FEM Models of Glued Laminated Timber Beams Enhanced by Bayesian Updating of Elastic Moduli

L. Melzerová, T. Janda, M. Šejnoha, J. Šejnoha

***Abstract*—**Two finite element (FEM) models are presented in this paper to address the random nature of the response of glued timber structures made of wood segments with variable elastic moduli evaluated from 3600 indentation measurements. This total database served to create the same number of ensembles as was the number of segments in the tested beam. Statistics of these ensembles were then assigned to given segments of beams and the Latin Hypercube Sampling (LHS) method was called to perform 100 simulations resulting into the ensemble of 100 deflections subjected to statistical evaluation. Here, a detailed geometrical arrangement of individual segments in the laminated beam was considered in the construction of two-dimensional FEM model subjected to in four-point bending to comply with the laboratory tests. Since laboratory measurements of local elastic moduli may in general suffer from a significant experimental error, it appears advantageous to exploit the full scale measurements of timber beams, i.e. deflections, to improve their prior distributions with the help of the Bayesian statistical method. This, however, requires an efficient computational model when simulating the laboratory tests numerically. To this end, a simplified model based on Mindlin's beam theory was established. The improved posterior distributions show that the most significant change of the Young's modulus distribution takes place in laminae in the most strained zones, i.e. in the top and bottom layers within the beam center region. Posterior distributions of moduli of elasticity were subsequently utilized in the 2D FEM model and compared with the original simulations.

***Keywords*—**Bayesian inference, FEM, four point bending test, laminated timber, parameter estimation, prior and posterior distribution, Young's modulus.

I. Introduction

Wood is an anisotropic and heterogeneous building material. Its local properties may thus vary quite significantly from point to point. On the contrary, computational models of wood require rather precise values of material parameters especially when dealing with structural elements of large dimensions or elements subjected to extreme load conditions. These elements are typically manufactured from glued laminated timber. This significantly increases variance of material parameters, because glued laminated timber is composed of many segments. The manufacturing process may result in cases where two neighboring segments differ more than twice in terms of their stiffness. This founding follows from the results of an extensive experimental program where each of the five timber beams was subjected to 3600 indentation measurements yielding local values of moduli of elasticity [1]. Point out that this non-destructive technique suffers from a relatively large measurement error. Therein, the measured indentation depth is transformed into a local value of the modulus of elasticity along the fiber direction using an empirical expression. Because it is possible to measure the depth of indentation with the accuracy of 0.5 mm only, the resulting error in the calculated values may exceed 0.25 GPa. The resulting ensembles of moduli of elasticity, statistically represented by their prior distributions, must therefore be improved. Here, a suitable method of attack is the Bayesian statistical method [2], [3].

Bayes' theorem has far-reaching implication but requires to shift our perception of probability. We need to attribute probability distribution also to quantities that are hidden from us and cannot be measured directly. While in case of the observable quantity the probability density expresses how often a given value occurs, in case of unobserved quantity the probability density expresses, how much we believe in that particular value. For example, in the present study of the laminated timber beam subjected to four point bending the observed quantity is the *reading* of the displacements at given points, while the unobserved quantities are the values of Young's modulus in each lamina, the *true value* of the displacement and the standard deviation of random measurement error.

If there is a stochastic relationship between the unobserved quantities and observed data we can reappraise our prior knowledge about the unobserved quantities, i.e. in our present study the prior probability densities of Young's moduli acquired from indentation measurements, and obtain a rationally updated knowledge, i.e. their posterior probability densities.

The main advantage of this approach is its mathematical soundness: if our model (stochastic relations) is correct then the posterior belief is the most rational one given the prior belief and the observed data. This is what Bayes' theorem says. On the other hand, one must be prepared that the integral statistics of posterior distribution such as mean value, standard deviation or quantiles can be computed analytically only for a limited category of models. Posterior distribution obtained with models of arbitrary structure has to be analyzed numerically via Markov chain Monte Carlo methods such as Metropolis–Hasting algorithm, Gibbs sampling or Hamiltonian Monte Carlo algorithm. Luckily the last two methods are implemented in open source programs JAGS [4]

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and Stan [5], respectively, and allow the user to specify the stochastic model declaratively in a dialect of BUGS language and generate the samples of unobserved model parameters distributed according their posterior distributions.

