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# Free Vibration Analysis of Non-Uniform Euler Beams on Elastic Foundation via Homotopy Perturbation Method

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***Abstract*****—**In this study Homotopy Perturbation Method (HPM) is employed to investigate free vibration of an Euler beam with variable stiffness resting on an elastic foundation. HPM is an easy-to-use and very efficient technique for the solution of linear or nonlinear problems. HPM produces analytical approximate expression which is continuous in the solution domain. This work shows that HPM is a promising method for free vibration analysis of nonuniform Euler beams on elastic foundation. Several case problems have been solved by using the technique and solutions have been compared with those available in the literature.

***Keywords*****—**Homotopy perturbation method, elastic foundation, vibration, beam.

## I. Introduction

In geotechnical engineering, beam on elastic foundation is widely seen in application. There are various types of foundation models such as Winkler, Pasternak, Vlasov, etc. that have been used in the analysis of structures on elastic foundations. The most frequently used foundation model in the analysis of beam on elastic foundation problems is the Winkler foundation model in which the soil is modeled as uniformly distributed linear elastic vertical springs which produce distributed reactions in the direction of the deflection of the beam.

There are also different beam types in theory. The mostly used one is the Euler-Bernoulli and it is suitable for slender beams. For moderately short and thick beams, Timoshenko beam model has to be used in the analysis. Vibration of a uniform Euler beam on elastic foundation was studied previously by Balkaya et al. [1] and Ozturk and Coskun [2]. Balkaya et al. [1] used Differential Transform Method while Ozturk and Coskun [2] used HPM in their studies. Avramidis and Morfidis [3] analyzed bending of beams on three-parameter elastic foundation. De Rosa [4] studied free vibration of Timoshenko beams on two-parameter elastic foundation. Matsunaga [5] studied vibration and buckling of deep beam-columns on two-parameter elastic foundations. El-Mously [6] determined fundamental frequencies of Timoshenko beams mounted on Pasternak foundation. Chen [7], [8] analyzed vibration of beam resting on an elastic foundation by the differential quadrature element method (DQEM). Coskun [9] investigated the response of a finite beam on a tensionless Pasternak foundation subjected to a harmonic load. Chen et al. [10] developed a mixed method for bending and free vibration of beams resting on a Pasternak elastic foundation. Maheshwari et al. [11] studied the response of beams on a tensionless extensible geosynthetic-reinforced earth bed subjected to moving loads. Auciello and De Rosa [12] developed different approaches to the dynamic analysis of beams on soils subjected to subtangential forces. Mutman [13] determined free vibration frequencies of rectangular Euler beams with linearly and exponentially varying width on elastic foundation.

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## II. The Equations of Motion and Boundary Conditions

An Euler beam resting on Winkler foundation shown in Fig. 1 is considered in this study. The equation of motion for this problem is given as follows.

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right)+k\left(x\right)w+\rho A\left(x\right)\left(\frac{\partial^2 w}{\partial t^2}\right)=0 \quad (1)$$

where k is the spring constant, w is deflection, $\rho$ is the mass density, A is the cross sectional area, EI is the beam stiffness and I is the area moment of inertia about the neutral axis. The deflection is a function of both space and time, i.e., $w = w(x,t)$ in which space variable x is measured along the length of the beam and t represents any particular instant of time.

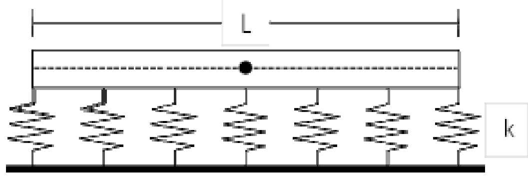


Fig. 1 Representation of a beam on Winkler foundation

Due to the support conditions at both ends of the beam, different conditions have to be imposed to the obtained solution to determine unknowns included in final approximation produced by HPM. These conditions are given as follows:

a) For clamped-clamped beam the end conditions are:

$$w = \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0, L \tag{2}$$

b) For cantilever beam (clamped-free) the end conditions are:

$$\begin{aligned} w = \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0 \\ \frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \quad \text{at} \quad x = L \end{aligned} \tag{3}$$

c) For simply supported beam (pinned-pinned) the end conditions are:

$$w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, L \tag{4}$$

d) For clamped-simply supported (pinned) beam the end conditions are:

$$\begin{aligned} w = \frac{\partial w}{\partial x} = 0 \quad \text{at} \quad x = 0 \\ w = \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = L \end{aligned} \tag{5}$$

Now, free vibration analysis of the beams with variable stiffness resting on elastic foundations will be formulated.

