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# Generalized Fuzzy Subalgebras and Fuzzy Ideals of BCI-Algebras with Operators

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***Abstract*—**The aim of this paper is to introduce the concepts of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, and to investigate their basic properties.

***Keywords*—**BCI-algebras with operators, generalized fuzzy subalgebras, generalized fuzzy ideals, generalized fuzzy quotient algebras.

## I. Introduction

THE fuzzy set is a generalization of the classical set. After the introduction of fuzzy sets, there have been a number of generalizations of this fundamental concept, especially, in the branches of mathematics. Imai and Iseki [1], [2] introduced the concept of BCK/BCI-algebras, which are generalizations of BCK-algebras. In 1980, Ming et al. [13] introduced the neighbourhood structure of a fuzzy point.

In 1991, Xi [3] applied the fuzzy sets to BCK-algebras; fuzzy BCK/BCI-algebras have been widely researched. Meng et al. [4] introduced the concept of fuzzy implicative ideals of BCK-algebras in 1997. Liu and Meng [6], [7] introduced the notions of fuzzy positive implicative ideals and fuzzy implicative ideals of BCI-algebras. Zheng [5] defined operators in BCK-algebras and raised the concept of BCI-algebras with operators and gave some isomorphism theorems of it. In 2002, Liu [8] introduced the concept of the fuzzy quotient algebras of BCI-algebras. In 2004, Jun [9] introduced the $(\alpha,\beta)$-fuzzy ideals of BCK/BCI-algebras and established the characterizations of $(\in,\in\vee q)$-fuzzy ideals. In 2006, Liao et al. [11] introduced the $\left(\in,\in\vee q_{(\lambda,\mu)}\right)$-fuzzy normal subgroup. In 2009, Jun et al. [12] introduced the concept of $(\in,\in\vee q)$-ideals of BCI-algebras. In 2011, Liu and Sun [10] introduced the concept of generalized fuzzy ideals of BCI-algebra and investigate some basic properties. In 2017, Hu et al. [14] introduced the fuzzy subalgebras and fuzzy ideals of BCI-algebras with operators.

In this paper, we give the notions of generalized fuzzy subalgebras, generalized fuzzy ideals and generalized fuzzy quotient algebras of BCI-algebras with operators, in particular, discuss the basic properties of generalized fuzzy BCI-algebras with operators and give several results about it.

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## II. Preliminaries

We recall some definitions and propositions which may be needed.

An algebra $\langle X;*,0\rangle$ of type (2,0) is called a BCI-algebra, if for all $x,y,z\in X$, it satisfies the following conditions:

1. $((x*y)*(x*z))*(z*y)=0;$
2. $(x*(x*y))*y=0;$
3. $x*x=0;$
4. $x*y=0$ and $y*x=0$ imply $x=y$.

We can define $x*y=0$ if and only if $x\le y$, then the above conditions can be written as:

1. $(x*y)*(x*z)\le z*y;$
2. $x*(x*y)\le y;$
3. $x\le x;$
4. $x\le y$ and $y\le x$ imply $x=y$.

If a BCI-algebra satisfies $0*x=0$, then it is called a BCK-algebra.

**Definition 1.** [5] $\langle X;*,0\rangle$ is a BCI-algebra, $M$ is a non-empty set, if there exists a mapping $(m,x)\to mx$ from $M\times X$ to $X$ which satisfies $m(x*y)=(mx)*(my),\forall x,y\in X,m\in M$. then $M$ is called a left operator of $X$, $X$ is called a BCI-algebra with left operator $M$, or $M-$BCI-algebra for short.

**Definition 2.** [13] $\langle X;*,0\rangle$ is a BCI-algebra, a fuzzy subset $A$ of $X$ of the form

$$A(y)=\begin{cases} t(\neq 0), y=x,\\ 0, y\neq x,\end{cases}$$

is said to be a fuzzy point with support $x$ and value $t$, and is denoted by $x_t$.

**Proposition 1.** [10] Let $\langle X;*,0\rangle$ be a BCI-algebra, if $A$ is a fuzzy generalized ideal of it, and $x*y\le z$, then

$$A(x)\vee\lambda\ge A(y)\wedge A(z)\wedge\mu, x,y,z\in X.$$

**Definition 3.** [5] Let $\langle X;*,0\rangle$ and $\langle \bar{X};*,0\rangle$ be two $M-$BCI-algebras, if $f$ is a homomorphism from $\langle X;*,0\rangle$ to $\langle \bar{X};*,0\rangle$, and $f(mx)=mf(x)$ for all $x\in X$, $m\in M$, then $f$ is called a homomorphism with operators.