In this paper, the direct use of prior probability distributions of Young's moduli will be used first in FEM simulations combined with LHS method to get statistics of the displacements evaluated at specific points. At this step, the 2D plane-stress FEM model will be constructed such as to represent the actual beam tested in laboratory as close as possible. Next, the Bayesian statistical method will be exploited in combination with the simplified beam-like FEM model based on Mindlin's beam theory to arrive at improved posterior distributions. These will in turn be adopted in the first step to judge the degree of improvement and significance of the Bayesian approach for this particular example.

II. Laboratory Measurements

Two sets of experiments are reviewed in this paper. First, the indentation measurements of local Young's moduli are summarized together with their statistical evaluation and construction of prior distributions. The full scale four-point bending test of glued laminated beam made of segments examined in the former tests is discussed providing values of vertical deflection at selected points. These then enter the Bayesian updating procedure to generate the desired posterior estimates of the prior distributions.

A. Prior Distributions of Young's Moduli from Indentation Measurements

Modulus of elasticity of wood can be measured by various methods. When considering wood segments already built into an existing structure it is necessary to adopt non-destructive testing methods, which cause either no or very low damage to the tested material. Owing to a considerable heterogeneity of laminated timber structures a large number of local measurements is needed. At present, only one such experimental method, which builds upon driving an indenter with the help of Pilodyn 6J device in Fig. 1 into the wood, is available.

In particular, a spike 2.5 mm in diameter is shot into the wood with the enforced energy of 6 J. The local elastic modulus in the fiber direction is then evaluated empirically based on the depth of indentation as, see also [1],

$$E = -564.1 t_p + 19367\,, \tag{1}$$

where E is the searched Young's modulus in MPa and t_p is the measured indentation depth in mm. The Pilodyn 6J device allows for reading with the accuracy of 0.5 mm. Clearly, if the measuring error is 0.5 mm then the corresponding error of computed modulus amounts to 2.5 GPa. Additional error follows from the material heterogeneity and uneven inclination of tangents to annual rings at the point of indentation with respect to the vertical surface of the structural element.

Fig. 1 Pilodyn 6J indentation device

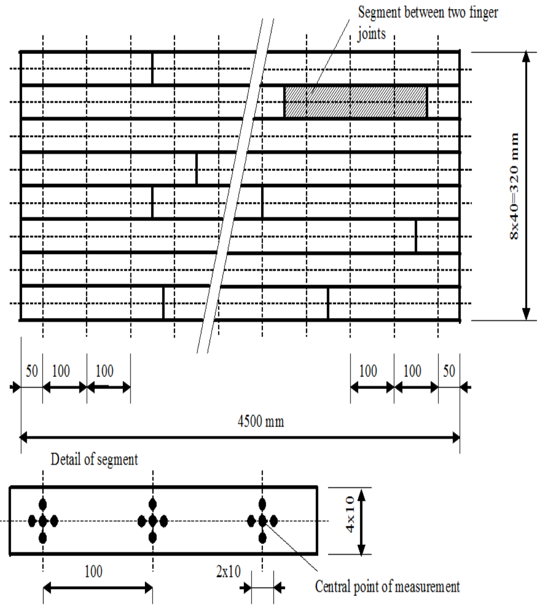

Fig. 2 Location of measuring points for the determination of local moduli of elasticity

The examined beam was subjected to 3600 measurements on both sides as shown in Fig. 2. The calculated elastic moduli were grouped into ensembles corresponding to individual segments. Fig. 3 provides graphical representation of prior distributions for all 21 segments together with one distribution pertinent to all 3600 values. These distributions were first used in initial LHS-based simulations, see Section III, and subsequently updated in Section IV.