A solution is assumed as the following form to formulate the analysis of the presented problem by the separation of variables:

$$w(x,t) = W(x)e^{i\omega t} \tag{6}$$

where$\omega$ is the circular frequency for the vibration. Substituting (6) into (1), equations of motion becomes as follows:

$$\frac{d^2}{dx^2}\left( EI(x)\frac{d^2 W}{dx^2} \right) + kW = \rho A \omega^2 W \tag{7}$$

This equation can be rearranged as:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{EI(x)'}{EI(x)}\frac{\partial^3 w}{\partial x^3} + \frac{EI(x)''}{EI(x)}\frac{\partial^2 w}{\partial x^2} + \frac{k(x)}{EI(x)}w + \frac{\rho A(x)}{EI(x)}\left( \frac{\partial^2 w}{\partial t^2} \right) = 0 \tag{8}$$

where$()'$ denotes total derivative with respect to x. The governing equation is now rewritten in a non-dimensional form. This procedure is provided from [1] in which a constant stiffness beam was analyzed. The notation is maintained in this study for the comparison purposes. The non-dimensional parameters for the Euler-beam on the Winkler foundation are defined as [1].

$$\bar{x} = \frac{x}{L}, \quad \bar{W} = \frac{W}{L}, \quad \lambda = \frac{kL^4}{EI}, \quad \bar{\omega} = \omega\sqrt{\frac{\rho A}{k}} \tag{9}$$

Using these parameters, non-dimensional form of the equations and formulation procedures are explained in the following sections.

## III. Homotopy Perturbation Method

Homotopy Perturbation Method [14]-[19] that is an analytical approximate solution technique can be considered as one of the most applied method for nonlinear problems. The HPM provides an analytical approximate expression as the solution for the problems which are continuous in the solution domain. The technique is applied to an equation of the form L(u)+N(u)=$f$(r), r ϵ Ω with boundary conditions B(u,∂u/∂n)=0, r ϵ Γ where L is a linear operator, N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ώ, and $f$(r)is a known analytic function. HPM, defines a homotopy as v(r,p)=Ω×[0,1]→R which satisfies the following inequalities:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0 \tag{10}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{11}$$

wherer ϵ Ω and p ϵ [0,1] is an imbedding parameter, $u_0$is an initial approximation which satisfies the boundary conditions. From (10), (11), we have:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{12}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0 \tag{13}$$

The changing process of p from zero to unity is that of v(r,p) from $u_0$ to u(r). In topology, this deformation L(v)-L($u_0$) and L(v)+N(v)-$f$(r) are called homotopic. The method expresses the solution of (10), (11) as a power series in p as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \dots \tag{14}$$

The approximate solution ofL(u)+N(u)=$f$(r),r ϵ Ω can be obtained as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{15}$$

The convergence of the series in (15) has been proven in[14]-[19].

## IV. HPM Formulation

In this study, α linearly varying stiffness is assumed for the beam considered. The linear variation of stiffness is due to the linearly varying and is formulated as:

$$b(x) = b_0(1-\alpha x) \tag{16}$$

where the dimension of α is [1/L]. By the use of variable width, both cross-sectional area and flexural stiffness becomes as:

$$A(x) = b_0 h(1-\alpha x) = A_0(1-\alpha x) \tag{17}$$

$$EI(x) = Eb_0\frac{h^3}{12}(1-\alpha x) = EI_0(1-\alpha x) \tag{18}$$

where $A_0$ and $I_0$ are the cross-sectional area and moment of inertia of the section at the origin, respectively. Inserting (17), (18) into (8):

$$\frac{\partial^4 w}{\partial x^4} - \frac{2\alpha EI_0}{EI_0(1-\alpha x)}\frac{\partial^3 w}{\partial x^3} + \frac{k(x)}{EI_0(1-\alpha x)}w + \frac{\rho A_0}{EI_0}\left(\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{19}$$