**Definition 4.** If $\langle X;*,0\rangle$ is a BCI-algebra, $A$ is a non-empty subset of $X$, and $mx \in A$ for all $x \in A, m \in M$, then $\langle A;*,0\rangle$ is called an $M-$subalgebra of $\langle X;*,0\rangle$.

In the following parts, $X$ always means a $M-$BCI-algebra unless otherwise specified.

## III. Generalized Fuzzy Subalgebras of BCI-Algebras with Operators

**Definition 5.** $\langle X;*,0\rangle$ is a BCI-algebra, let $A$ be a fuzzy subset of $X$, $t,\lambda,\mu \in [0,1]$ and $\lambda < \mu$. if $A(x) \ge t$, we denoted $x_t \in A$; if $t > \lambda$ and $A(x)+t > 2\mu$, we denoted $x_t q_{(\lambda,\mu)} A$; if $x_t \in A$ or $x_t q_{(\lambda,\mu)} A$, we denoted $x_t \in \vee q_{(\lambda,\mu)} A$.

**Definition 6.** $\langle X;*,0\rangle$ is an $M-$BCI-algebra, let $A$ be a fuzzy subset of $X$, if it satisfies:

1. $x_t \in A$ and $y_r \in A$ implies $(x*y)_{t\wedge r} \in \vee q_{(\lambda,\mu)} A$, $\forall x,y \in X$, $t,r \in [0,1]$;
2. $x_t \in A$ implies $(mx)_t \in \vee q_{(\lambda,\mu)} A, \forall x \in X, t \in [0,1]$.

Then $A$ is called an $M-\left(\in,\in \vee q_{(\lambda,\mu)}\right)-$fuzzy subalgebra or a generalized $M-$fuzzy subalgebra for short.

**Proposition 2.** A fuzzy subset $A$ of $X$ is a generalized $M-$fuzzy subalgebra of $X$ if and only if it satisfies:

1. $A(x*y) \vee \lambda \ge A(x) \wedge A(y) \wedge \mu, \forall x,y \in X$;
2. $A(mx) \vee \lambda \ge A(x) \wedge \mu, \forall x \in X$.

**Proof.** Suppose that $A$ is a generalized $M-$fuzzy subalgebra of $X$. We first verify that

$$A(x*y) \vee \lambda \ge A(x) \wedge A(y) \wedge \mu, \forall x,y \in X.$$

Suppose there exists $x_0, y_0 \in X$ such that $A(x_0 * y_0) \vee \lambda < A(x_0) \wedge A(y_0) \wedge \mu$, choose $t$ such that $A(x_0 * y_0) \vee \lambda < t < A(x_0) \wedge A(y_0) \wedge \mu$, then $A(x_0 * y_0) < t$, $\lambda < t < \mu$, $A(x_0) > t$ and $A(y_0) > t$, therefore $(x_0)_t \in A, (y_0)_t \in A$. Based on Definition 6, $(x_0 * y_0)_t \in \vee q_{(\lambda,\mu)} A$, but we have $A(x_0 * y_0) < t$, therefore $A(x_0 * y_0) + t \le t + t < 2\mu$, this is a contradiction, therefore we have $A(x*y) \vee \lambda \ge A(x) \wedge A(y) \wedge \mu, \forall x,y \in X$. We shall now show that $A(mx) \vee \lambda \ge A(x) \wedge \mu, \forall x \in X$.

Suppose there exists $x_0 \in X$ such that $A(mx_0) \vee \lambda < A(x_0) \wedge \mu$, choose $t$ such that $A(mx_0) \vee \lambda < t < A(x_0) \wedge \mu$, then $A(x_0) > t$, therefore $(x_0)_t \in A$. Based on Definition 6, $(mx_0)_t \in \vee q_{(\lambda,\mu)} A$, but we have $A(mx_0) < t$, therefore $A(mx_0) + t \le t + t < 2\mu$, this is a contradiction, therefore we have $A(mx) \vee \lambda \ge A(x) \wedge \mu, \forall x \in X$.

Conversely, assume that $A$ satisfies condition 1, 2.