B. Laboratory Measurements of Four Point Bending Test

Note that Bayesian updating requires the knowledge of at least one additional measured parameter. In this study we chose the beam deflections measured during the four point bending test at three different locations, see Fig. 4. The beam was loaded gradually up to the final failure. At each load step the two applied forces were increased by 2 kN each, kept constant for a short period of time followed by collecting the deflection values. The three vertical displacements were all measured at the bottom surface of the beam below the two forces (w_1, w_3) and at the beam center (w_2). The resulting values are listed in Table I.

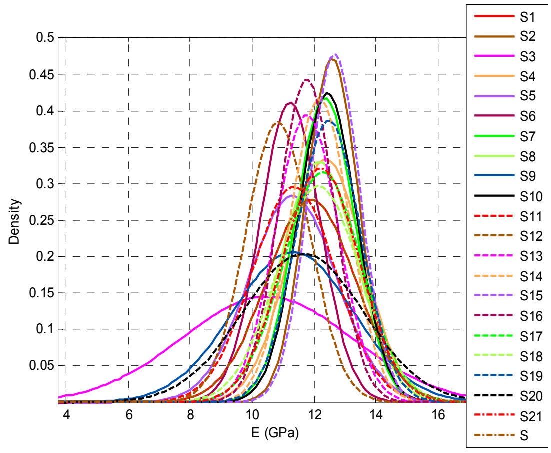

Fig. 3 Prior distributions of probability density functions assuming Gaussian distribution at all segments (S1-S21) of the examined beam and for the entire beam (S)

Fig. 4 Four point bending test

III. Prediction of Beam Deflection Using Prior Distributions

In the first computational step the prior distributions acquired in Section II-A were adopted in stochastic FEM simulations. The 2D computational model was created such as to fully comply with the tested beam, i.e. prior to meshing the model into rectangular elements it was subdivided into sections being represented by parts of individual lamellas between two finger joints, recall Fig. 2. As typical of Bayesian approach the prior distributions were considered as statistically independent events. Following our previous work [1], the LHS method was employed to generate individual realizations. In our particular case, the stochastic analysis considered 100 runs which attributed to the subdivision of the distribution function of Young's modulus pertaining to segment i into 100 equally distributed intervals to randomly select its central value for a given run associated with a given value of E_i, i = 1,...,21. Each of the 100 realizations appears with the same probability and only once. This thus provides 100 fictitious beams that can be analyzed to render the values of deflections at points corresponding to their measured counterparts. Finally, the recorded deflections were statistically evaluated to render their mean value and standard deviation stored in the 1st row of Table V.

TABLE I
Measured Vertical Displacements for a Given Load Level

F, kN	w_1, mm	w_2, mm	w_3, mm
3.97	1.15	1.34	1.19
7.98	2.54	2.85	2.57
11.96	3.84	4.38	3.95
15.96	5.21	5.92	5.33
19.97	6.59	7.47	6.74
24.01	7.93	9.01	8.10
28.02	9.34	10.58	9.51
32.02	10.68	12.11	10.92
36.05	12.13	13.67	12.32
40.06	13.51	15.31	13.84
44.03	14.95	16.93	15.30
48.03	16.39	18.53	16.74
52.05	17.76	20.10	18.18
56.04	19.20	21.73	19.60
60.03	20.63	23.37	21.08
64.02	22.01	24.96	22.53
68.00	23.47	26.68	24.05
72.02	24.87	28.30	25.50
76.02	26.27	29.94	26.93
80.07	27.64	31.57	28.35
84.18	28.99	33.17	29.74
88.20	30.41	34.87	31.15
92.19	31.80	36.56	32.62
96.15	33.21	38.32	34.20

IV. Bayesian Statistical Method and Updating

This section presents theoretical grounds of the Bayesian statistical method with particular application to glued laminated beams. Individual steps are described in the following paragraphs leading to improved posterior probability density distributions of Young's moduli for individual segments. These are then introduced into the stochastic simulations outlined in the previous section to compare with the predictions based on prior distributions and experimental measurements, see ahead Table V.

