

# Fusion Filters Weighted by Scalars and Matrices for Linear Systems

Seok Hyoung Lee, and Vladimir Shin

**Abstract**—An optimal mean-square fusion formulas with scalar and matrix weights are presented. The relationship between them is established. The fusion formulas are compared on the continuous-time filtering problem. The basic differential equation for cross-covariance of the local errors being the key quantity for distributed fusion is derived. It is shown that the fusion filters are effective for multi-sensor systems containing different types of sensors. An example demonstrating the reasonable good accuracy of the proposed filters is given.

**Keywords**—Kalman filtering, fusion formula, multi-sensor, mean-square error.

## I. INTRODUCTION

IN recent years, there has been growing interest to fuse multi-sensor data in order to increase the accuracy of parameter estimates and system states. This interest is motivated by the availability of different types of sensors which use various characteristics of the optical, infrared, and electromagnetic spectra. In many situations, system states or targets are measured by multi-sensors. Many techniques for distributed estimation fusion are presented in [1]-[4] and references therein. The measurements used in the estimation process are assigned to a common target as a result of the association process. There is a problem of how to combine local estimates obtained from different types of sensors.

Several distributed fusion architectures were discussed in [3], [4]. Some algorithms for distributed estimation fusion, which are for finding the “best” linear combination of the local estimates have been developed in [4], [6]-[8]. The Bar-Shalom and Campo fusion formula (FF) for two-sensors systems [5] has been generalized for an arbitrary number of sensors in [7]-[9]. The FF represents an optimal mean-square linear combination of the local estimates with the matrix weights satisfying linear algebraic equations [7], [8]. The explicit and recursive formulas for the matrix weights have been derived in [9]. The FF with scalar weights has been presented in [10]-[12].

The main purpose of this paper is to establish relation between the FFs with matrix and scalar weights, and compare their accuracy on the filtering problems.

This paper is organized as follows. In Section 2, we present

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the FFs with matrix and scalar weights and demonstrate some interesting particular cases, including relationship with the Bar-Shalom and Campo formula. In Section 3, we theoretically establish the relation between the FFs. In Section 4, we apply the FFs to the filtering problem in multi-sensor continuous-time linear systems. The equation for local cross-covariance being the key for usage of the FFs is derived. In Section 5, the fusion filters based on the FFs are verified and compared via simulations. In Section 6, the conclusions are made.

## II. FUSION FORMULAS

Suppose that we have  $N$  local estimates of an unknown random vector  $\mathbf{x} \in \mathcal{R}^n$ ,

$$\hat{\mathbf{x}}^{(1)}, \dots, \hat{\mathbf{x}}^{(N)} \quad (1)$$

where  $\mathcal{R}^n$  is an  $n$ -dimensional Euclidean space. The associated error cross-covariances are given,

$$\mathbf{P}^{(ij)} = \text{cov}(\mathbf{e}^{(i)}, \mathbf{e}^{(j)}), \quad \mathbf{e}^{(i)} = \mathbf{x} - \hat{\mathbf{x}}^{(i)}, \quad i, j = 1, \dots, N. \quad (2)$$

Let consider two linear combinations of the local estimates (1) with matrix and scalar weights. We have

$$\hat{\mathbf{x}}^{\text{FFM}} = \sum_{i=1}^N \mathbf{A}^{(i)} \hat{\mathbf{x}}^{(i)}, \quad \sum_{i=1}^N \mathbf{A}^{(i)} = \mathbf{I}_n, \quad (3)$$

$$\hat{\mathbf{x}}^{\text{FFS}} = \sum_{i=1}^N \mathbf{a}^{(i)} \hat{\mathbf{x}}^{(i)}, \quad \sum_{i=1}^N \mathbf{a}^{(i)} = 1, \quad (4)$$

where  $\mathbf{I}_n$  is an  $n \times n$  identity matrix,  $\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(N)}$  are  $n \times n$  constant weight matrices, and  $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(N)}$  are scalars. The matrices  $\mathbf{A}^{(i)}$  and scalars  $\mathbf{a}^{(i)}$  are determined from the mean-square criterions

$$\mathbf{J}^{\mathbf{A}} = \mathbb{E} \|\mathbf{x} - \hat{\mathbf{x}}^{\text{FFM}}\|^2 \rightarrow \min_{\mathbf{A}^{(i)}}, \quad \mathbf{J}^{\mathbf{a}} = \mathbb{E} \|\mathbf{x} - \hat{\mathbf{x}}^{\text{FFS}}\|^2 \rightarrow \min_{\mathbf{a}^{(i)}}, \quad (5)$$

respectively.