1). If $(x)_{t_1} \in A, (y)_{t_2} \in A, \forall x,y \in X, t_1, t_2 \in [0,1]$, then $A(x) \ge t_1, A(y) \ge t_2$, choose $T = t_1 \wedge t_2$, since $A$ is a generalized $M-$fuzzy subalgebra of $X$, we have

$$A(x*y) \vee \lambda \ge A(x) \wedge A(y) \wedge \mu > t_1 \wedge t_2 \wedge \mu,$$

if $T \le \mu$, then $A(x*y) \ge T$, so we have $(x*y)_T \in A$, if $T > \mu$, then $A(x*y) \ge \mu$, thus $A(x*y) + T \ge \mu + T > 2\mu$, then $(x*y)_T\, q_{(\lambda,\mu)} A$, therefore we have $(x*y)_T \in \vee q_{(\lambda,\mu)} A$.

2). If $x_t \in A, \forall x \in X, t \in [0,1]$, then $A(x) \ge t$, since $A$ is a generalized $M-$fuzzy subalgebra of $X$, then $A(mx) \vee \lambda \ge A(x) \wedge \mu$, if $t \le \mu$, then $A(mx) \vee \lambda \ge t$, since $\lambda < t$, so we have $A(mx) \ge t$, hence $(mx)_t \in A$, if $t > \mu$, then $A(mx) \vee \lambda \ge \mu$, since $\lambda < \mu$, so we have $A(mx) \ge \mu$, hence $A(mx) + t \ge \mu + t > 2\mu$, thus $(mx)_t\, q_{(\lambda,\mu)} A$, therefore we have $(mx)_t \in \vee q_{(\lambda,\mu)} A$. So $A$ is a generalized $M-$fuzzy subalgebra of $X$.

**Example 1.** If $A$ is a generalized $M-$fuzzy subalgebra of $X$, then $X_A$ is a generalized $M-$fuzzy subalgebra of $X$, define $X_A$ by

$$X_A : X \to [0,1], X_A(x) = \begin{cases} 1, x \in A \\ 0, x \notin A. \end{cases}$$

**Proof.** (1) For all $x,y \in X$, if $x,y \in A$, then $x*y \in A$, thus

$$X_A(x*y) \vee \lambda = 1 \ge X_A(x) \wedge X_A(y) \wedge \mu,$$

if there exists at least one which does not belong to $A$ between $x$ and $y$, for example $x \notin A$, thus

$$X_A(x*y) \vee \lambda \ge 0 = X_A(x) \wedge X_A(y) \wedge \mu.$$

(2) For all $x \in X$, $m \in M$, if $x \in A$, then $mx \in A$, therefore

$$X_A(mx) \vee \lambda = 1 \ge X_A(x) \wedge \mu,$$

if $x \notin A$, then $X_A(mx) \vee \lambda \ge 0 = X_A(x) \wedge \mu$, therefore $X_A$ is a generalized $M-$fuzzy subalgebra of $X$.

**Proposition 3.** $A$ is a generalized $M-$fuzzy subalgebra of $X$ if only if $A_t$ is a $M-$subalgebra of $X$, where $A_t$ is a non-empty set, define $X_A$ by $A_t = \{x \mid x \in X, A(x) \ge t\}, \forall t \in (\lambda, \mu]$.

**Proof.** Suppose $A$ is a generalized $M-$fuzzy subalgebra of $X$, $A_t$ is a non-empty set, $t \in (\lambda,\mu]$, then we have $A(x*y) \vee \lambda \ge A(x) \wedge A(y) \wedge \mu$. If $x \in A_t, y \in A_t$, then $A(x) \ge t, A(y) \ge t$, thus $A(x*y)\lambda \ge A(x) \wedge A(y) \wedge \mu \ge t$, thus we have $x*y \in A_t$.

For all $x \in X, m \in M$, if $A$ is a generalized $M-$fuzzy subalgebra of $X$, hence $A(mx) \vee \lambda \ge A(x) \wedge \mu \ge t$, thus $mx \in A_t$, therefore $A_t$ is an $M-$subalgebra of $X$. Conversely, suppose $A_t$ is an $M-$subalgebra of $X$, then we have $x*y \in A_t$. Let

$A(x)=t$, then $A(x*y)\vee\lambda\ge t=A(x)\ge A(x)\wedge A(y)\wedge\mu.$ For all $x\in X$, $m\in M$, if $A_t$ is an $M-$subalgebra of $X$, then we have $A(mx)\vee\lambda\ge t=A(x)\ge A(x)\wedge\mu,$ therefore $A$ is a generalized $M-$fuzzy subalgebra of $X$.