A. Beam Element of Laminated Timber

It has already been advocated that the application of Bayesian statistical method requires an efficient tool for the evaluation of the objective function, i.e. a tool to predict the values of measured displacements numerically. To that end, we propose to replace the original 2D model with a beam-like model developed on the basis of Mindlin's beam theory.

This approach requires replacing the heterogeneous cross-section by an equivalent one with the effective (homogenized) material properties. In our particular case, the cross-section consists of eight layers (lamellas) with different values of Young's modulus E, see Fig. 5.

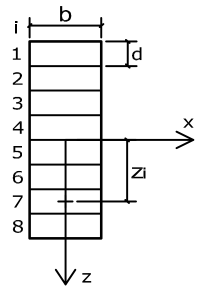

Fig. 5 Description of laminated timber beam cross-section

We index the lamellas in the cross-section by $i = 1,...,8$. Thus in the i-th lamella E_i denotes the value of Young's modulus and z_i is the distance between the i-th ply center and the center of the entire cross-section. The homogenized bending stiffness of the cross-section becomes

$$D_b = bd^3 \frac{1}{12}\sum_{i=1}^{8} E_i + db\sum_{i=1}^{8} E_i\left(z_i - z_T\right)^2, \tag{2}$$

where d and b are the lamina's width and thickness, respectively. The value z_T represents the position of the centroid of the homogenized cross-section

$$z_T = \frac{\sum_{i=1}^{8} E_i z_i}{\sum_{i=1}^{8} E_i}. \tag{3}$$

The homogenized shearing stiffness of the cross-section reads

$$kGA = \frac{D_b^2 b^2}{bd\sum_{i=1}^{8}\frac{(ES)_i^2}{G_i}}, \tag{4}$$

where

$$(ES)_i = bd\sum_{j=i}^{8} E_j\left(z_j - z_T\right), \tag{5}$$

$$G_i = \frac{E_i}{2(1+\nu)}. \tag{6}$$

Conventionally, the vector of nodal forces of the finite element is expressed as a product of the element stiffness matrix and the vector of nodal displacements

$$\mathbf{R}_e = \mathbf{K}_e \mathbf{r}_e. \tag{7}$$

Omitting the axial forces, which are zero during pure bending, we express the vectors of nodal forces and displacements as

$$\mathbf{R}_e = \{Z_1, M_1, Z_2, M_2\}^T, \tag{8}$$

$$\mathbf{r}_e = \{w_1, \varphi_1, w_2, \varphi_2\}^T, \tag{9}$$

and the stiffness matrix of the element with the length l assumes this form

$$\mathbf{K}_e = \frac{2D_b}{(1+2\kappa)l}\begin{bmatrix} \frac{6}{l^2} & -\frac{3}{l} & -\frac{6}{l^2} & -\frac{3}{l} \\ -\frac{3}{l} & 2+\kappa & \frac{3}{l} & 1-\kappa \\ -\frac{6}{l^2} & \frac{3}{l} & \frac{6}{l^2} & \frac{3}{l} \\ -\frac{3}{l} & 1-\kappa & \frac{3}{l} & 2+\kappa \end{bmatrix}, \tag{10}$$