This equation can be rewritten as:

$$\frac{\partial^4 w}{\partial x^4} - 2\alpha\left(\frac{1}{1-\alpha x}\right)\frac{\partial^3 w}{\partial x^3} + \frac{k(x)}{EI_0}\left(\frac{1}{1-\alpha x}\right)w + \frac{\rho A_0}{EI_0}\left(\frac{\partial^2 w}{\partial t^2}\right) = 0 \tag{20}$$

employing (6);

$$W^{iv} - 2\alpha\xi(x)W''' + \frac{k(x)}{EI_0}\xi(x)W - \frac{\rho A_0}{EI_0}\omega^2 W = 0 \tag{21}$$

Equation (24) can be made non-dimensional in view of (9) as follows:

$$\overline{W}^{iv} - 2\overline{\alpha}\xi(\overline{x})\overline{W}''' + \lambda\left(\xi(\overline{x}) - \overline{\omega}^2\right)W = 0 \tag{22}$$

where

$$\overline{x} = \frac{x}{L},\quad \overline{W} = \frac{W}{L},\quad \lambda = \frac{kL^4}{EI_0},\quad \overline{\omega} = \omega\sqrt{\frac{\rho A_0}{k}} \tag{23}$$

$$\xi(\overline{x}) = \frac{1}{1-\overline{\alpha}\overline{x}} \quad , \quad \overline{\alpha} = \alpha L \tag{24}$$

By the application of HPM, following iteration algorithm is obtained:

$$\overline{W}_0^{iv} - u_0^{iv} = 0 \tag{25}$$

$$\overline{W}_1^{iv} + u_0^{iv} - 2\overline{\alpha}\xi(\overline{x})\overline{W}_0''' + \lambda\left(\xi(\overline{x}) - \overline{\omega}^2\right)\overline{W}_0 = 0$$

$$\overline{W}_n^{iv} - 2\overline{\alpha}\xi(\overline{x})\overline{W}_{n-1}''' + \lambda\left(\xi(\overline{x}) - \overline{\omega}^2\right)\overline{W}_{n-1} = 0 \quad , \quad n \geq 2$$

## V. Solution Procedure

A cubic polynomial with four unknown coefficients can be chosen as initial approximation. There exist four boundary conditions, i.e., two at each end of the column, due to the end supports of the beam in the presented problem. Hence, the initial approximation is:

$$\overline{W}_0 = A\overline{x}^3 + B\overline{x}^2 + C\overline{x} + D \tag{26}$$

Twenty iterations are conducted through the analysis procedure and four boundary conditions for each case are rewritten by using the solution for displacement of the beam. Each boundary condition produces an equation containing four unknowns due to the initial approximation. These boundary conditions in non-dimensional form are:

Clamped-Clamped beam:

$$\overline{W} = \frac{d\overline{W}}{d\overline{x}} = 0 \quad \text{at} \quad \overline{x} = 0,1 \tag{27}$$

Clamped-Free (Cantilever) beam:

$$\overline{W} = \frac{d\overline{W}}{d\overline{x}} = 0 \quad \text{at} \quad \overline{x} = 0$$
$$\frac{d^2\overline{W}}{d\overline{x}^2} = \frac{d^3\overline{W}}{d\overline{x}^3} = 0 \quad \text{at} \quad \overline{x} = 1 \tag{28}$$

Pinned-Pinned (Simply supported) beam:

$$\overline{W} = \frac{d^2\overline{W}}{d\overline{x}^2} = 0 \quad \text{at} \quad \overline{x} = 0,1 \tag{29}$$

Clamped-Pinned beam:

$$\overline{W} = \frac{d\overline{W}}{d\overline{x}} = 0 \quad \text{at} \quad \overline{x} = 0$$
$$\overline{W} = \frac{d^2\overline{W}}{d\overline{x}^2} = 0 \quad \text{at} \quad \overline{x} = 1 \tag{30}$$

Hence, four equations in four unknowns may be written with respect to the boundary conditions of the problem. These equations can be represented in matrix form as follows:

$$\left[M\left(\bar{\omega}\right)\right]\left\{A\right\}=\left\{0\right\} \tag{31}$$

where {A}=‹ABCD›$^T$. For a nontrivial solution, determinant of coefficient matrix must be zero. Determinant of coefficient matrix yields a characteristic equation in terms of $\bar{\omega}$. Positive real roots of this equation are the normalized free vibration frequencies for the case considered.