**Proposition 4.** Suppose $X,Y$ are $M-$BCI-algebras, $f$ is a mapping from $X$ to $Y$, if $A$ is a generalized $M-$fuzzy subalgebra of the $Y$, then $f^{-1}(A)$ is a generalized $M-$fuzzy subalgebra of $X$.

**Proof.** Let $y\in Y$, suppose $f$ is a epimorphism, then there exists $x$ in $X$, we have $y=f(x)$. If $A$ is a generalized $M-$fuzzy subalgebra of $Y$, then we have

$$A(x*y)\vee\lambda\ge A(x)\wedge A(y)\wedge\mu; A(mx)\vee\lambda\ge A(x)\wedge\mu.$$

For all $x,y\in X, m\in M$, we have

(1) $f^{-1}(A)(x*y)\vee\lambda=A(f(x)*f(y))\vee\lambda$
$\ge A(f(x))\wedge A(f(y))\wedge\mu=f^{-1}(A)(x)\wedge f^{-1}(A)(y)\wedge\mu;$

(2) $f^{-1}(A)(mx)\vee\lambda=A(f(mx))\vee\lambda=A(mf(x))\vee\lambda$
$\ge A(f(x))\wedge\mu=f^{-1}(A)(x)\wedge\mu.$

Therefore $f^{-1}(A)$ is a generalized $M-$fuzzy subalgebra of $X$.

## IV. Generalized Fuzzy Ideals of BCI-Algebras with Operators

**Definition 7.** $\langle X;*,0\rangle$ is an $M-$BCI-algebra, let $A$ be a fuzzy subset of $X$, if it satisfies:

1. $x_t\in A$ implies $0_t\in\vee q_{(\lambda,\mu)}A,\forall x\in X,t\in[0,1]$;
2. $(x*y)_t\in A$ and $y_r\in A$ implies $x_{t\wedge r}\in\vee q_{(\lambda,\mu)}A,\forall x,y\in X$, $t,r\in[0,1]$;
3. $x_t\in A$ implies $(mx)_t\in\vee q_{(\lambda,\mu)}A,\forall x\in X,t,\in[0,1]$.

Then $A$ is called a $M-(\in,\in\vee q_{(\lambda,\mu)})-$fuzzy subalgebra or a generalized $M-$fuzzy subalgebra for short.

**Proposition 5.** A fuzzy subset $A$ of $X$ is a generalized $M-$fuzzy ideal of $X$ if and only if it satisfies:

1. $A(0)\vee\lambda\ge A(x)\wedge\mu,\forall x\in X$;
2. $A(x)\vee\lambda\ge A(x*y)\wedge A(y)\wedge\mu,\forall x,y\in X$;
3. $A(mx)\vee\lambda\ge A(x)\wedge\mu,\forall x\in X.$

**Proof.** Suppose that $A$ is a generalized $M-$fuzzy ideal of $X$. We first verify that $A(0)\vee\lambda\ge A(x)\wedge\mu,\forall x\in X$. Suppose there exists $x_0\in X$ such that $A(0)\vee\lambda<A(x_0)\wedge\mu,$ choose $t$ such that $A(0)\vee\lambda<t<A(x_0)\wedge\mu,$ then $A(x_0)>t$ and $\lambda<t<\mu,$ therefore $(x_0)_t\in A.$ Based on Definition 7, $0_t\in\vee q_{(\lambda,\mu)}A,$ but we have $A(0)<t\le\mu,$ therefore $A(0)+t\le t+t\le 2\mu,$ this is a contradiction, therefore we have $A(0)\vee\lambda\ge A(x)\wedge\mu,\forall x\in X$. We shall now show that

$$A(x)\vee\lambda\ge A(x*y)\wedge A(y)\wedge\mu,\forall x,y\in X.$$

Suppose there exists $x_0,y_0\in X$ such that $A(x_0)\vee\lambda<A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ choose $t$ such that $A(x_0)\vee\lambda<t<A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ then $A(x_0)<t,$ $\lambda<t<\mu,$ $A(x_0*y_0)>t$ and $A(y_0)>t,$ therefore $(x_0*y_0)_t\in A,(y_0)_t\in A.$ Based on Definition 7, $(x_0)_t\in\vee q_{(\lambda,\mu)}A,$ but we have $A(x_0)<t,$ therefore $A(x_0)+t\le t+t\le 2\mu,$ this is a contradiction, therefore we have $A(x)\vee\lambda\ge A(x*y)\wedge A(y)\wedge\mu,\forall x,y\in X.$

