Frictionless Contact Problem Between Two Orthotropic Elastic Layers

V. Kahya, A. Birinci, and R. Erdol

Abstract—A frictionless contact problem for a two-layer orthotropic elastic medium loaded through a rigid flat stamp is considered. It is assumed that tensile tractions are not allowed and only compressive tractions can be transmitted across the interface. In the solution, effect of gravity is taken into consideration. If the external load on the rigid stamp is less than or equal to a critical value, continuous contact between the layers is maintained. The problem is expressed in terms of a singular integral equation by using the theory of elasticity and the Fourier transforms. Numerical results for initial separation point, critical separation load and contact stress distribution are presented.

Keywords—Frictionless contact, Initial separation, Orthotropic material, Singular integral equation.

I. INTRODUCTION

CONTACT problems for an elastic layer resting on a foundation which may be either elastic [1-8] or rigid [9-11] have been extensively studied by several authors because of their possible application to a variety of structures of practical interest. In addition, there are few studies on the contact problem for layered composites consisting of two or more elastic layers with different heights and material constants [12-13]. Some authors studied the contact problems for a single layer or a layered medium resting on supports [13-16].

In most of the previous studies, isotropic material properties were used. In recent years, the use of composites in many engineering structures is increased rapidly. Modern composites are preferred increasingly in point of some properties such as strength, lightness, etc. So, this has brought up the need for more extensive analyses of anisotropic materials and has prompted the solution to many problems involving various composite geometries.

In this study, frictionless contact problem for a layered medium consisting of two orthotropic elastic layers is considered. The medium is loaded by a concentrated force

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through a rigid flat stamp. It is assumed that tensile tractions are not allowed and only compressive tractions can be transmitted across the interfaces. In the solution, the effect of gravity is taken into account. If the external load on the rigid stamp is less than or equal to a critical value, continuous contact between the layers is maintained. The problem is reduced to a singular integral equation. Numerical results for initial separation point, critical separation load and contact stress are presented by depending on material properties.

II. FORMULATION OF THE PROBLEM

Consider a layered elastic medium which consists of two orthotropic elastic layers with heights h_1 and h_2 as seen in Fig. 1. The layered medium is loaded by a concentrated force P on its top surface by means of a rigid flat stamp width of which is $2a$. Also, the medium is perfectly bonded to a rigid plane at its bottom surface. It is assumed that the contact is frictionless and no tensile tractions are allowed at interfaces. In the solution, the effect of gravity is taken into account. If magnitude of the external load exceeds a critical value, a separation will occur in a finite region at interface between the layers. Separation is avoided in most of engineering applications, especially in Geotechnics and foundation engineering. Thus, determination of initial separation point and corresponding critical load value which causes initial separation is a very important problem.
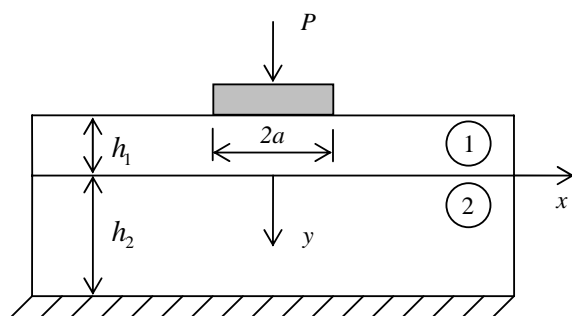

Fig. 1 Geometry of the continuous contact problem

In plane elasticity, governing equations for an orthotropic elastic layer can be written as follows.

$$\beta_1 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \beta_3 \frac{\partial^2 v}{\partial x \partial y} = 0 , \tag{1a}$$

$$\frac{\partial^2 v}{\partial x^2} + \beta_2 \frac{\partial^2 v}{\partial y^2} + \beta_3 \frac{\partial^2 u}{\partial x \partial y} = \frac{\rho g}{G_{xy}} , \tag{1b}$$

where $u = u(x,y)$ and $v = v(x,y)$ represent displacement components in x and y- directions, respectively. G_{xy}, ρ and g are shear modulus in xy- plane, mass density and gravitational acceleration, respectively. Material constants β_1, β_2 and β_3 shown in (1) are defined as

$$\beta_1 = \frac{E_x}{G_{xy}(1 - \nu_{xy}\nu_{yx})}, \quad \beta_2 = \frac{E_y}{E_x}\beta_1, \quad \beta_3 = 1 + \nu_{xy}\beta_2 , \tag{2}$$

where E_x and E_y represent Young's moduli in x and y-directions, respectively. ν_{xy} and ν_{yx} are Poisson's ratios. Among these elastic constants, $E_x \nu_{yx} = E_y \nu_{xy}$ relationship is satisfied.