## VI. Numerical Results

### A. Constant Stiffness Case

As the first example Euler beam of constant stiffness, (i.e. EI is constant), with different boundary conditions is investigated. For the sake of comparison, all the values are set to unity such as I=E=A=ρ=1, hence λ=1, according to previous studies [7]. Both algorithms given for linear and exponential variations lead to constant stiffness when α=0.

In Table I, first three normalized free vibration frequencies for simply supported (pinned-pinned) beam are compared with HPM results in the literature and the exact solution. Excellent agreement is observed for HPM with the exact solution.

TABLE I
Normalized Free Vibration Frequencies of Simply Supported Beam Resting on Winkler Foundation

| Method | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|
| HPM | 9.92014 | 39.4911 | 88.8321 | 157.9168 | 246.7421 |
| DTM [1] | 9.92014 | 39.4911 | 88.8321 | - | - |
| DQEM[7] | 9.92014 | 39.4913 | 89.4002 | - | - |
| Exact Solution [1] | 9.92014 | 39.4911 | 88.8321 | - | - |

The first five natural frequencies for clamped-clamped beam and cantilever (clamped-free) beam are presented in Tables II and III, respectively.

TABLE II
Normalized Free Vibration Frequencies of Clamped-Clamped Beam Resting on Winkler Foundation

| Method | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|
| HPM [9] | 22.3956 | 61.6809 | 120.908 | 199.862 | 298.557 |
| DTM[1] | 22.3733 | 61.6728 | 120.903 | 199.859 | 298.556 |
| DQEM [7] | 22.3956 | 61.6811 | 120.910 | 199.885 | 298.675 |

TABLE III
Normalized Free Vibration Frequencies of Cantilever Beam Resting on Winkler Foundation

| Method | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|
| HPM [9] | 3.65546 | 22.0572 | 61.7053 | 120.906 | 199.862 |
| DTM [1] | 3.65546 | 22.0572 | 61.7053 | 120.906 | 199.862 |
| DQEM [7] | 3.65544 | 22.0572 | 61.7057 | 120.911 | 199.894 |

Excellent agreement is observed for HPM with previous available results for both cantilever and clamped-clamped beams. Clamped-pinned beam was only included in [9]. Hence, only HPM results are tabulated for this case in Table IV.

TABLE IV
Normalized Free Vibration Frequencies of Clamped-Pinned Beam Resting on Winkler Foundation

| Method | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ |
|---|---|---|---|---|---|
| HPM [9] | 15.451 | 49.975 | 104.253 | 178.273 | 272.033 |

As one can see, perfect agreement is obtained for constant stiffness case. This issue is mainly due to constant coefficient governing equation. In the following sections, variable stiffness cases are investigated.

### B. Linearly Varying Stiffness

A number of case studies are conducted with respect to parameter a, and the results are given in Tables V-VIII below.

TABLE V
Normalized Free Vibration Frequencies of Cantilever Beam Resting on Winkler Foundation with Linearly Varying Flexural Stiffness

| α | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
|---|---|---|---|---|---|---|
| $\omega_1$ | 3.65546 | 3.77785 | 3.91814 | 4.08113 | 4.27363 | 4.50571 |
| $\omega_2$ | 22.0572 | 22.2779 | 22.5271 | 22.8129 | 23.1475 | 23.5506 |
| $\omega_3$ | 61.7053 | 61.9182 | 62.1616 | 62.4458 | 62.7867 | 63.2104 |
| $\omega_4$ | 120.9061 | 121.1194 | 121.3645 | 121.6528 | 122.0024 | 122.4434 |
| $\omega_5$ | 199.8620 | 200.0755 | 200.3213 | 200.6117 | 200.9661 | 201.4172 |