Next, we shall show that $A(mx)\vee\lambda\ge A(x)\wedge\mu,\forall x\in X.$ Suppose there exists $x_0\in X$ such that $A(mx_0)\vee\lambda<A(x_0)\wedge\mu,$ choose $t$ such that $A(mx_0)\vee\lambda<t<A(x_0)\wedge\mu,$ then $A(x_0)>t,$ therefore $(x_0)_t\in A.$ Based on Definition 7, $(mx_0)_t\in\vee q_{(\lambda,\mu)}A,$ but we have $A(mx_0)<t,$ therefore $A(mx_0)+t\le t+t<2\mu,$ this is a contradiction, therefore we have $A(mx)\vee\lambda\ge A(x)\wedge\mu,\forall x\in X.$

Conversely, assume that $A$ satisfies condition 1, 2, 3.

1). If $x_t\in A,\forall x\in X,t\in(0,1],$ then $A(x)\ge t,$ since $A$ is a generalized $M-$fuzzy ideal of $X$, we have $A(0)\vee\lambda\ge A(x)\wedge\mu\ge t\wedge\mu,$ if $t\le\mu,$ then $A(0)\ge t,$ so we have $0_t\in A,$ if $t>\mu,$ then $A(0)\ge\mu,$ thus $A(0)+t\ge t+\mu>2\mu,$ then $0_t q_{(\lambda,\mu)}A,$ therefore we have $0_t\in\vee q_{(\lambda,\mu)}A.$

2). If $(x*y)_{t_1}\in A, y_{t_2}\in A,\forall x,y\in X,t_1,t_2\in(\lambda,1],$ then $A(x*y)\ge t_1,A(y)\ge t_2,$ choose $T=t_1\wedge t_2,$ since $A$ is a generalized $M-$fuzzy ideal of $X$. We have $A(x)\vee\lambda\ge A(x*y)\wedge A(y)\wedge\mu>t_1\wedge t_2\wedge\mu,$ if $T\le\mu,$ then $A(x)\ge T,$ so we have $x_T\in A,$ if $T>\mu,$ then $A(x)\ge\mu,$ thus $A(x)+T\ge\mu+T>2\mu,$ then $x_T q_{(\lambda,\mu)}A,$ therefore we have $x_T\in\vee q_{(\lambda,\mu)}A.$

3). If $x_t\in A,\forall x\in X,t\in(\lambda,1],$ then $A(x)\ge t,$ since $A$ is a generalized $M-$fuzzy ideal of $X$. We have $A(mx)\vee\lambda\ge A(x)\wedge\mu,$ if $t\le\mu,$ then $A(mx)\vee\lambda\ge t,$ since $\lambda<t,$ so we have $A(mx)\ge t,$ hence $(mx)_t\in A,$ if $t>\mu,$ then $A(mx)\vee\lambda\ge\mu,$ since $\lambda<\mu,$ so we have $A(mx)\ge\mu,$ hence $A(x)+t\ge\mu+t>2\mu,$ thus $(mx)_t q_{(\lambda,\mu)}A,$ therefore we have $(mx)_t\in\vee q_{(\lambda,\mu)}A.$ So, $A$ is a generalized $M-$fuzzy ideal of $X$.

**Example 2.** If $A$ is a generalized $M-$fuzzy ideal of $X$, then $X_A$ is a generalized $M-$fuzzy ideal of $X$, define $X_A$ by

$$X_A: X\to[0,1], X_A(x)=\begin{cases}1, x\in A\\ 0, x\notin A.\end{cases}$$

**Proof.** (1) For all $x,y\in X$, if $x,y\in A,$ then $x*y\in A,$ thus

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$$X_A(0)\vee\lambda=1\geq X_A(x)\wedge\mu,$$
$$X_A(x)\vee\lambda=1\geq X_A(x*y)\wedge X_A(y)\wedge\mu,$$

if there exists at least one which does not belong to $A$ between $x$ and $y$, for example $x\notin A$, thus

$$X_A(0)\vee\lambda=1\geq X_A(x)\wedge\mu,$$

$$X_A(x)\vee\lambda\geq X_A(x*y)\wedge X_A(y)\wedge\mu=0;$$

(2)For all $x\in X$, $m\in M$, if $x\in A$, then $mx\in A$, thus $X_A(mx)\vee\lambda=1\geq X_A(x)\wedge\mu.$ If $x\notin A$, then $X_A(mx)\vee\lambda\geq 0=X_A(x)\wedge\mu,$ therefore $X_A$ is a generalized $M-$fuzzy ideal of $X$.