One can write the stress-displacement relationships for an orthotropic elastic layer as follows.

$$\sigma_x / G_{xy} = \beta_1 \frac{\partial u}{\partial x} + (\beta_3 - 1)\frac{\partial v}{\partial y} , \tag{3a}$$

$$\sigma_y / G_{xy} = (\beta_3 - 1)\frac{\partial u}{\partial x} + \beta_2 \frac{\partial v}{\partial y} , \tag{3b}$$

$$\tau_{xy} / G_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} . \tag{3c}$$

Let us assume that solutions for u and v are as follows.

$$u = u_0(x,y) + u_p(x) , \qquad v = v_0(x,y) + v_p(y) , \tag{4}$$

where subscripts 0 and p represent homogeneous and particular solutions of (1), respectively.

For a single layer with height h under the effect of its own weight, particular solutions to (1) corresponding to the non-homogeneous term ρg may be obtained as

$$u_p = \frac{(\beta_3 - 1)\rho g h x}{2 G_{xy}[(\beta_3 - 1)^2 - \beta_1 \beta_2]} , \tag{5a}$$

$$v_p = -\frac{\rho g y}{2 G_{xy}\beta_2}\left[y + \frac{(\beta_3 - 1)^2 - 2\beta_1 \beta_2}{(\beta_3 - 1)^2 - \beta_1 \beta_2} h \right] . \tag{5b}$$

Substituting (5) into (3b), one may readily obtain the particular solution for the stress component, σ_y, as follows.

$$\sigma_{yp} = -\rho g(y + h) . \tag{6}$$

In order to obtain the homogeneous solutions of (1) for u and v, the displacements and stresses may be assumed as the Fourier transforms of unknown functions such as

$$\{U(\xi,y), V(\xi,y)\} = \int_{-\infty}^{\infty} \{i u_0(x,y), v_0(x,y)\}e^{-i\xi x}dx , \tag{7a}$$

$$\{S_x(\xi,y), S_y(\xi,y), S_{xy}(\xi,y)\} = \int_{-\infty}^{\infty} \{\sigma_{x0}(x,y), \sigma_{y0}(x,y), i\tau_{xy0}(x,y)\}e^{-i\xi x}dx . \tag{7b}$$

Inverse transforms of (7) are

$$\{u_0(x,y), v_0(x,y)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{-iU(\xi,y), V(\xi,y)\}e^{i\xi x}d\xi , \tag{8a}$$

$$\{\sigma_{x0}(x,y), \sigma_{y0}(x,y), \tau_{xy0}(x,y)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{S_x(\xi,y), S_y(\xi,y), -iS_{xy}(\xi,y)\}e^{i\xi x}d\xi . \tag{8b}$$

In (7) and (8), ξ represents transform variable.

Application of (7a) to (1) by neglecting body forces and solution of the resulting ordinary differential equation gives the displacement expressions in the transform domain as follows.

$$U(\xi,y) = \sum_{j=1}^{2} A_j e^{\alpha_j \xi y} + B_j e^{-\alpha_j \xi y} , \tag{9a}$$

$$V(\xi,y) = \sum_{j=1}^{2} R_j(A_j e^{\alpha_j \xi y} - B_j e^{-\alpha_j \xi y}) , \tag{9b}$$

where

$$R_j = -\frac{\beta_1 - \alpha_j^2}{\beta_3 \alpha_j}, \qquad (j = 1,2) . \tag{10}$$

In (9), α_j $(j = 1,2)$ are real roots of the following characteristic equation.

$$\alpha^4 + \frac{\beta_3^2 - \beta_1\beta_2 - 1}{\beta_2}\alpha^2 + \frac{\beta_1}{\beta_2} = 0 . \tag{11}$$

Applying (7b) to (3) and substituting (9) into the resulting expressions, one may readily obtain the stress expressions in the transform domain as follows.