We call the formula (3) as *the fusion formula with matrix weights (FFM)*. And analogously the formula (4) is called *the fusion formula with scalar weights (FFS)*.

The following results completely define the FFM and FFS, and also the fusion error covariances,

$$\mathbf{P}^{\text{FFM}} = \mathbf{E} \left( \mathbf{e}^{\text{FFM}} \left( \mathbf{e}^{\text{FFM}} \right)^T \right), \quad \mathbf{e}^{\text{FFM}} = \mathbf{x} - \hat{\mathbf{x}}^{\text{FFM}}, \quad (6)$$

$$\mathbf{P}^{\text{FFS}} = \mathbf{E} \left( \mathbf{e}^{\text{FFS}} \left( \mathbf{e}^{\text{FFS}} \right)^T \right), \quad \mathbf{e}^{\text{FFS}} = \mathbf{x} - \hat{\mathbf{x}}^{\text{FFS}}. \quad (7)$$

respectively.

**Theorem 1** [7]-[9]. Let  $\hat{\mathbf{x}}^{(1)}, \dots, \hat{\mathbf{x}}^{(N)}$  be the local estimates (1) of an unknown vector  $\mathbf{x}$ . Then the matrix weight matrices  $\mathbf{A}^{(1)}, \dots, \mathbf{A}^{(N)} \in \mathfrak{R}^{n \times n}$  satisfy the following linear algebraic equations:

$$\sum_{i=1}^N \mathbf{A}^{(i)} (\mathbf{P}^{(ij)} - \mathbf{P}^{(in)}) = 0, \quad j=1, \dots, N-1, \quad \sum_{i=1}^N \mathbf{A}^{(i)} = \mathbf{I}_n \quad (8)$$

and they can be explicitly written down as

$$\mathbf{A}^{(i)} = \sum_{k=1}^N (\mathbf{P}^{(ik)})^{-1} \left( \sum_{j=1}^N (\mathbf{P}^{(ij)})^{-1} \right)^{-1}, \quad i=1, \dots, N. \quad (9)$$

**Theorem 2** [10]-[12]. Let  $\hat{\mathbf{x}}^{(1)}, \dots, \hat{\mathbf{x}}^{(N)}$  be the local estimates (1) of an unknown vector  $\mathbf{x}$ . Then the scalar weights  $a^{(1)}, \dots, a^{(N)} \in \mathfrak{R}$  satisfy the following linear algebraic equations:

$$\sum_{i=1}^N a^{(i)} \text{tr}(\mathbf{P}^{(ij)} + \mathbf{P}^{(ji)} - \mathbf{P}^{(in)} - \mathbf{P}^{(ni)}) = 0, \quad \sum_{i=1}^N a^{(i)} = 1, \quad (10)$$

$j=1, \dots, N-1$

and they can be explicitly written down as

$$\mathbf{a} = \frac{\mathbf{C}^{-1} \mathbf{e}}{\mathbf{e}^T \mathbf{C}^{-1} \mathbf{e}}, \quad \mathbf{a} = [a^{(1)} \dots a^{(N)}]^T, \quad \mathbf{e} = [1 \ 1 \ \dots \ 1]^T, \quad (11)$$

$$\mathbf{C} = [\text{tr}(\mathbf{P}^{(ij)})]_{i,j=1, \dots, N} \in \mathfrak{R}^{N \times N}, \quad \mathbf{a}, \mathbf{e} \in \mathfrak{R}^N.$$

In (10), (11)  $\text{tr}(\mathbf{X})$  is the trace of matrix  $\mathbf{X}$ .

**Corollary 1.** If the local estimates  $\hat{\mathbf{x}}^{(1)}, \dots, \hat{\mathbf{x}}^{(N)}$  are unbiased then the fusion estimates  $\hat{\mathbf{x}}^{\text{FFM}}$  and  $\hat{\mathbf{x}}^{\text{FFS}}$  are also unbiased.