**Proposition 6.** $A$ is a generalized $M-$fuzzy ideal of $X$ if only if $A_t$ is an $M-$ideal of $X$, where $A_t$ is non-empty set, define $A_t$ by $A_t=\{x\mid x\in X, A(x)\geq t\},\forall t\in(\lambda,\mu]$.

**Proof.** Suppose $A$ is a generalized $M-$fuzzy ideal of $X$, $A_t$ is non-empty set, $t\in(\lambda,\mu]$, then we have $A(0)\vee\lambda\geq A(x)\wedge\mu\geq t,$ thus $0\in A_t$. If $x*y\in A_t, y\in A_t$, then $A(x*y)\geq t, A(y)\geq t,$ thus $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu\geq t,$ thus we have $x\in A_t$. For all $x\in X$, $m\in M$, if $A$ is a generalized $M-$ fuzzy ideal of $X$, hence $A(mx)\vee\lambda\geq A(x)\wedge\mu\geq t,$ thus $mx\in A_t$, therefore $A_t$ is an $M-$ideal of $X$. Conversely, suppose $A_t$ is an $M-$ideal of $X$, then we have $0\in A_t, A(0)\geq t.$ Let $A(x)=t,$ thus $x\in A_t$, we have $A(0)\vee\lambda\geq t=A(x)\wedge\mu,$ suppose there is no $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu,$ then there exist $x_0,y_0\in X$, we have $A(x_0)\vee\lambda<A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ let $t_0=A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ then $A(x_0)\vee\lambda<t_0=A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ if $x_0*y_0\in A_{t_0}, y_0\in A_{t_0}$, then we have $x_0\in A_{t_0}$, then $A(x_0)\geq t_0,$ which is inconsistent with $A(x_0)\vee\lambda<t_0=A(x_0*y_0)\wedge A(y_0)\wedge\mu,$ then we have $A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu.$ For all $x\in X, m\in M$, if $A_t$ is an $M-$ideal of $X$, then we have $A(mx)\vee\lambda\geq t=t\wedge\mu=A(x)\wedge\mu,$ therefore $A$ is a generalized $M-$fuzzy ideal of $X$.

**Proposition 7.** Suppose $X,Y$ are $M-$BCI-algebras, $f$ is a mapping from $X$ to $Y$, $A$ is a generalized $M-$fuzzy ideal of $Y$, then $f^{-1}(A)$ is a generalized $M-$fuzzy ideal of $X$.

**Proof.** Let $y\in Y$, suppose $f$ is an epimorphism, then there exists $x\in X$, we have $y=f(x)$. If $A$ is a generalized $M-$fuzzy ideal of $Y$, then we have

$$A(0)\vee\lambda\geq A(y)\wedge\mu; A(x)\vee\lambda\geq A(x*y)\wedge A(y)\wedge\mu;$$
$$A(mx)\vee\lambda\geq A(x)\wedge\mu.$$

For all $x,y\in X, m\in M$, we have

$$(1)\, f^{-1}(A)(0)\vee\lambda=A(f(0))\vee\lambda=A(0)\vee\lambda$$
$$\geq A(f(x))\wedge\mu=f^{-1}(A)(x)\wedge\mu;$$
$$(2)\, f^{-1}(A)(x)\vee\lambda=A(f(x))\vee\lambda\geq A(f(x)*f(y))\wedge A(f(y))\wedge\mu$$
$$=A(f(x*y))\wedge A(f(y))\wedge\mu=f^{-1}(A)(x*y)\wedge f^{-1}(A)(y)\wedge\mu;$$
$$(3)\, f^{-1}(A)(mx)\vee\lambda=A(f(mx))\vee\lambda=A(mf(x))\vee\lambda$$
$$\geq A(f(x))\wedge\mu=f^{-1}(A)(x)\wedge\mu.$$

Therefore $f^{-1}(A)$ is a generalized $M-$fuzzy ideal of $X$.

