized Min-W-Regret method accomplishes the same properties as the original OWA operator such as commutativity, monotonicity, idempotency and boundedness.

In order to adequate the generalized Min-W-Regret approach to a degree of optimism with the weighting vector used in the regret matrix, Yager defined R-OPT(W) = 1 – $\alpha(W)$. Here $\alpha(W)$ represents the attitudinal character introduced in [2] for the original OWA operator, and R-OPT(W) is the adapted version for the Min-W-Regret approach. We see that for $w_1 = 1$ and $w_j = 0$, $\forall\ j \neq 1$; $\alpha(W) = 1$ and hence R-OPT(W) = 0, while for $w_n = 1$ and $w_j = 0$, $\forall\ j \neq n$; $\alpha(W) = 0$ and hence R-OPT(W) = 1.

Analysing the attitudinal character, we see that Yager developed a method that adapted the generalized Min-W-Regret approach to the degree of optimism of the weighting vector but it could be simplified by using the AOWA operator. Then, the aggregation would reflect automatically the attitudinal character. The reason for this problem could come from a theoretical point of view where we could say that the OWA operator is appropriate to use in situations involving benefits while the AOWA operator is appropriate to use in situations involving costs. From a more generalized perspective, we could say that we should use the OWA operator in situations where the highest value of the payoff matrix is the best result while we should use the AOWA operator in situations where the smallest value is the best result.

The procedure to follow with the AOWA operator is very similar with the difference that now the reordering step is developed in ascending order. We can summarize it as follows:

Step 1: Calculate the payoff matrix.

Step 2: Calculate $C_j = \text{Max}\{c_{ij}\}$ for each S_j.
Step 3: Calculate $r_{ij} = C_j - c_{ij}$; for each pair A_i and S_j.
Step 4: Calculate $R_i = \text{AOWA}(r_{i1}, \ldots, r_{in})$, for each A_i.
Step 5: Select A_{i*} such that $R_{i*} = \text{Min}\{R_i\}$.

As we can see, by choosing a different manifestation in the weighting vector of step 4, we can obtain different criteria such as the original work developed by Savage, the maximum, the average, etc. Note that the weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DOWG (or OWG) operator and w^*_{n+1-j} the jth weight of the AOWG operator.

In this case, we can see that we obtain directly the degree of optimism. For example, if $w_n = 1$ and $w_j = 0$, $\forall\ j \neq n$; $\alpha(W) = 0$; and if $w_1 = 1$ and $w_j = 0$, $\forall\ j \neq 1$; $\alpha(W) = 1$. If we consider the properties of this generalized Min-W-Regret method with the AOWA operator, we also find that it is commutative, monotonic, bounded and idempotent.

IV. Using the OWG Operator in Decision Making with Minimization of Regret

The use of the OWG operator in decision making with minimization of regret is an alternative when taking decisions with regret methods. It consists in introduce the OWG operator in the aggregation step of the regret matrix. The motivation for using the OWG operator is because there are some cases where we could prefer to aggregate with a geometric operator instead of the traditional methods used previously. Here, the procedure will be the same as for the case with the OWA operator with the difference that now we will use the OWG operator in the aggregation phase. Then, we can summarize the procedure as follows:

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{S_1, \ldots, S_n\}$. c_{ij} is the payoff to the decision maker if he selects alternative A_i and the state of nature is S_j. The matrix R whose components are the r_{ij}, is the regret matrix. The objective of the problem is to select the alternative which best satisfies the payoff to the decision maker. In order to do that, we should follow the following steps:

Step 1: Calculate the payoff matrix.
Step 2: Calculate $C_j = \text{Max}\{c_{ij}\}$ for each S_j.
Step 3: Calculate $r_{ij} = C_j / c_{ij}$ for each pair A_i and S_j.
Step 4: Calculate $R_i = \text{OWG}(r_{i1}, \ldots, r_{in})$ using (8), for each A_i.
Step 5: Select A_{i*} such that $R_{i*} = \text{Min}\{R_i\}$.

Here, we should note that in the construction of the regret matrix, we divide the values because if we do not do this, we would not get consistent results as the OWG operator cannot aggregate arguments with value 0. The reason is because when aggregating with 0, the whole aggregation automatically becomes 0. Analysing this change, we see that now the aggregation is stable because for the best cases, when multiplying by 1, the result continues to be the same. Then, the result obtained is similar as in the previous cases where the best value of each state of nature did not add any regret in the whole aggregation.

In this case, we could also obtain different aggregations in step 4 by choosing a different weighting vector such as the original regret work developed by Savage:

1) When $w_1 = 1$ and $w_j = 0$, $\forall\ j \neq 1$; we get the traditional Min-Max regret method with the difference that now the result has one unit more. Thus, the original work developed by Savage can be considered as a particular case of this generalization.
2) When $w_n = 1$ and $w_j = 0$, $\forall\ j \neq n$; we associate with each alternative the minimal regret.
3) When $w_j = 1/n$, $\forall\ j$; we are aggregating the regret matrix with the geometric mean.