$$\kappa = \frac{6D_b}{kGAl^2}. \tag{11}$$

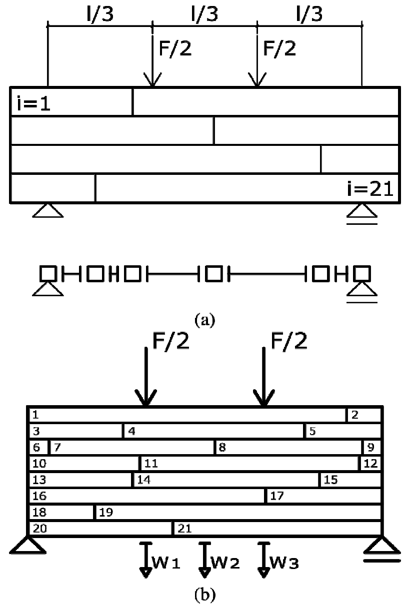

Fig. 6 (a) Model of laminated timber beam, (b) Actual location of all 21 segments in the examined beam

The FEM model of the laminated timber is established by localizing the stiffness matrices of individual elements into the global stiffness matrix of the structure. The positions of elements nodes were chosen to coincide with the finger joints of all segments in the beam as shown in Fig. 6 (a). The cross-section parameters, thus, do not change along the element length, but are different from element to element depending on actual values of Young's moduli in given element layup. The particular timber consists of 21 segments of different virtually random lengths arranged in 8 layers which divide the beam length into 14 elements, see Fig. 6 (b).

The calculated nodal displacements were finally introduced into the same elements shape functions as adopted in the formulation of the element stiffness in (10) to yield the vertical displacements $w_1 = w(x = l/3)$, $w_2 = w(x = l/2)$ and $w_3 = w(x = 2l/3)$ at the measurement points, see also Fig. 6 (b). Table II compares the numerical predictions provided by both the original 2D and the present beam-like FEM models clearly showing the sufficient accuracy and thus supporting the applicability of the simplified model.

TABLE II
COMPARISON OF NUMERICAL RESULTS PROVIDED BY 2D AND BEAM-LIKE FEM MODELS FOR MEAN VALUES OF PRIOR DISTRIBUTIONS OF YOUNG'S MODULI

Value	w_1 mm	w_2 mm	w_3 mm
2D model	24.21	27.71	24.09
Beam model	23.96	27.43	23.87

B. Definition of Response Function

Although for this particular study the beam-like model is sufficiently simple and efficient to enter the Bayesian updating procedure, it may still be more convenient to introduced further simplification particularly in view of the employed simulation softwares mentioned later in this section.

To this end, we shall consider the FEM computations as a simple function that maps an arbitrary set of values E_m and the load force levels F_j, j = 1,...,17 to the displacements at measured points i = 1,...,3 as

$$w_{ij} = f_i(\mathbf{E})F_j. \tag{12}$$

The function $f_i(\mathbf{E})$ thus represents the FEM calculations of a displacement at the i-th point for a unit load (F = 1 kN) and F_j represents the actual force applied in the j-th loading step.

The formulation of function $f_i(\mathbf{E})$ to be used within the Markov chain Monte Carlo (MCMC) program goes a one step further. Although the MCMC programs are capable of calling a user defined function in external module, this approach can be limiting for two reasons. First, the function (the finite element model) has to be implemented exclusively in C++, compiled only with recommended compiler and included in a certain module class implementing specific interfaces. Even though the process of creating a custom module is documented to a certain level it can be quite demanding to wire things together correctly. The second reason is potential performance limits. The custom function is evaluated every time the MCMC algorithm is called to generate one of typically thousands of samples.

To treat the above mentioned drawback we promote an alternative approach that replaces the function $f_i(\mathbf{E})$ with its approximation. Such approximation is easily implemented directly in the model definition and also faster to evaluate. In case of the FEM model of a laminated timber beam we approximated functions f_i linearly as[1]

$$\bar{f}_i(\mathbf{E}) = f_i(\mu_E) + (\nabla f)_{\mu_E} . (\mathbf{E} - \mu_E). \tag{13}$$