## V. Generalized Fuzzy Quotient BCI-Algebras with Operators

**Definition 8.** Let $A$ be an $M-(\in,\in\vee q_{(\lambda,\mu)})$-fuzzy ideal of $X$, for all $a\in X$, fuzzy set $A_a$ on $X$ defined as: $A_a: X\to[0,1]$ $\quad A_a(x)=A(a*x)\wedge A(x*a)\wedge\mu,\forall x\in X.$ Denote $X/A=\{A_a: a\in X\}; A(x)\geq\lambda.$

**Proposition 8.** Let $A_a,A_b\in X/A$, then $A_a=A_b$ if only if $A(a*b)\wedge A(b*a)\wedge\mu=A(0)\wedge\mu.$

**Proof.** Let $A_a=A_b$, then we have $A_a(b)=A_b(b)$, thus $A(a*b)\wedge A(b*a)\wedge\mu=A(b*b)\wedge A(b*b)\wedge\mu=A(0)\wedge\mu,$ that is $A(a*b)\wedge A(b*a)\wedge\mu=A(0)\wedge\mu.$ Conversely, suppose that $A(a*b)\wedge A(b*a)\wedge\mu=A(0)\wedge\mu.$ For all $x\in X$, since $(a*x)*(b*x)\leq a*b,(x*a)*(x*b)\leq b*a.$ It follows from Proposition 1 that

$$A(a*x)=A(a*x)\vee\lambda\geq A(b*x)\wedge A(a*b)\wedge\mu,$$
$$A(x*a)=A(x*a)\vee\lambda\geq A(x*b)\wedge A(b*a)\wedge\mu.$$

Hence

$$\begin{aligned}A_a(x)&=A(a*x)\wedge A(x*a)\wedge\mu\\&\geq A(b*x)\wedge A(x*b)\wedge A(a*b)\wedge A(b*a)\wedge\mu\\&=A(b*x)\wedge A(x*b)\wedge A(0)\wedge\mu=A(b*x)\wedge A(x*b)\wedge\mu\\&=A_b(x),\end{aligned}$$

that is $A_a\geq A_b$. Similarly, for all $x\in X$, since

$$(b*x)*A(a*x)\leq b*a,(x*b)*A(x*a)\leq a*b.$$

It follows from Proposition 1 that

$$A(b*x)=A(b*x)\vee\lambda\geq A(a*x)\wedge A(b*a)\wedge\mu,$$
$$A(x*b)=A(x*b)\vee\lambda\geq A(x*a)\wedge A(a*b)\wedge\mu.$$

Hence

$$\begin{aligned}A_b(x)&=A(b*x)\wedge A(x*b)\wedge\mu\\&\geq A(a*x)\wedge A(x*a)\wedge A(b*a)\wedge A(a*b)\wedge\mu\\&=A(a*x)\wedge A(x*a)\wedge A(0)\wedge\mu\\&=A(a*x)\wedge A(x*a)\wedge\mu\\&=A_a(x),\end{aligned}$$

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that is $A_b \geq A_a$. Therefore, $A_a = A_b$. We complete the proof.

**Proposition 9.** Let $A_a = A_{a'}, A_b = A_{b'}$, then $A_{a*b} = A_{a'*b'}$.

**Proof.** Since

$$\begin{aligned}&((a*b)*(a'*b'))*(a*a') = ((a*b)*(a*a'))*(a'*b')\\ &\leq (a'*b)*(a'*b') \leq b'*b,\\ &((a'*b')*(a*b))*(b*b') = ((a'*b')*(b*b'))*(a*b)\\ &\leq (a'*b)*(a*b) \leq a'*a.\end{aligned}$$

Hence

$$\begin{aligned}&A((a*b)*(a'*b')) = A((a*b)*(a'*b')) \vee \lambda\\ &\geq A(a*a') \wedge A(b'*b) \wedge \mu,\\ &A((a'*b')*(a*b)) = A((a'*b')*(a*b)) \vee \lambda\\ &\geq A(b*b') \wedge A(a'*a) \wedge \mu.\end{aligned}$$

Therefore

$$\begin{aligned}&A((a*b)*(a'*b')) \wedge A((a'*b')*(a*b)) \wedge \mu\\ &= A(a*a') \wedge A(a'*a) \wedge \mu \wedge A(b*b') \wedge A(b'*b) \wedge \mu \wedge \mu\\ &= A(0) \wedge \mu,\end{aligned}$$

it follows from Proposition 8 that $A_{a*b} = A_{a'*b'}$, we completed the proof. Let $A$ be a generalized $M-$fuzzy ideal of $X$, the operation "*" of $R/A$ is defined as follows: $\forall A_a, A_b \in R/A, A_a * A_b = A_{a*b}$. By Proposition 8, the above operation is reasonable.