$$S_x(\xi,y) = \sum_{j=1}^{2} L_j \xi(A_j e^{\alpha_j \xi y} + B_j e^{-\alpha_j \xi y}) , \tag{12a}$$

$$S_y(\xi,y) = \sum_{j=1}^{2} M_j \xi(A_j e^{\alpha_j \xi y} + B_j e^{-\alpha_j \xi y}) , \tag{12b}$$

$$S_{xy}(\xi,y) = \sum_{j=1}^{2} N_j \xi(A_j e^{\alpha_j \xi y} - B_j e^{-\alpha_j \xi y}) , \tag{12c}$$

where

$$L_j = G_{xy}[\beta_1 + (\beta_3 - 1)\alpha_j R_j],$$
$$M_j = G_{xy}[(\beta_3 - 1) + \beta_2 \alpha_j R_j],$$
$$N_j = G_{xy}[\alpha_j - R_j], \qquad (j = 1,2) . \tag{13}$$

In above expressions given in (9) and (12), A_j and B_j $(j = 1,2)$ are unknown constant coefficients which will be determined from boundary conditions of the problem.

The problem must be solved under the following boundary conditions.

$$\sigma_{y1}(x,-h_1) = \begin{cases} -p(x) & -a < x < a \\ 0 & -\infty < x < -a \ \text{and} \ a < x < \infty \end{cases} , \tag{14a}$$

$$\tau_{xy1}(x,-h_1) = 0 , \qquad -\infty < x < \infty , \tag{14b}$$

$$\tau_{xy1}(x,0) = \tau_{xy2}(x,0) = 0 , \qquad -\infty < x < \infty , \tag{14c,d}$$

$$\sigma_{y1}(x,0) = \sigma_{y2}(x,0) , \qquad -\infty < x < \infty , \tag{14e}$$

$$v_1(x,0) = v_2(x,0) , \qquad -\infty < x < \infty , \tag{14f}$$

$$u_2(x,h_2) = 0 , \qquad v_2(x,h_2) = 0 , \qquad -\infty < x < \infty , \tag{14g,h}$$

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$$\frac{\partial}{\partial x} v_1(x, -h_1) = 0, \quad -a < x < a, \qquad (14i)$$

where $p(x)$ and a represent unknown contact pressure under the rigid stamp and contact half-width, respectively.

Applying (7) to the boundary conditions from (14a) to (14h) and substituting the expressions given in (9) and (12) into the resulting expressions, the eight unknown coefficients may readily be obtained in terms of unknown contact pressure, $p(x)$.

III. SINGULAR INTEGRAL EQUATION

For displacements and stresses at any point of the medium, it is first needed to obtain unknown contact pressure, $p(x)$ under the rigid stamp. Applying (7a) to the boundary condition (14i) and substituting the expressions given in (9) and (12) into the resulting expression and then using inverse transform given in (8a), after some manipulations, a singular integral equation may be obtained in terms of the unknown contact pressure, $p(x)$ as follows.

$$\frac{1}{\pi} \int_{-a}^{a} p(t) \left[\frac{1}{t-x} + K(x,t) \right] dt = 0, \quad -a < x < a, \quad (15)$$

where

$$K(x,t) = \int_0^\infty \left(\frac{V_1(\xi, -h_1)}{R_1 B_{1s} + R_2 B_{2s}} + 1 \right) \sin[\xi(x-t)] d\xi. \quad (16)$$

The kernel $K(x,t)$ is bounded in the closed interval $-a \le x, t \le a$. Expressions of $V_1(\xi, -h_1)$ and B_{js} ($j = 1,2$) are given in Appendix. From equilibrium, one can also write

$$\int_{-a}^{a} p(t) dt = P. \qquad (17)$$

In order to simplify solution of the singular integral equation, the following dimensionless quantities are introduced.

$$t = as, \quad x = aw, \quad g(s) = \frac{p(as)h_1}{P}, \quad z = \xi h_1. \quad (18)$$

Substituting expressions given in (18) into (15) to (17), one may write the following expressions.

$$\frac{1}{\pi} \int_{-1}^{1} g(s) \left[\frac{1}{s-w} + \frac{a}{h_1} K(w,s) \right] ds = 0, \quad -1 < w < 1, \quad (19)$$

$$\frac{a}{h_1} \int_{-1}^{1} g(s) ds = 1. \qquad (20)$$

Since the contact pressure under the stamp goes to infinity at the corners, i.e. $g(\pm 1) \to \infty$, index of the singular integral equation is +1 [17]. Writing the solution

$$g(s) = G(s)(1 - s^2)^{-1/2}, \qquad (21)$$

and using appropriate Gauss-Chebyshev polynomials [17], (19) and (20) may be replaced by the following expressions.