**Corollary 2.** The fusion error covariances  $\mathbf{P}^{\text{FFM}}$  and  $\mathbf{P}^{\text{FFS}}$  are given by

$$\mathbf{P}^{\text{FFM}} = \sum_{i,j=1}^N \mathbf{A}^{(i)} \mathbf{P}^{(ij)} \mathbf{A}^{(j)T}, \quad \mathbf{P}^{\text{FFS}} = \sum_{i,j=1}^N a^{(i)} a^{(j)} \mathbf{P}^{(ij)}. \quad (12)$$

**Example 1: Fusion of Two Vector Estimates**

In the particular case with  $N=2$ , the FFM (3), (8) is reduced to the Bar-Shalom and Campo formula [5]:

$$\hat{\mathbf{x}}^{\text{FFM}} = \mathbf{A}^{(1)} \hat{\mathbf{x}}^{(1)} + \mathbf{A}^{(2)} \hat{\mathbf{x}}^{(2)}, \quad \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathfrak{R}^n, \quad (13)$$

$$\mathbf{A}^{(1)} = (\mathbf{P}^{(22)} - \mathbf{P}^{(21)})(\mathbf{P}^{(11)} + \mathbf{P}^{(22)} - \mathbf{P}^{(12)} - \mathbf{P}^{(21)})^{-1},$$

$$\mathbf{A}^{(2)} = (\mathbf{P}^{(11)} - \mathbf{P}^{(12)})(\mathbf{P}^{(11)} + \mathbf{P}^{(22)} - \mathbf{P}^{(12)} - \mathbf{P}^{(21)})^{-1}.$$

and the FFS (4), (10) takes the form

$$\hat{\mathbf{x}}^{\text{FFS}} = a^{(1)} \hat{\mathbf{x}}^{(1)} + a^{(2)} \hat{\mathbf{x}}^{(2)}, \quad \mathbf{x}^{(1)}, \mathbf{x}^{(2)} \in \mathfrak{R}^n, \quad (14)$$

$$a^{(1)} = \frac{\text{tr}(\mathbf{P}^{(22)} - \mathbf{P}^{(12)} - \mathbf{P}^{(21)})}{2\text{tr}(\mathbf{P}^{(11)} + \mathbf{P}^{(22)} - \mathbf{P}^{(12)} - \mathbf{P}^{(21)})},$$

$$a^{(2)} = \frac{\text{tr}(\mathbf{P}^{(11)} - \mathbf{P}^{(12)} - \mathbf{P}^{(21)})}{2\text{tr}(\mathbf{P}^{(11)} + \mathbf{P}^{(22)} - \mathbf{P}^{(12)} - \mathbf{P}^{(21)})}.$$

**Example 2: Fusion of the Uncorrelated Vector N-estimates**

Suppose that the local vector estimates  $\hat{\mathbf{x}}^{(1)}, \dots, \hat{\mathbf{x}}^{(N)} \in \mathfrak{R}^n$  are uncorrelated, i.e.,  $\mathbf{P}^{(ij)} = 0, \quad i \neq j$ . Then the linear systems (8) and (10) have the following explicit solutions:

$$\mathbf{A}^{(i)} = \left[ \mathbf{P}^{(ii)} \sum_{j=1}^N (\mathbf{P}^{(jj)})^{-1} \right]^{-1}, \quad \mathbf{a}^{(i)} = \left[ \text{tr}(\mathbf{P}^{(ii)}) \sum_{j=1}^N \frac{1}{\text{tr}(\mathbf{P}^{(jj)})} \right]^{-1}, \quad i=1, \dots, N. \quad (15)$$

For the scalar uncorrelated estimates  $\hat{x}^{(1)}, \dots, \hat{x}^{(N)} \in \mathfrak{R}$  the formulas (15) coincide, i.e.,

$$\mathbf{A}^{(i)} = a^{(i)} = \frac{1}{\mathbf{P}^{(ii)}} \left( \frac{1}{\mathbf{P}^{(11)}} + \dots + \frac{1}{\mathbf{P}^{(NN)}} \right)^{-1}, \quad \mathbf{P}^{(ii)} = \mathbf{E}(\mathbf{x} - \hat{\mathbf{x}}^{(i)})^2, \quad i=1, \dots, N. \quad (16)$$

### III. RELATIONSHIP BETWEEN FFM AND FFS

Here we establish relationship between the mean-square errors (MSEs)  $\mathbf{J}^A = \text{tr}(\mathbf{P}^{\text{FFM}})$  and  $\mathbf{J}^a = \text{tr}(\mathbf{P}^{\text{FFS}})$ .

Let the matrix weights  $\mathbf{A}^{(i)}$  have the diagonal form with identical elements  $a^{(i)}$ , i.e.,  $\mathbf{A}^{(i)} = \text{diag}[a^{(i)}, \dots, a^{(i)}], \quad i=1, \dots, N$ . Then it is evident that the FFM coincides with FFS and

$$\mathbf{J}^A = \mathbf{J}^a. \quad (17)$$

Further, let consider one example in which  $\mathbf{J}^A < \mathbf{J}^a$ .