TABLE VI
Normalized Free Vibration Frequencies of Clamped-Pinned Beam Resting on Winkler Foundation with Linearly Varying Flexural Stiffness

| α | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
|---|---|---|---|---|---|---|
| $\omega_1$ | 15.4506 | 15.5615 | 15.6801 | 15.8069 | 15.9427 | 16.0879 |
| $\omega_2$ | 49.9749 | 50.0781 | 50.1878 | 50.3052 | 50.4316 | 50.5685 |
| $\omega_3$ | 104.2525 | 104.3556 | 104.4655 | 104.5836 | 104.7121 | 104.8544 |
| $\omega_4$ | 178.2725 | 178.3756 | 178.4853 | 178.6036 | 178.7333 | 178.8785 |
| $\omega_5$ | 272.0328 | 272.1358 | 272.2455 | 272.3640 | 272.4944 | 272.6418 |

TABLE VII
Normalized Free Vibration Frequencies of Clamped-Clamped Beam Resting on Winkler Foundation with Linearly Varying Flexural Stiffness

| α | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
|---|---|---|---|---|---|---|
| $\omega_1$ | 22.3956 | 22.3922 | 22.3777 | 22.3477 | 22.2958 | 22.2120 |
| $\omega_2$ | 61.6809 | 61.6752 | 61.6541 | 61.6114 | 61.5380 | 61.4183 |
| $\omega_3$ | 120.9075 | 120.9009 | 120.8775 | 120.8302 | 120.7485 | 120.6144 |
| $\omega_4$ | 199.8620 | 199.8549 | 199.8302 | 199.7803 | 199.6939 | 199.5512 |
| $\omega_5$ | 298.5572 | 298.5499 | 298.5243 | 298.4729 | 298.3834 | 298.2351 |

TABLE VIII
Normalized Free Vibration Frequencies of Simply-Supported Beam Resting on Winkler Foundation with Linearly Varying Flexural Stiffness

| α | 0.00 | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 |
|---|---|---|---|---|---|---|
| $\omega_1$ | 9.9201 | 9.9217 | 9.9210 | 9.9170 | 9.9084 | 9.8932 |
| $\omega_2$ | 39.4911 | 39.4928 | 39.4970 | 39.5047 | 39.5166 | 39.5340 |
| $\omega_3$ | 88.8321 | 88.8340 | 88.8398 | 88.8511 | 88.8697 | 88.8986 |
| $\omega_4$ | 157.9168 | 157.9189 | 157.9257 | 157.9389 | 157.9612 | 157.9966 |
| $\omega_5$ | 246.7421 | 246.7443 | 246.7516 | 246.7661 | 246.7907 | 246.8302 |

Variations of normalized free vibration frequencies ($\bar{\omega}$) with respect to non-dimensional variation parameter α for each beam are also given in Figs.2-6.

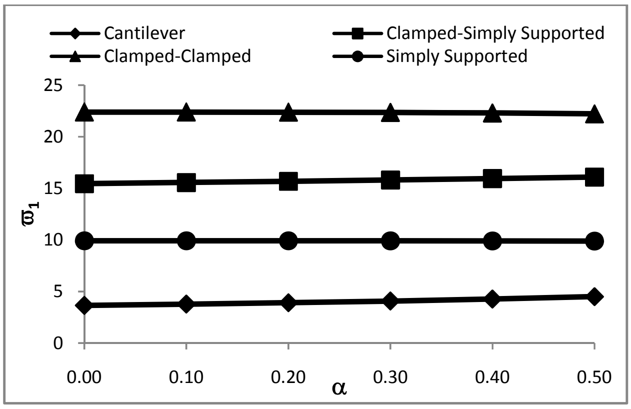


Fig. 2 Variation of normalized first mode frequency with respect to normalized variation coefficient

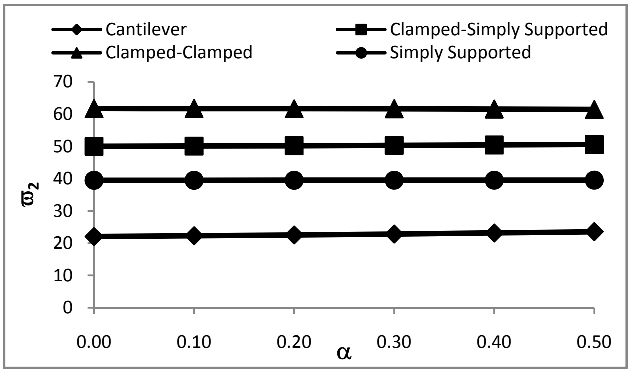


Fig. 3 Variation of normalized second mode frequency with respect to normalized variation coefficient