$$\sum_{i=1}^{n} C_i \left[\frac{1}{s_i - w_j} + \frac{a}{h_1} K(w_j, s_i) \right] G(s_i) = 0, \quad j = 1, \dots, n-1,$$

$$(22)$$

$$\frac{a}{h_1} \sum_{i=1}^{n} C_i G(s_i) = \frac{1}{\pi}, \qquad (23)$$

where

$$C_1 = C_n = \frac{1}{2n-2}, \quad C_i = \frac{1}{n-1} \ i = 2, \dots, n-1,$$

$$s_i = \cos\left(\frac{i-1}{n-1} \pi \right), \quad i = 1, \dots, n,$$

$$w_j = \cos\left(\frac{2j-1}{2n-2} \pi \right), \quad j = 1, \dots, n-1. \quad (24)$$

The system of (22) and (23) constitutes n algebraic equations for n unknowns $G(s_i)$. Solution of this system and use of (21) gives the normalised contact pressure distribution under the rigid stamp, $p(x)h_1 / P$.

IV. INITIAL SEPARATION POINT AND CRITICAL SEPARATION LOAD

Once $g(s)$ is obtained, the contact stress between the layers can readily be calculated by using stress expression $\sigma_y(x,y)$ at $y = 0$. Applying (8b) to (12b) and adding the particular solution given in (6) to the resulting expression, it may be written the stress expression for the top layer at $y = 0$ as follows.

$$\sigma_{y1}(x,0) = -\rho_1 g h_1 - \frac{1}{\pi} \int_{-a}^{a} p(t) \int_0^\infty S_{y1}(\xi, 0) \cos[\xi(x-t)] d\xi dt. \quad (25)$$

Explicit expression of $S_{y1}(\xi, 0)$ is given in the Appendix.

By making use of dimensionless quantities given in (18), (25) may be rewritten as

$$\frac{\sigma_{y1}(x,0) h_1}{P} = -\frac{1}{\lambda} - \frac{1}{\pi} \int_{-1}^{1} g(s) \cdot \frac{a}{h_1} \int_0^\infty S_{y1}(z,0) \cos\left[\left(\frac{x}{h_1} - \frac{a}{h_1} s \right) z \right] dz ds, \quad (26)$$

where λ is called as load factor and defined as

$$\lambda = \frac{P}{\rho_1 g h_1^2}. \qquad (27)$$

By using appropriate Gauss-Chebyshev polynomials, (26) may be replaced by

$$\frac{\sigma_{y1}(x,0) h_1}{P} = -\frac{1}{\lambda} - \sum_{i=1}^{n} C_i G(s_i) \cdot \frac{a}{h_1} \int_0^\infty S_{y1}(z,0) \cos\left[\left(\frac{x}{h_1} - \frac{a}{h_1} s \right) z \right] dz. \quad (28)$$

In order to be valid the singular integral equation given in (19) the contact stress between the layers must be compressive everywhere and no sign changing is allowed. Thus, the critical load value can be calculated numerically by equating (28) to zero and the following expression can be obtained.

$$\lambda_{cr} = \left\{ \sum_{i=1}^{n} C_i G(s_i) \frac{a}{h_1} \int_0^\infty S_{y1}(z,0) \cos\left[\left(\frac{x_{cr}}{h_1} - \frac{a}{h_1} s \right) z \right] dz \right\}^{-1}, \quad (29)$$

where λ_{cr} is called as critical load factor for which initial separation will occur.

V. NUMERICAL RESULTS

The frictionless continuous contact problem for a two-layer elastic medium is solved numerically. In the numerical analyses, the same material properties are used for both layers. Since orthotropic materials have six independent elastic constants, a parametric study must be carried out by depending on material constants given by (2). Assuming E_y/E_x ratio is variable while the constants β_1 and ν_{xy} are kept constant, only the effect of E_y/E_x on numerical results for the normalised contact stresses under the stamp and between the layers, the initial separation point and the corresponding critical separation load is investigated.

Variation of the initial separation point and the corresponding critical load with E_y/E_x for various a/h_1 values are given in Figs. 2 and 3, respectively. As E_y/E_x increases, the values of initial separation points decrease while the corresponding critical load values increase. Figs. 2 and 3 also show that both x_{cr}/h_1 and λ_{cr} increase with increasing of a/h_1. In Table 1, values of the initial separation point and the corresponding critical loads are given for various a/h_1 and E_y/E_x values.