**Example 3: Fusion Least Square Estimation of a Constant Vector Unknown**

If  $\mathbf{x} \in \mathfrak{R}^2$  is a constant vector measured in additive white noise by using two independent sensors then the measurement model is

$$y_1 = \mathbf{x} + \mathbf{v}_1, \quad y_2 = \mathbf{x} + \mathbf{v}_2, \quad \mathbf{x}, y_1, y_2 \in \mathfrak{R}^2, \quad (18)$$

$$\mathbf{v}_1 \sim \mathcal{N}(0, \mathbf{Q}_1), \quad \mathbf{v}_2 \sim \mathcal{N}(0, \mathbf{Q}_2), \quad \mathbf{E}(\mathbf{v}_1 \mathbf{v}_2^T) = 0,$$

$$\mathbf{Q}_1 = \text{diag}[\sigma_1^2 \quad \sigma_2^2], \quad \mathbf{Q}_2 = \text{diag}[\sigma_3^2 \quad \sigma_4^2].$$

Then the local least square estimates  $\hat{\mathbf{x}}^{(1)}$  and  $\hat{\mathbf{x}}^{(2)}$  based on the single measurements  $y_1$  and  $y_2$ , respectively, take the following form:  $\hat{\mathbf{x}}^{(1)} = y_1$  and  $\hat{\mathbf{x}}^{(2)} = y_2$ . With the assumption that  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are uncorrelated, the local cross-covariances  $\mathbf{P}^{(ij)} = \mathbf{E}[(\mathbf{x} - \hat{\mathbf{x}}^{(i)})(\mathbf{x} - \hat{\mathbf{x}}^{(j)})]$ ,  $i=1, 2$  are

$$\mathbf{P}^{(11)} = \mathbf{E}(\mathbf{v}_1 \mathbf{v}_1^T) = \mathbf{Q}_1, \quad \mathbf{P}^{(22)} = \mathbf{E}(\mathbf{v}_2 \mathbf{v}_2^T) = \mathbf{Q}_2, \quad (19)$$

$$\mathbf{P}^{(12)} = \mathbf{E}(\mathbf{v}_1 \mathbf{v}_2^T) = 0, \quad \mathbf{P}^{(21)} = \mathbf{P}^{(12)T} = 0.$$

Substituting (19) into (12)-(14), we obtain precise formulas for the FFM's and FFS's weights, and the corresponding fusion MSEs:

$$\begin{aligned} A^{(1)} &= P^{(22)}(P^{(11)} + P^{(22)})^{-1} = \begin{bmatrix} \sigma_3^2/(\sigma_1^2 + \sigma_3^2) & 0 \\ 0 & \sigma_4^2/(\sigma_2^2 + \sigma_4^2) \end{bmatrix}, \\ A^{(2)} &= P^{(11)}(P^{(11)} + P^{(22)})^{-1} = \begin{bmatrix} \sigma_1^2/(\sigma_1^2 + \sigma_3^2) & 0 \\ 0 & \sigma_2^2/(\sigma_2^2 + \sigma_4^2) \end{bmatrix}, \\ a^{(1)} &= \text{tr}(P^{(22)})/\text{tr}(P^{(11)} + P^{(22)}) = (\sigma_3^2 + \sigma_4^2)/(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2), \\ a^{(2)} &= \text{tr}(P^{(11)})/\text{tr}(P^{(11)} + P^{(22)}) = (\sigma_1^2 + \sigma_2^2)/(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2), \\ J^A &= \text{tr}[P^{\text{FFM}}] = \text{tr}[A^{(1)^2}P^{(11)} + A^{(2)^2}P^{(22)}] = \frac{\sigma_1^2\sigma_3^2}{\sigma_1^2 + \sigma_3^2} + \frac{\sigma_2^2\sigma_4^2}{\sigma_2^2 + \sigma_4^2}, \\ J^a &= \text{tr}[P^{\text{FFS}}] = a^{(1)^2}\text{tr}(P^{(11)}) + a^{(2)^2}\text{tr}(P^{(22)}) = \frac{(\sigma_1^2 + \sigma_2^2)(\sigma_3^2 + \sigma_4^2)}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}. \end{aligned} \quad (20)$$

In general, comparing  $J^A$  and  $J^a$ , we obtain

$$J^A < J^a. \quad (21)$$

Note that the precise equality  $J^A = J^a$  is achieved in the case that  $\sigma_2^2\sigma_3^2 = \sigma_1^2\sigma_4^2$ .