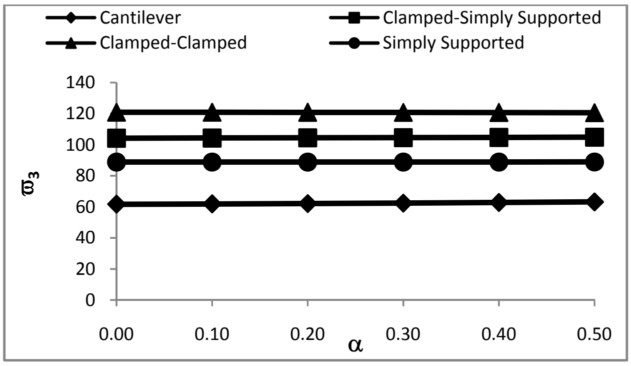


Fig. 4 Variation of normalized third mode frequency with respect to normalized variation coefficient

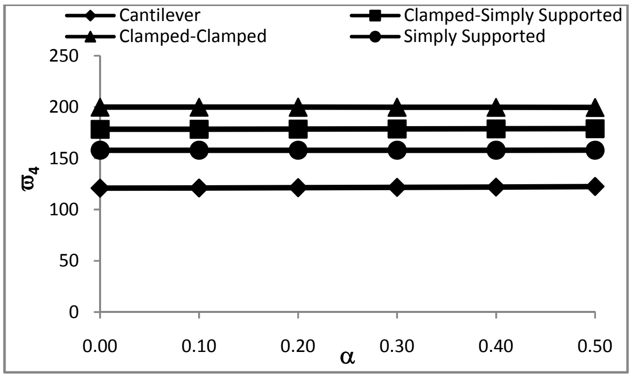


Fig. 5 Variation of normalized fourth mode frequency with respect to normalized variation coefficient

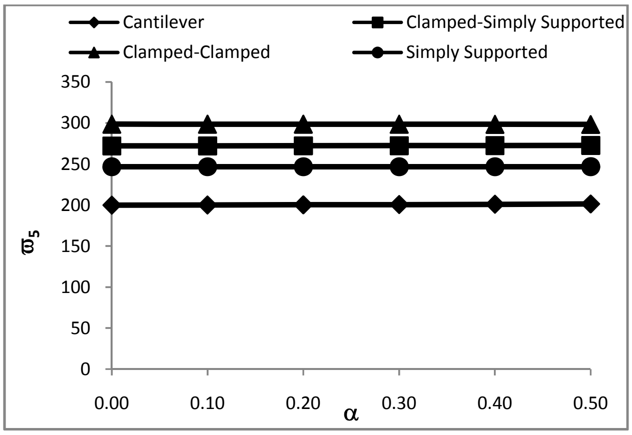


Fig. 6 Variation of normalized fifth mode frequency with respect to normalized variation coefficient

## VII. Conclusion

In this study, HPM is introduced for the free vibration analysis of variable stiffness non-uniform Euler beams on elastic foundations. As a demonstration of application of the method, firstly constant stiffness uniform Euler beam is considered and HPM results are comparison with the available results. HPM has produced results in excellent agreement with the previously available solutions that encourage the application of the method for variable stiffness non-uniform Euler beams. To represent a variation in stiffness, a rectangular beam with varying width is considered. The analyses areexpanded for variable stiffness cases. HPM also produced reasonable results for the vibration of variable stiffness Euler beams which show the efficiency of the method. In the case of variable stiffness, the governing equation becomes a differential equation with variable coefficients, and it is not easy to obtain analytical solutions for these types of problems. However, HPM would produce reasonable results after performing some iterations with the method. The results obtained in this study point out that the proposed method is a powerful and reliable method in the analysis of the presented problem.

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