Figs. 4 and 5 show variation of the initial separation point and the corresponding critical load with E_y/E_x for various h_2/h_1 values, respectively. It is reached to the same results as mentioned above for variation of the initial separation point and the corresponding critical load values with E_y/E_x. As h_2/h_1 increases, both x_{cr}/h_1 and λ_{cr} increase, too.

Fig. 3 Variation of λ_{cr} with E_y/E_x for various a/h_1 values
($h_2/h_1 = 1.00$, $G_{xy1}/G_{xy2} = 1.00$)

TABLE I

VARIATION OF THE INITIAL SEPARATION POINT AND CORRESPONDING CRITICAL LOAD FOR VARIOUS a/h_1 VALUES BY DEPENDING ON E_y/E_x

($h_2/h_1 = 1.00$, $G_{xy1}/G_{xy2} = 1.00$)

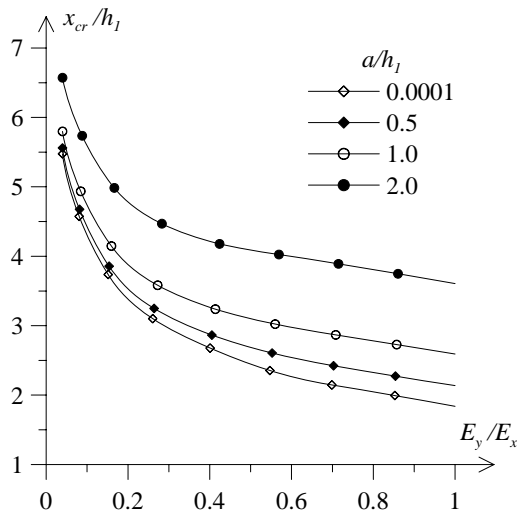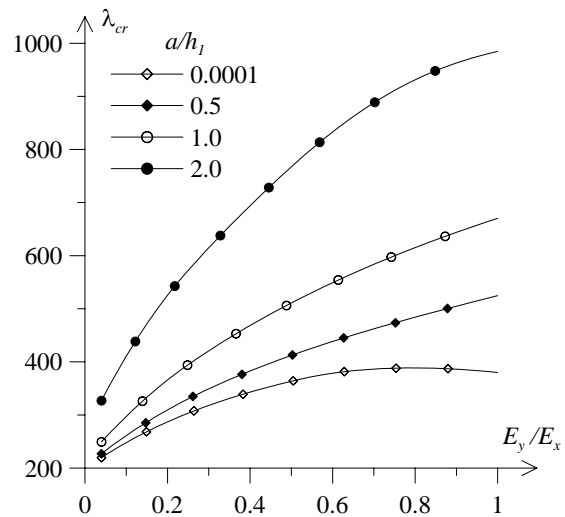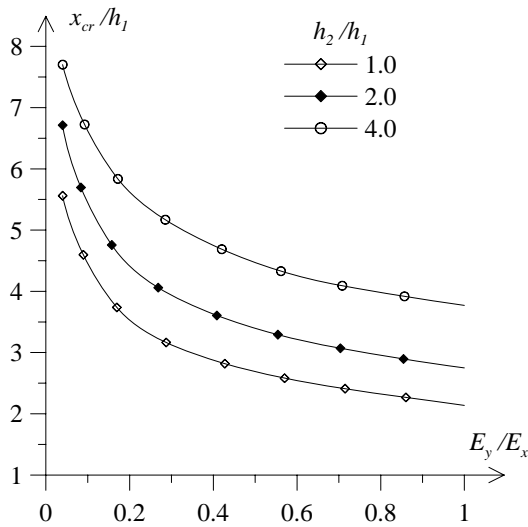| a/h_1	E_y/E_x	x_{cr}/h_1	λ_{cr}
0.0001	0.04	5.4762	219.437
	0.20	3.3809	287.458
	0.40	2.6790	342.947
	0.60	2.2649	378.837
	0.80	2.0479	388.615
	$\rightarrow 1.0$	1.8385	380.077
0.50	0.04	5.5605	227.071
	0.20	3.5296	309.982
	0.40	2.8754	382.818
	0.60	2.5405	438.657
	0.80	2.3253	484.082
	$\rightarrow 1.0$	2.1365	524.833
1.00	0.04	5.7983	249.277
	0.20	3.8793	365.952
	0.40	3.2643	468.579
	0.60	2.9762	549.235
	0.80	2.7818	615.281
	$\rightarrow 1.0$	2.5911	670.156
2.00	0.04	6.5717	326.976
	0.20	4.7809	525.533
	0.40	4.2138	694.122
	0.60	3.9972	833.138
	0.80	3.8079	931.439
	$\rightarrow 1.0$	3.6057	984.904