Thus the results (17) and (21) yield the following:

**Theorem 3.** Let FFM and FFS be fusion formulas determined by (8), (9) and (10), (11), respectively. Then

$$J^A \leq J^a. \quad (22)$$

#### IV. FUSION OF LOCAL KALMAN ESTIMATES

Consider a continuous-time linear dynamic system with additive white Gaussian noise

$$\dot{x}_t = F_t x_t + G_t v_t, \quad t \geq 0, \quad (23)$$

where  $x_t \in \mathbb{R}^n$  is a state vector, and  $v_t \in \mathbb{R}^r$  is a zero-mean white Gaussian noise with intensity  $Q_t$ ,  $E(v_t v_s^T) = Q_t \delta(t-s)$ .

Suppose that the measurement system involves  $N$  sensors

$$\begin{aligned} y_t^{(1)} &= H_t^{(1)} x_t + w_t^{(1)}, \quad y_t^{(1)} \in \mathbf{R}^{m_1}, \\ &\dots \dots \\ y_t^{(N)} &= H_t^{(N)} x_t + w_t^{(N)}, \quad y_t^{(N)} \in \mathbf{R}^{m_N}, \end{aligned} \quad (24)$$

where  $w_t^{(i)}$  are a zero mean white Gaussian noises with intensities  $R_t^{(i)}$ ,  $i=1, \dots, N$ . We assume that the initial state  $x_0 \sim N(\bar{x}_0, P_0)$  and the white noises  $v_t, w_t^{(1)}, \dots, w_t^{(N)}$  are mutually uncorrelated.

Then the Kalman filter (KF) can be used to produce the optimal mean square state estimate based on the overall sensor

measurements  $Y_t = \{y_t^{(1)}, \dots, y_t^{(N)}\}$ ,

$$Y_t = H_t x_t + w_t, \quad (25)$$

where

$$Y_t = [y_t^{(1)T}, \dots, y_t^{(N)T}]^T, \quad H_t = [H_t^{(1)T}, \dots, H_t^{(N)T}]^T, \quad (26)$$

$$w_t = [w_t^{(1)T}, \dots, w_t^{(N)T}]^T.$$

The KF is the centralized filter, where all measured sensor data are communicated to the central site for processing. The advantage of this filter is that there is a minimal information loss. However, the centralized filter may be unreliable or suffer from poor accuracy and stability when there is severe data fault. The second method is the decentralized, where information from local filters can yield the optimal or suboptimal fusion filter according to some information fusion criterion. The advantages of this method are that the requirement of memory space to the fusion center is broadened, and the parallel structures can increase the input data rates. However, the precision of the decentralized filter is generally lower than that of the centralized filter when there is not data fault. Recently, various decentralized and parallel versions of the KF have been reported [3], [4], [7]-[12].

However, the purpose of the paper to compare the accuracy of two decentralized KFs based on the FFM and FFS.

According to (23) and (24), we have  $N$  dynamic subsystems ( $i=1, \dots, N$ ) with the common state  $x_t$  and individual (local) sensor  $y_t^{(i)}$ :

$$\begin{aligned} \dot{x}_t &= F_t x_t + G_t v_t, \quad v_t \sim (0, Q_t), \\ y_t^{(i)} &= H_t^{(i)} x_t + w_t^{(i)}, \quad w_t^{(i)} \sim N(0, R_t^{(i)}), \end{aligned} \quad (27)$$

where the number  $i$  of the subsystem is fixed.

Further, denote a local estimate of the state  $x_t$  based on the local sensor measurement  $y_t^{(i)}$  by  $\hat{x}_t^{(i)}$ . To find  $\hat{x}_t$ , we apply the KF to the subsystem (27) and obtain

$$\begin{aligned} \dot{\hat{x}}_t^{(i)} &= F_t \hat{x}_t^{(i)} + P_t^{(ii)} H_t^{(i)T} R_t^{(i)-1} (y_t^{(i)} - H_t^{(i)} \hat{x}_t^{(i)}), \\ \dot{P}_t^{(ii)} &= F_t P_t^{(ii)} + P_t^{(ii)} F_t^T - P_t^{(ii)} H_t^{(i)T} R_t^{(i)-1} P_t^{(ii)} + \tilde{Q}_t, \\ \tilde{Q}_t &= G_t Q_t G_t^T, \quad \hat{x}_0^{(i)} = \bar{x}_0, \quad P_0^{(ii)} = P_0, \\ P_t^{(ii)} &= E(\tilde{x}_t^{(i)} \tilde{x}_t^{(i)T}), \quad \tilde{x}_t = x_t - \hat{x}_t^{(i)}. \end{aligned} \quad (28)$$

Thus we have  $N$  local Kalman estimates and corresponding error covariances.