Other families of geometric operators could be obtained for the Min-OWG-Regret method by choosing different manifestations of the weighting vector. For example, when $w_k = 1$ and $w_j = 0$ for all $j \neq k$ we are using the step-OWG [27] in the regret matrix. Note that if $k = 1$, the step-OWG is transformed in the maximum and if $k = n$, the step-OWG becomes the minimum. Also note that the results obtained for the step-OWG are the same than the results obtained for the step-OWA.

Other aggregations such as the OWG median and the weighted OWG median can also be used in the Min-OWG-Regret method. For the OWG median, that it is based on the OWA median [10], if n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and if n is even we assign for example $w_{n/2} = w_{(n/2)+1} = 0.5$. Note that if n is odd, the result obtained in the OWG median is the same than the result found in the OWA median.

For the weighted OWG median, we follow the same procedure as used for the weighted OWA median. We select the kth largest argument of the r_i such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5.

Another family is the centered-OWG operator. We can define it in a similar way as Yager [17] defined the centered-OWA operator. An OWG operator is a centered aggregation if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$, then $w_i < w_j$ and when $i > j \geq (n + 1)/2$, then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a relaxation of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. These cases are known as softly decaying centered-OWG operator. A particular case of this situation is the geometric mean because all the weights are equal. Another special case of centered-OWG appears when the third condition is not accomplished. This type is known as non-inclusive centered-OWG operator. A particular case of this situation is the OWG-median.

If $w_1 = w_n = 0$ and for all others $w_j = 1/(n-2)$, we are using the olympic-OWG operator in the regret matrix. Note that if $n = 3$ or $n = 4$, the olympic-OWG is transformed in the OWG-median and if $m = n - 2$ and $k = 2$, the window-OWG is transformed in the olympic-OWG.

A further family of Min-OWG-Regret methods is the Min-window-OWG-Regret method. This family is found when $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$. Note that in this case k and m must also be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, the Min-window-OWG-Regret is transformed in the maximum, if $m = 1$ and $k = n$, the Min-window-OWG-Regret becomes the minimum and if $m = n$ and $k = 1$, the Min-window-OWG-Regret is transformed in the geometric mean.

A further type of geometric operator that could be used in the regret matrix is the E-Z OWG weights. In this type of aggregation, we find two different classes. In the first class, we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$, and in the second class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n.

Another interesting issue to consider is the properties of this type of generalized Min-W-Regret method. As we can see, it accomplishes the same properties than the OWA version.

1) Commutativity: any permutation of the arguments has the same evaluation.
2) Monotonicity: If $r_i \geq d_i$ for all $i \Rightarrow \text{OWG}(r_1,\ldots, r_n) \geq \text{OWG}(d_1,\ldots, d_n)$.
3) Boundedness: $\text{Min}\{r_i\} \leq \text{OWG}(r_1,\ldots, r_n) \leq \text{Max}\{r_i\}$.
4) Idempotency: If $r_i = r$, for all $i \Rightarrow \text{OWG}(r_1,\ldots, r_n) = r$.

Another alternative that we could use in the aggregation of the regret matrix is the AOWG operator. The motivation for use an ascending order appears in situations where the smallest value is the best result because then, the weighting vector will consider first the best result and so on. The procedure to follow with the AOWG operator is very similar with the difference that now the reordering step is developed in ascending order. We can summarize it as follows:

Step 1: Calculate the payoff matrix.
Step 2: Calculate $C_j = \text{Max}\{c_{ij}\}$ for each S_j.
Step 3: Calculate $r_{ij} = C_j / c_{ij}$ for each pair A_i and S_j.
Step 4: Calculate $R_i = \text{AOWG}(r_{i1}, \ldots, r_{in})$, for each A_i.
Step 5: Select A_{i*} such that $R_{i*} = \text{Min}\{R_i\}$.

Again, in this case we also add one unit in order to keep stable the aggregation. By choosing a different weighting vector we could also obtain different aggregations in step 4 such as the original regret work developed by Savage, the maximum, the average, the median, the step-AOWG, the window-AOWG, the olympic-AOWG, the centered-AOWG, the E-Z AOWG, the S-AOWG, etc. Analysing the properties of this type of generalized Min-W-Regret method with the AOWG operator, we find the same properties as with the original OWG operator.

V. Illustrative Example

In the following, we are going to develop an example in order to understand numerically all the procedures commented above. We will distinguish between two general cases. In the first case, we will construct the regret matrix in the original form as it was developed by Savage [30], [31] and we will consider the aggregation with the arithmetic mean (AM), with the weighted average (WA), with the OWA operator and with the AOWA operator. In the second case, we will construct the regret matrix as it has been explained in the Min-OWG-Regret method and we will consider the aggregation with the GM, with the WGM, with the OWG and the AOWG operator.