The derivatives of f_i were precomputed numerically by

$$\left.\frac{\partial f_i}{\partial E_j}\right|_{\mathbf{E}} \approx \frac{f_i(\mathbf{E}_j^+) - f_i(\mathbf{E}_j^-)}{2h}, \tag{14}$$

$$\mathbf{E}_j^{[\bullet]} = \{E_{1j}^{[\bullet]}, \ldots, E_{ij}^{[\bullet]}, \ldots, E_{nj}^{[\bullet]}\}^T, \tag{15}$$

[1]This simplified approach gives admissible results with the coefficient of variation in the range of 10–20%.

$$E_{ij}^{[\bullet]} = E_{ij}[\bullet]\delta_{ij}h, \tag{16}$$

where [•] denotes either addition or subtraction, δ_{ij} is Kronecker delta and h represents the small change in particular value E_i. The numerical test of the linear approximation showed that the error is below 2% for $E_i = \mu_{Ei} \pm 2\sigma_{Ei}$. Having the precomputed function values and derivatives on hand the evaluation of the approximated function (13) only involves vector subtraction and dot product which are trivial to express in both open source programs for generating the MCMC chains used in our study, namely the Stan and JAGS software products [4], [6].

C. Hierarchical Model

A hierarchical model of a stochastic system relates the involved quantities in either stochastic or deterministic manner. The model of bending test of a laminated timber beam can possibly be formulated as

$$w_{ij} \sim \cdot\,(\mu_{w_{ij}}, \sigma_w), \tag{17}$$

$$\mu_{w_{ij}} \leftarrow f_i(\mathbf{E})F_j, \tag{18}$$

$$E_i \sim \cdot\,(\mu_{E_i}, \sigma_{E_i}), \tag{19}$$

$$\sigma_w \sim \text{Inv-Gamma}(\alpha, \beta). \tag{20}$$

Relation (17) states that the measured values of displacements are normally distributed around the true values. The standard deviation is identical for all load steps and all measured points. Deterministic relation (18) includes the FEM model and provides the theoretical mean value based on the values of Young's moduli and load. Relation (19) specifies the distribution of our prior belief in different values of Young's moduli in different segments. We assume that each quantity E_i is normally distributed with the mean value and the standard deviation estimated previously from the indentation tests. Finally, we have to specify our prior belief in values of σ_w. In general, σ_w can either be exactly known or, as in our model, it can be estimated from the data. In the later case, we still have to provide the prior distributions. The inverse gamma distribution is commonly chosen for this purpose since it is a conjugate prior distribution for the parameter σ of the normal distribution $\mathrm{N}(\mu,\sigma)$. Parameters $\alpha = \beta = 0.1$ are fixed and their values are chosen to form a virtually uninformative distribution. Eventually, the present model results in 21 uncertain parameters (E_i and σ_w).

Having the model formulated, we utilize the Bayes theorem to express the posterior probability density of the parameters

$$p(\mathbf{E}, \sigma_w \mid w_{ij}) \propto p(\mathrm{w_{ij}} \mid \mathbf{E}, \sigma_w)\, p(\mathbf{E}, \sigma_w). \tag{21}$$

The likelihood can be expressed, with the help of (18), as

$$p(w_{ij} \mid \mathbf{E}, \sigma_w) = f_N(\mathrm{w_{ij}}, f_i(\mathbf{E})F_j, \sigma_w), \tag{22}$$

and since the variables E_i and σ_w are mutually independent we write the prior distribution as

$$p(\mathbf{E},\sigma_w) = f_{IG}(\sigma_w,\alpha,\beta)\prod_{i=1}^{21} f_N(E_i,\mu_{E_i},\sigma_{E_i}), \qquad (23)$$

where $f_N(y,\mu,\sigma)$ is the probability density function of normal distribution and $f_{IG}(y,\alpha,\beta)$ is the probability density function of the inverse gamma distribution.