$$\hat{x}_t^{(1)}, P_t^{(11)}, \dots, \hat{x}_t^{(N)}, P_t^{(NN)}. \quad (29)$$

To express the final fusion estimate of the state on terms of the local Kalman estimates  $\hat{x}_t^{(1)}, \dots, \hat{x}_t^{(N)}$ , we use the FFM and FFS. From (3) and (4) we have

$$\hat{\mathbf{x}}_t^{\text{FFM}} = \sum_{i=1}^N \mathbf{A}_t^{(i)} \hat{\mathbf{x}}_t^{(i)}, \quad \hat{\mathbf{x}}_t^{\text{FFS}} = \sum_{i=1}^N \mathbf{a}_t^{(i)} \hat{\mathbf{x}}_t^{(i)}. \quad (30)$$

The linear equations (8) and (10) for the unknown weights  $\mathbf{A}^{(i)}$  and  $\mathbf{a}^{(i)}$  are given by

$$\begin{aligned} \sum_{i=1}^N \mathbf{A}_t^{(i)} (\mathbf{P}_t^{(ij)} - \mathbf{P}_t^{(in)}) &= 0, \quad \sum_{i=1}^N \mathbf{A}_t^{(i)} = \mathbf{I}_n, \\ \sum_{i=1}^N \mathbf{a}_t^{(i)} \text{tr}(\mathbf{P}_t^{(ij)} + \mathbf{P}_t^{(ji)} - \mathbf{P}_t^{(in)} - \mathbf{P}_t^{(ni)}) &= 0, \quad \sum_{i=1}^N \mathbf{a}_t^{(i)} = 1, \\ j &= 1, \dots, N-1. \end{aligned} \quad (31)$$

Note that (31) depend on the local error covariances  $\mathbf{P}_t^{(ii)}$ ,  $i = 1, \dots, N$  determined by the Riccati equations (28), and the cross-covariances  $\mathbf{P}_t^{(ij)} = E(\hat{\mathbf{x}}_t^{(i)} \hat{\mathbf{x}}_t^{(j)T})$ ,  $i \neq j$ . Knowledge of the cross-covariance is needed for distributed fusion. It is a key quantity for the linear fusion estimation. The cross-covariance  $\mathbf{P}_t^{(ij)}$  satisfies the following differential equation:

$$\begin{aligned} \dot{\mathbf{P}}_t^{(ij)} &= [\mathbf{F}_t - \mathbf{K}_t^{(i)} \mathbf{H}_t^{(i)}] \mathbf{P}_t^{(ij)} + \mathbf{P}_t^{(ij)} [\mathbf{F}_t - \mathbf{K}_t^{(j)} \mathbf{H}_t^{(j)}]^T + \tilde{\mathbf{Q}}_t, \\ \mathbf{K}_t^{(i)} &= \mathbf{P}_t^{(ii)} \mathbf{H}_t^{(i)T} \mathbf{R}_t^{(i)-1}, \quad \mathbf{P}_0^{(ij)} = \mathbf{P}_0, \quad i, j = 1, \dots, N, \quad i \neq j. \end{aligned} \quad (32)$$

The derivation of (32) is given in Appendix.

Thus, the local estimates and covariances ( $\hat{\mathbf{x}}_t^{(i)}$ ,  $\mathbf{P}_t^{(ii)}$ ), the local cross-covariances  $\mathbf{P}_t^{(ij)}$ ,  $i \neq j$ , and the equations for weights (31) completely establish the fusion estimates (30).

*Remark 1.* The local Kalman estimates (28) are separated for different sensors, i.e., each estimate  $\hat{\mathbf{x}}_t^{(i)}$  is found independently of other estimates. Therefore, they can be evaluated in parallel.

*Remark 2.* The fusion estimates  $\hat{\mathbf{x}}_t^{\text{FFM}}$  and  $\hat{\mathbf{x}}_t^{\text{FFS}}$  can be corrected if one of the parallel local estimate  $\hat{\mathbf{x}}_t^{(i)}$  diverges. In this case, the corresponding weight  $\mathbf{A}_t^{(i)}$  (or  $\mathbf{a}_t^{(i)}$ ) tends to zero, thereby indicating that the diverging estimate  $\hat{\mathbf{x}}_t^{(i)}$  is discarded in the weighting sum (30).

*Remark 3.* All local error covariances  $\mathbf{P}_t^{(ij)}$ , and the weights  $\mathbf{A}_t^{(i)}$ ,  $\mathbf{a}_t^{(i)}$  can be pre-computed, since they do not depend on the current measurements, but only on the noise statistics, and the system matrices, which are the part of system model (23), (24). Thus, once the measurement schedule has been settled, the real-time implementation of the fusion filters requires only the computation of the local Kalman estimates  $\hat{\mathbf{x}}_t^{(1)}, \dots, \hat{\mathbf{x}}_t^{(N)}$  and the final fusion estimates  $\hat{\mathbf{x}}_t^{\text{FFM}}$  or  $\hat{\mathbf{x}}_t^{\text{FFS}}$ .