Fig. 2 Variation of x_{cr}/h_1 with E_y/E_x for various a/h_1 values
($h_2/h_1 = 1.00$, $G_{xy1}/G_{xy2} = 1.00$)

Fig. 4 Variation of x_{cr} / h_1 with E_y / E_x for various h_2 / h_1 values
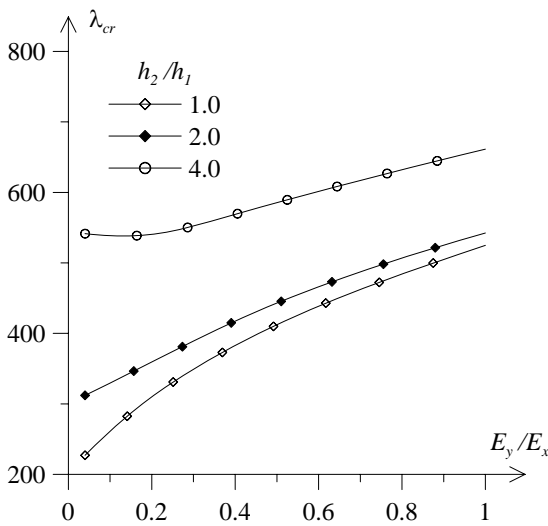($a / h_1 = 0.50$, $G_{xy1} / G_{xy2} = 1.00$)

Fig. 5 Variation of λ_{cr} with E_y / E_x for various h_2 / h_1 values
($a / h_1 = 0.50$, $G_{xy1} / G_{xy2} = 1.00$)

Fig. 6 shows normalised contact stress distribution under the rigid stamp for various E_y / E_x values. As expected, the contact stresses become infinite at corners of the stamp. The normalised contact stress increases with increasing of E_y / E_x .

In Fig. 7, the normalised contact stress distribution at interface of the layers for various E_y / E_x values is given. It is seen in the figure that the normalised contact stress increases with increasing of E_y / E_x in the region close to $x = 0$. Contact points of the curves to the x- axis represent initial separation points. From this figure, it is seen also that initial

separation points decreases with increasing of E_y / E_x . The contact stress between the layers has maximum value at $x = 0$ and decreases along x- axis until the external load effect vanishes. From the point at which the external load effect vanishes, only the weight of the upper layer has effect in occurring of the contact stresses. Therefore, the normalised contact stress value becomes equal to 1 and remains constant.
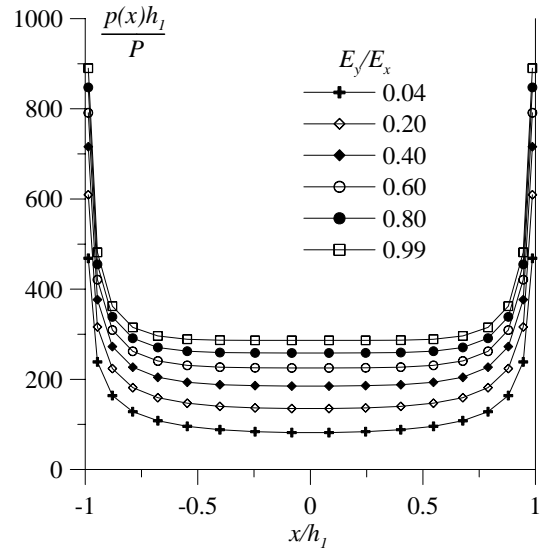