The symbol ∝ means that the posterior distribution does not equal the right hand side but is merely proportional to it. Therefore, it does not integrate to 1 as is common for the probability density functions. Luckily, the family of MCMC algorithms allows us to generate samples of parameters distributed according to the posterior distribution in such a proportional form. Once we have enough samples of the updated model parameters, we compute their statistics, compare them to the prior distribution and thus learn how the observed data changed our prior belief in model parameters.

TABLE III
PRIOR AND POSTERIOR MEAN VALUES OF YOUNG'S MODULUS [GPA] OF I-TH LAMINA COMPUTED FROM 10000 SAMPLES GENERATED WITH STAN AND JAGS

i	$\mu_{E,pri}$	$\mu_{E,post}$		$\mu_{E,post}$ - $\mu_{E,pri}$	
		Stan	JAGS	Stan	JAGS
1	11.91	12.17	12.44	0.26	0.53
2	12.58	12.54	12.53	-0.04	-0.05
3	10.48	11.04	10.92	0.56	0.44
4	12.39	12.74	12.64	0.35	0.25
5	11.32	10.75	10.88	-0.57	-0.44
6	11.23	11.22	11.22	-0.01	-0.01
7	12.36	12.76	12.85	0.40	0.49
8	12.16	11.52	11.59	-0.64	-0.57
9	11.33	11.31	11.30	-0.02	-0.03
10	12.43	12.44	12.46	0.01	0.03
11	11.36	11.28	11.29	-0.08	-0.07
12	10.84	10.84	10.82	0	-0.02
13	11.77	11.80	11.80	0.03	0.03
14	12.18	12.17	12.17	-0.01	-0.01
15	12.66	12.64	12.63	-0.02	-0.03
16	11.76	12.04	12.11	0.28	0.35
17	12.28	11.69	11.81	-0.59	-0.47
18	12.15	12.23	12.22	0.08	0.07
19	12.47	12.51	12.51	0.04	0.04
20	11.65	14.34	13.83	2.69	2.18
21	12.23	11.31	11.17	-0.92	-1.06

D. Resulting Estimates of Posterior Distributions

As already mentioned in Section IV-B we have compared the application of two source programs for generating the MCMC chains of samples in Bayesian updating procedure. The resulting shifts in mean values of E_i generated by Stan and JAGS softwares are shown in Table III.

The results suggest that the mean values are corrected mostly in the segments that are located near the top and bottom parts of the timber beam, e.g. segments 1, 20 and 21, see Fig. 6 (b). This corresponds to the obvious fact that the material at these zones influences the most the resulting vertical displacements. On the other hand, the stiffness of segments which hardly influences the observed overall behavior is not updated at all, e.g. segments 6, 9 and 12.

It can also be seen from the data that there is no clear trend in the updated mean values. This can as well be expected, because the displacements computed with the prior mean values μ_{Ei} fit the measured values quite accurately, recall Table II and compare with laboratory measurements given in the last row of Table V. Therefore the mean values of Young's moduli are not either just increased or just decreased throughout the entire timber beam but are merely tuned to each other according to the likelihood given by the observed data.

The last stochastic parameter refined by the updating procedure is the standard deviation σ_w of the measured values w_{ij}. The samples generated from its posterior distributions by Stan have a mean value $\mu_{\sigma w}$ = 0.3406 (0.3459 for samples generated by JAGS) and standard deviation $\sigma_{\sigma w}$ = 0.02893 (0.03026 for JAGS). This suggests that the deterministic model represents the actual four point bending test fairly well and the assumption of linear relation between the load and the displacements is acceptable.