With these eight types of aggregations we will see the different results obtained by using a different aggregation in the decision. Note that as the geometric construction of the regret matrix is completely different than the arithmetic one, the results will also be different. The interesting point to analyse is to see which results give the same decision about the selection of an alternative. In this example, we will assume the following weighting vector: W = (0.1, 0.2, 0.2, 0.3, 0.2).

Step 1: Assume that an enterprise wants to increase its volume of activities. In order to do this, the board of directors has established five possible investments that the enterprise could develop in the future.

(1) A_1 is a food company called *V*.
(2) A_2 is a chemical company called *W*.
(3) A_3 is a car company called *X*.
(4) A_4 is a TV company called *Y*.
(5) A_5 is a computer company called *Z*.

After careful review of the information, the experts have given the following general information. They have summarized the information of the investments giving the expected results depending on the five states of nature S_j that could happen in the future. The results are shown in table I.

TABLE I
PAYOFF MATRIX

	S_1	S_2	S_3	S_4	S_5
A_1	60	20	10	40	50
A_2	80	50	20	10	20
A_3	30	40	40	30	40
A_4	20	30	20	30	80
A_5	70	40	40	10	20

Step 2 – Step 3: For the first case, that affects the AM, the WA, the OWA operator and the AOWA operator, we will calculate $C_j = \text{Max}\{c_{ij}\}$ for each S_j and $r_{ij} = C_j - c_{ij}$, for each pair A_i and S_j. For the second case, that affects the GM, the WGM, the OWG operator and the AOWG operator, we will calculate $C_j = \text{Max}\{c_{ij}\}$ for each S_j and $r_{ij} = C_j / c_{ij}$, for each

pair A_i and S_j. The results for the first case are shown in table II and the results for the second case are shown in table III.

TABLE II
REGRET MATRIX

	S_1	S_2	S_3	S_4	S_5
A_1	20	30	30	0	30
A_2	0	0	20	30	60
A_3	50	10	0	10	40
A_4	60	20	20	10	0
A_5	10	10	0	30	60

TABLE III
REGRET MATRIX FOR THE GEOMETRIC OPERATORS

	S_1	S_2	S_3	S_4	S_5
A_1	1.33	2.5	4	1	1.6
A_2	1	1	2	4	4
A_3	2.66	1.25	1	1.33	2
A_4	4	1.66	2	1.33	1
A_5	1.14	1.25	1	4	4

Step 4: Aggregate the regret matrix with each aggregation operator according to its formulation. For the first case, we will aggregate table II with the AM, with the WA, with the OWA and with the AOWA operators. The OWA operator and the AOWA operator are defined by (1). Note that the AM is a special case of the OWA operator when $w_j = 1/n$, for all j. For the WA, we will associate each weight j with its corresponding regret argument j. For the second case, we will aggregate table III with the GM, with the WGM, with the OWG and with the AOWG operators. The GM is defined by (6), the WGM is defined by (7), and the OWG and the AOWG operator by (8). The results are shown in tables IV and V.

TABLE IV
AGGREGATED REGRET FOR THE FIRST CASE

	AM	*WA*	*OWA*	*AOWA*
A_1	22	20	21	25
A_2	22	25	16	25
A_3	22	18	18	26
A_4	22	17	17	24
A_5	22	24	17	25

TABLE V
AGGREGATED REGRET FOR THE SECOND CASE

	GM	*WGM*	*OWG*	*AOWG*
A_1	1.84	1.79	1.65	2.02
A_2	2	2.29	1.74	2.29
A_3	1.54	1.44	1.43	1.65
A_4	1.77	1.59	1.59	1.90
A_5	1.86	2.11	1.64	2.14

Step 5: As we can see, with the AM we cannot select an alternative as we get the same result for all of them. With the WA and with the AOWA operator we select alternative 4 as it gives the lowest expected cost. With the OWA operator we will select alternative 2 as in this case, this one gets the lowest expected value. For the GM, the WGM, the OWG and the AOWG operators, we select alternative 3 as in these cases this alternative is the one with the lowest value.

VI. Conclusion

In this paper, we have suggested the use of the OWG operator in situations of decision making with minimization of regret. For doing this, we have made some changes in the construction of the regret matrix in order to adapt it to the aggregation characteristics of the OWG operator. With this new construction, we have shown that it is possible to deal with negative numbers in the OWG operator by transforming the initial results in positive numbers. We have developed the decision making process distinguishing in the aggregation step between the use of the OWA operator, the AOWA operator, the OWG operator and the AOWG operator. Finally, an illustrative example has been given where we have shown the process to follow in a decision making problem with minimization of regret.

In future research, we expect to develop new approaches about using different types of aggregation operators in decision making problems with minimization of regret and we will apply it in other decision making problems such as human resource selection, strategic management, etc.

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