Fig. 6 Normalised contact stress distribution under the rigid stamp for various E_y / E_x values ($a / h_1 = 1.00$, $h_2 / h_1 = 1.00$,
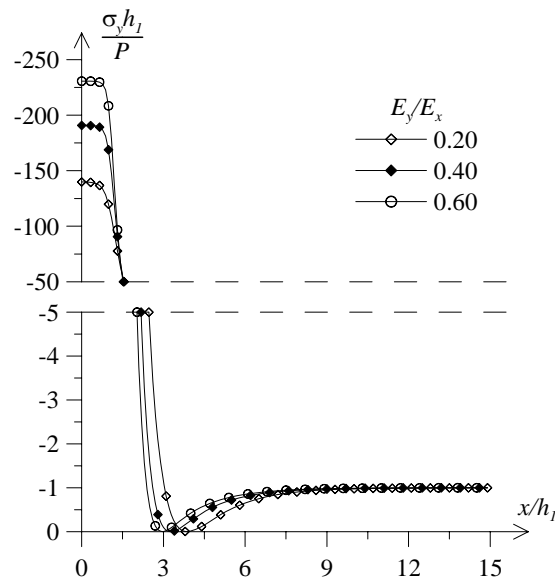$G_{xy1} / G_{xy2} = 1.00$)

Fig. 7 Normalised contact stress distribution between the layers for various E_y / E_x values

($a / h_1 = 1.00$, $h_2 / h_1 = 1.00$, $G_{xy1} / G_{xy2} = 1.00$)

VI. Conclusions

The continuous contact problem for a two-layer orthotropic elastic medium loaded by a concentrated force through a rigid flat stamp is solved numerically. Obtained results show that intensity of the applied load, heights and material properties of the layers have considerable effect on the contact stress distribution at the contact surfaces and, thus, the initial separation point and the critical separation load.

Appendix

Here, explicit expressions of $V_1(\xi, -h_1)$, $S_{y1}(\xi, 0)$ and related quantities appearing in (16) and (25) are given.

$$V_1(\xi, -h_1) = R_1(A_1 e^{-\alpha_1 \xi h_1} - B_1 e^{\alpha_1 \xi h_1}) + R_2(A_2 e^{-\alpha_2 \xi h_1} - B_2 e^{\alpha_2 \xi h_1}), \quad (A.1)$$

$$S_{y1}(\xi, 0) = M_1(A_1 + B_1) + M_2(A_2 + B_2). \quad (A.2)$$

Eight unknown constant coefficients appearing in the displacement and stress expressions can readily be obtained by solving the following matrix equation.

$$[K]\{a\} = \{f\}, \quad (A.3)$$

where,

$$\{a\} = \{A_1 \quad A_2 \quad B_1 \quad B_2 \quad A_3 \quad A_4 \quad B_3 \quad B_4\}^T, \quad (A.4)$$

$$\{f\} = \{P(\xi) \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\}^T, \quad (A.5)$$

$$[K] = \begin{bmatrix}
M_1 e^{-\alpha_1 \xi h_1} & M_2 e^{-\alpha_2 \xi h_1} & M_1 e^{\alpha_1 \xi h_1} & M_2 e^{\alpha_2 \xi h_1} \\
N_1 e^{-\alpha_1 \xi h_1} & N_2 e^{-\alpha_2 \xi h_1} & -N_1 e^{\alpha_1 \xi h_1} & -N_2 e^{\alpha_2 \xi h_1} \\
N_1 & N_2 & -N_1 & -N_2 \\
R_1 & R_2 & -R_1 & -R_2 \\
M_1 & M_2 & M_1 & M_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

$$\begin{matrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-R_3 & -R_4 & R_3 & R_4 \\
-M_3 & -M_4 & -M_3 & -M_4 \\
N_3 e^{\alpha_3 \xi h_2} & N_4 e^{\alpha_4 \xi h_2} & -N_3 e^{-\alpha_3 \xi h_2} & -N_4 e^{-\alpha_4 \xi h_2} \\
e^{\alpha_3 \xi h_2} & e^{\alpha_4 \xi h_2} & e^{-\alpha_3 \xi h_2} & e^{-\alpha_4 \xi h_2} \\
R_3 e^{\alpha_3 \xi h_2} & R_4 e^{\alpha_4 \xi h_2} & -R_3 e^{-\alpha_3 \xi h_2} & -R_4 e^{-\alpha_4 \xi h_2}
\end{matrix}, \quad (A.6)$$

$$P(\xi) = -\int_{-a}^{a} \frac{p(t)}{\xi} e^{-i\xi t} dt, \quad (A.7)$$

where, subscripts *1* and *2* represent quantities for the upper layer while *3* and *4* represent quantities for the lower layer.

The terms B_{js} $(j = 1, 2)$ appearing in (16) can readily be obtained by using asymptotic behaviour of the coefficients B_j $(j = 1, 2)$ as $\xi \to \infty$.

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