**Proposition 10.** Let $A$ be a generalized $M-$fuzzy ideal of $X$, then $R/A = \{R/A; *, A_0\}$ is an $M-$BCI-algebra.

**Proof.** For all $A_x, A_y, A_z \in R/A$,

$$((A_x * A_y) * (A_x * A_z)) * (A_z * A_y) = A_{((x*y)*(x*z))*(z*y)} = A_0;$$

$$(A_x * (A_x * A_y)) * A_y = A_{(x*(x*y))*y} = A_0;$$
$$A_x * A_x = A_{x*x} = A_0;$$

if $A_x * A_y = A_0, A_y * A_x = A_0$, then $A_{x*y} = A_0, A_{y*x} = A_0$, it follows from Proposition 8 that $A(x*y) = A(0), A(y*x) = A(0)$, hence $A(x*y) \wedge A(y*x) \wedge \mu = A(0) \wedge \mu$, then $A_x = A_y$. Therefore $R/A = \{R/A; *, A_0\}$ is a BCI-algebra. For all $A_x \in R/A, m \in M$, we define $mA_x = A_{mx}$. Firstly, we verify that $mA_x = A_{mx}$ is reasonable. If $A_x = A_y$, then we verify $mA_x = mA_y$, that is to verify $A_{mx} = A_{my}$. We have

$$A(mx*my) \wedge \mu = A(m(x*y)) \wedge \mu \geq A(x*y) \wedge \mu = A(0) \wedge \mu,$$

$$A(my*mx) \wedge \mu = A(m(y*x)) \wedge \mu \geq A(y*x) \wedge \mu = A(0) \wedge \mu,$$

so we have $A(mx*my) \wedge A(my*mx) \wedge \mu = A(0) \wedge \mu$, that is, $A_{mx} = A_{my}$. In addition, for all $m \in M, A_x, A_y \in R/A$,

$$\begin{aligned}m(A_x * A_y) &= mA_{x*y} = A_{m(x*y)} = A_{(mx)*(my)}\\ &= A_{mx} * A_{my} = mA_x * mA_y.\end{aligned}$$

Therefore $R/A = \{R/A; *, A_0\}$ is an $M-$BCI-algebra.

**Definition 9.** Let $\mu$ be a generalized $M-$fuzzy subalgebra of $X$, and $A$ be a generalized $M-$fuzzy ideal of $X$, we define a fuzzy set of $X/A$ as follows:

$$\mu/A: X/A \to [0,1],$$
$$\mu/A(A_i) \vee \lambda = \sup_{A_x = A_i} \mu(x) \wedge \mu, \forall A_i \in X/A.$$

**Proposition 11.** $\mu/A$ is a generalized $M-$fuzzy subalgebra of $X/A$.

**Proof.** For all $A_x, A_y \in X/A$, we have

$$\begin{aligned}&\mu/A(A_x * A_y) \vee \lambda = \mu/A(A_{x*y}) \vee \lambda = \sup_{A_z = A_{x*y}} \mu(z) \wedge \mu\\ &\geq \sup_{A_s = A_x, A_t = A_y} \mu(s*t) \wedge \mu \geq \sup_{A_s = A_x, A_t = A_y} \mu(s) \wedge \mu(t) \wedge \mu\\ &= \sup_{A_s = A_x,} \mu(s) \wedge \sup_{A_t = A_y,} \mu(t) \wedge \mu = \mu/A(A_x) \wedge \mu/A(A_y) \wedge \mu.\end{aligned}$$

For all $m \in M, A_x \in R/A$, we have

$$\begin{aligned}&\mu/A(A_{mx}) \vee \lambda = \sup_{A_{mz} = A_{mx}} \mu(mz) \wedge \mu\\ &\geq \sup_{A_z = A_x} \mu(z) \wedge \mu = \mu/A(A_x) \wedge \mu.\end{aligned}$$

Therefore $\mu/A$ is a generalized $M-$fuzzy subalgebra of $X/A$.

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