TABLE IV
PRIOR AND POSTERIOR STANDARD DEVIATIONS OF YOUNG'S MODULUS [GPA] OF I-TH LAMINA COMPUTED FROM 10000 SAMPLES GENERATED WITH STAN AND JAGS

i	$\sigma_{E,pri}$	$\sigma_{E,post}$		$\sigma_{E,post}$ - $\sigma_{E,pri}$	
		Stan	JAGS	Stan	JAGS
1	1.44	0.98	0.85	-0.46	-0.59
2	0.84	0.84	0.92	0	0.08
3	2.75	2.69	1.63	-0.06	-1.12
4	1.19	1.13	1.03	-0.06	-0.16
5	1.41	1.41	1.17	0	-0.24
6	0.97	0.99	1.00	0.02	0.03
7	0.96	0.96	0.99	0	0.03
8	1.21	1.21	1.09	0	-0.12
9	1.93	1.94	1.37	0.01	-0.56
10	0.94	0.94	0.96	0	0.02
11	1.35	1.35	1.17	0	-0.18
12	1.04	1.04	1.03	-0.01	-0.01
13	1.01	1.00	1.00	-0.01	-0.01
14	0.96	0.95	0.98	-0.01	0.02
15	0.84	0.83	0.92	-0.01	0.08
16	0.90	0.88	0.91	-0.02	0.01
17	1.26	1.25	1.11	-0.01	-0.15
18	1.34	1.34	1.17	0	-0.17
19	1.03	1.01	0.98	-0.02	-0.05
20	1.96	1.15	0.95	-0.81	-1.01
21	1.24	0.95	0.88	-0.29	-0.36

V. CONCLUSIONS AND FUTURE WORK

In this paper, the material properties were treated as random variables characterized by their distributions. The prior estimates were updated utilizing Bayesian statistical theory and exploiting the Stan and JAGS software products. The methodology of this approach was validated against the measurements of displacements.

It is fair to acknowledge that in this specific case of the linear response of the beam the Bayesian statistical method renders just a petty improvement compared to direct deterministic calculation of deflection using the measured values of E_i. Though the proposed methodology undoubtedly tends to decrease the standard deviation of deflection, see Table V.

TABLE V
COMPARISON OF FINAL PREDICTIONS BASED ON PRIOR AND POSTERIOR DISTRIBUTIONS WITH MEASUREMENTS

Method	w_1 (mm)		w_2 (mm)		w_3 (mm)	
	μ	σ	μ	σ	μ	σ
Prior Distrib.	24.21	1.198	27.86	1.383	24.34	1.232
Posterior Distrib. Stan	23.63	0.786	26.99	0.886	23.30	0.751
Posterior Distrib. JAGS	23.60	0.707	26.97	0.794	23.30	0.666
Measurements	23.47		26.68		24.05	

All the same, a more pronounced effect may be anticipated in predicting the bearing capacity of the beam, considering the fracture phenomena as well as the orthotropic properties of wood. Further improvement is contemplated by treating the material properties as random fields instead of random variables.

The mean values and standard deviations of posterior distribution slightly differ for samples generated with Stan and JAGS softwares. This indicates that one of the two generated chains may not be long enough to achieve steady state. In practical applications the convergence of Markov chain should be checked either visually or by appropriate convergence criteria.

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REFERENCES

[1] L. Melzerová and M. Šejnoha, “Interpretation of results of penetration tests performed on timber structures in bending,” in *Applied Mechanics and Materials*, vol. 486, 2014, pp. 347–352.

[2] N. Siu and D. Kelly, “Bayesian parameter estimation in probabilistic risk assessment,” in *Reliability Engineering and System Safety*, vol. 62, 1998, pp. 89–116.

[3] D Kelly and C. Smith, “Bayesian inference in probabilistic risk assessment – the current state of the art,” in *Reliability Engineering and System Safety*, vol. 94, 2009, pp. 628–643.

[4] M. Plummer, “JAGS: A program for analysis of Bayesian graphical models using Gibbs sampling,” (Online) *Available: http://mcmc-jags.sourceforge.net/* 2003.

[5] Stan Development Team, “Stan: A c++ library for probability and sampling, version 2.5.0,” (Online) *Available: http://mc-stan.org/* 2014.

[6] “Stan Modeling Language Users Guide and Reference Manual, Version 2.5.0,” (Online) *Available: http://mc-stan.org* 2014.