## V. EXAMPLE: A PIPELINE

Consider an industrial complex including two factories which need water for production. In order to supply required water for the factories, a special structure of pipelines should be used, because there are situations when one of the factories needs more water than another (see Fig. 1).

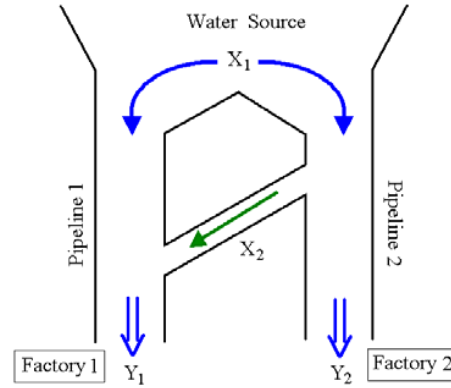


Fig. 1 Structure of pipelines

For supply water control, we need know the amount of water in each pipeline at any time instant. Assuming that the function of the water flow is like that  $x_{\text{water}} = e^{-at}$ , then the system model for  $x_{1,t}$  and  $x_{2,t}$  can be following

$$\dot{\mathbf{x}}_t = \begin{bmatrix} \dot{x}_{1,t} \\ \dot{x}_{2,t} \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} v_{1,t} \\ v_{2,t} \end{bmatrix}, \quad t \geq 0, \quad (33)$$

where we have added zero-mean white Gaussian noise  $\mathbf{v}_t = [v_{1,t} \ v_{2,t}]^T$  to compensate system uncertainties,  $E(\mathbf{v}_t \mathbf{v}_s^T) = \mathbf{Q}_t \delta(t-s)$ ,  $\mathbf{Q}_t = \text{diag}[q_1 \ q_2]$ ,  $\alpha, \beta > 0$  are known constants,  $\mathbf{x}_0 \sim N(\bar{\mathbf{x}}_0, \mathbf{P}_0)$ .

The amount of water in each pipeline is observable through the measurements. According to Fig. 1 we have

$$\begin{aligned} y_t^{(1)} &= x_{1,t} + x_{2,t} + w_t^{(1)}, \\ y_t^{(2)} &= x_{1,t} - x_{2,t} + w_t^{(2)}, \end{aligned} \quad (34)$$

where  $\mathbf{w}_t = [w_{1,t} \ w_{2,t}]^T$  is zero-mean white Gaussian noise with intensity  $\mathbf{R}_t = \text{diag}[r_1 \ r_2]$ .

The two fusion filters (FFM and FFS) for the system model (33), (34) are considered and compared with the optimal KF. To study the behavior of the KF, FFM and FFS error covariances, set

$$\alpha = 0.1, \quad \beta = 0.7, \quad \bar{\mathbf{x}}_0 = [15 \ 4]^T, \quad \mathbf{P}_0 = \text{diag}[0.7 \ 0.7],$$

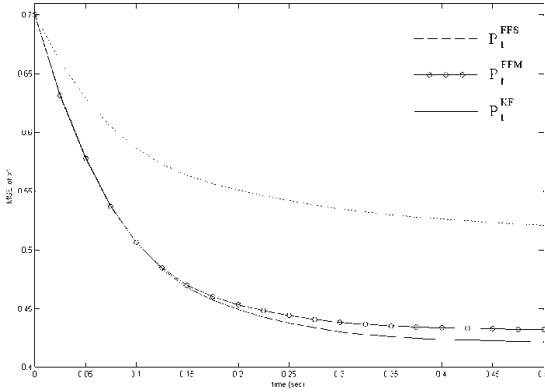
$$q_1 = 0.3, \quad q_2 = 0.2, \quad r_1 = 2, \quad r_2 = 3.$$

The point of interest is the MSEs of the state components  $x_{1,t}$  and  $x_{2,t}$ ,

$$P_t^{KF} = E[(x_t - \hat{x}_t^{KF})(x_t - \hat{x}_t^{KF})^T] = \begin{bmatrix} P_{11,t}^{KF} & P_{12,t}^{KF} \\ P_{21,t}^{KF} & P_{22,t}^{KF} \end{bmatrix},$$

$$P_t^{FFM} = E[(x_t - \hat{x}_t^{FFM})(x_t - \hat{x}_t^{FFM})^T] = \begin{bmatrix} P_{11,t}^{FFM} & P_{12,t}^{FFM} \\ P_{21,t}^{FFM} & P_{22,t}^{FFM} \end{bmatrix},$$

$$P_t^{FFS} = E[(x_t - \hat{x}_t^{FFS})(x_t - \hat{x}_t^{FFS})^T] = \begin{bmatrix} P_{11,t}^{FFS} & P_{12,t}^{FFS} \\ P_{21,t}^{FFS} & P_{22,t}^{FFS} \end{bmatrix}.$$

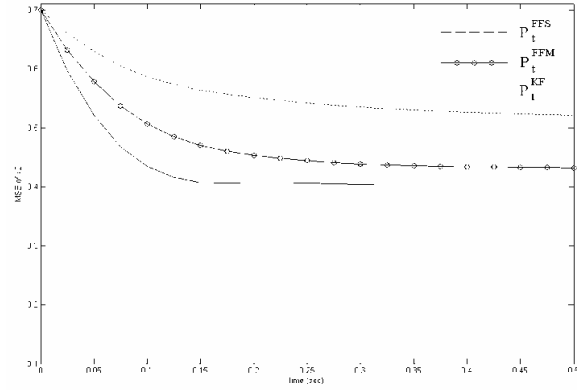
Fig. 2 Comparison of MSEs for  $x_{1,t}$ 

These quantities are shown in Figs. 2 and 3. From these Figs. it follows that the differences between the optimal MSEs ( $P_{11,k}^{KF}, P_{22,k}^{KF}$ ) and the fusion MSEs ( $P_{11,k}^{FFM}, P_{22,k}^{FFM}$ ) and ( $P_{11,k}^{FFS}, P_{22,k}^{FFS}$ ) are not negligible both for the first amount  $x_{1,t}$  and for the second one  $x_{2,t}$ .

As the relationship between FFM and FFS has been already discussed theoretically in chapter 3, we can see that the fusion MSEs follow the relationship of the result (22).

$$P_t^{FFM} \leq P_t^{FFS}. \quad (35)$$

Besides, we can also observe that the fusion filter with matrix weights yields a good estimate accuracy as compared to the optimal Kalman filter, while the fusion filter with scalar weights produces a poor estimate accuracy relatively

Fig. 3 Comparison of MSEs for  $x_{2,t}$ 

## VI. CONCLUSION

In this paper we compare two fusion formulas with matrix and scalar weights. The rigorous relationship between their mean square errors is established (Theorem 3).

Two suboptimal fusion filters for continuous-time linear systems with multi-sensor environment are proposed. The key differential equation for the local cross-covariance which influence on the accuracy of distributed fusion is derived (Eq. (32)).

However it is not sufficient to show the only theoretical results. Even though the relationship is clear, sometimes theoretical result can be thought of as pedantic or a little bit vague. So, a numerical example is given to support the theoretical results, and through them the theoretical relationship between two fusion formulas has confirmed. It was also shown that the fusion filters yield a reasonably good estimation accuracy, especially for steady-state regime. The obtained fusion filters with matrix weights are slightly suboptimal as compare with the optimal centralized KF.

## APPENDIX: DERIVATION OF EQ. (32)

The KF equations (28) yield the following differential equation for the local error  $\tilde{x}_t^{(i)} = x_t - \hat{x}_t^{(i)}$ :

$$\begin{aligned} \dot{\tilde{x}}_t^{(i)} &= \dot{x}_t - \dot{\hat{x}}_t^{(i)} = F_t x_t + G_t v_t - F_t \hat{x}_t^{(i)} - K_t^{(i)}(y_t^{(i)} - H_t^{(i)} \hat{x}_t^{(i)}) \\ &= F_t \tilde{x}_t^{(i)} + G_t v_t - K_t^{(i)}(H_t^{(i)} x_t + w_t^{(i)} - H_t^{(i)} \hat{x}_t^{(i)}) \\ &= (F_t - K_t^{(i)} H_t^{(i)}) \tilde{x}_t^{(i)} + G_t v_t - K_t^{(i)} w_t^{(i)}. \end{aligned}$$

Substituting this expression into the Lyapunov equation for the cross-covariance  $P_t^{(ij)} = E(\tilde{x}_t^{(i)} \tilde{x}_t^{(j)T})$  [13]. By virtue to the assumptions that the system and sensor noises  $v_t$ ,  $w_t^{(i)}$  and  $w_t^{(j)}$ ,  $i \neq j$ , are mutually uncorrelated, we obtain (32). This completes the derivation of (32)

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