

Fekete-Szegö Problem for Subclasses of Analytic Functions Defined by New Integral Operator

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Abstract—The author introduced the integral operator, by using this operator a new subclasses of analytic functions are introduced. For these classes, several Fekete-Szegö type coefficient inequalities are obtained.

Keywords—Integral operator, Fekete-Szegö inequalities, Analytic functions.

I. INTRODUCTION AND DEFINITION

LET \mathcal{A} denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Also let \mathcal{S} denote the subclasses of \mathcal{A} consisting of functions which are univalent in \mathbb{U} .

In [2] Fekete and Szegö proved a noticeable result that the estimate

$$|a_3 - \mu a_2^2| \leq 1 + 2 \exp\left(\frac{-2\mu}{1-\mu}\right)$$

holds for $f \in \mathcal{S}$ and for $0 \leq \mu \leq 1$. This inequality is sharp for each μ . The coefficient functional

$$\phi_{\mu}(f) = a_3 - \mu a_2^2 = \frac{1}{6} \left(f'''(0) - \frac{3\mu}{2} (f''(0))^2 \right)$$

on $f \in \mathcal{A}$ represents various geometric quantities as well as in the sense that this behaves well with respect to the rotation, namely

$$\phi_{\mu}(e^{-i\theta} f(e^{i\theta} z)) = e^{2i\theta} \phi_{\mu}(f), \quad (\theta \in \mathbb{R}).$$

In fact, other than the simplest case when

$$\phi_0(f) = a_3,$$

we have several important ones. For example,

$$\phi_1(f) = a_3 - a_2^2,$$

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represent $S_f(0)/6$, where S_f denotes the Schwarzian derivative

$$S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

Moreover, the first two non-trivial coefficients of the n -th root transform

$$(f(z^n))^{\frac{1}{n}} = z + c_{n+1} z^{n+1} + c_{2n+1} z^{2n+1} + \dots$$

of f with the power series (1), are written by

$$c_{n+1} = \frac{a_2}{n}$$

and

$$c_{2n+1} = \frac{a_3}{n} + \frac{(n-1)a_2^2}{2n^2}$$

so that

$$a_3 - \mu a_2^2 = n(c_{2n+1} - \lambda c_{n+1}^2),$$

where

$$\lambda = \mu n + \frac{n-1}{2}$$

Thus, it is quite natural to ask about inequalities for ϕ_{μ} corresponding to subclasses of \mathcal{S} . This is called Fekete-Szegö problem. Actually many authors have considered this problem for typical classes of univalent functions.

Recently, in [1] the author introduced a certain integral operator $\mathcal{I} - c^{\delta}$ defined by :

$$\mathcal{I}_c^{\delta} f(z) = \frac{(1+c)^{\delta}}{\Gamma(\delta)} \int_0^1 t^{c-1} (\log 1/t)^{\delta-1} f(tz) dt, \quad (2)$$

where $c > 0$, $\delta > 1$ and $z \in \mathbb{U}$.

We also note that the operator $\mathcal{I}_c^{\delta} f(z)$ defined by (1) can be expressed by the series expansion as following:

$$\mathcal{I}_c^\delta f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+c}{k+c}\right)^\delta a_k z^k, \tag{3}$$

Obviously, we have, for $(\delta, \lambda \geq 0)$

$$\mathcal{I}_c^\delta (\mathcal{I}_c^\lambda f(z)) = \mathcal{I}_c^{\delta+\lambda} f(z). \tag{4}$$

and

$$\mathcal{I}_c^\delta (z f'(z)) = z (\mathcal{I}_c^\delta f(z))'. \tag{5}$$

Moreover, from (3), it follows that

$$z(\mathcal{I}_c^{\delta+1} f(z))' = (c+1)\mathcal{I}_c^\delta f(z) - c\mathcal{I}_c^{\delta+1} f(z) \tag{6}$$

We note that :

- For $c = 0$ and $\delta = n$ (n is any integer), the multiplier transformation $\mathcal{I}_0^n f(z) = I^n f(z)$ was studied by Flett [3] and Salagean [4];
- For $c = 0$ and $\delta = -n$ ($n \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$), the differential operator $\mathcal{I}_0^{-n} f(z) = D^n f(z)$ was studied by Salagean [4];
- For $c = 1$ and $\delta = n$ (n is any integer), the operator $\mathcal{I}_1^n f(z) = \mathcal{I}^n f(z)$ was studied by Uralegaddi and Somanatha [5];
- For $c = 1$, the multiplier transformation $\mathcal{I}_1^\delta f(z) = \mathcal{I}^\delta f(z)$ was studied by Jung et al. [6];
- For $c = a - 1$ ($a > 0$), the integral operator $\mathcal{I}_{a-1}^\delta f(z) = \mathcal{I}_{a-1}^\delta f(z)$ was studied by Komatu [7];

Using the operator \mathcal{I}_c^δ , we now introduce the following classes:

Definition 1: we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{S}_{c,\delta}(b)$ if

$$\Re \left\{ 1 + \frac{1}{b} \left(\frac{z(\mathcal{I}_c^\delta f(z))'}{\mathcal{I}_c^\delta f(z)} - 1 \right) \right\} > 0, \tag{7}$$

$(c > 0, \delta \geq 0, b \in \mathbb{C} \setminus \{0\}, z \in \mathbb{U}).$

Definition 2: we say that a function $f \in \mathcal{A}$ is in the class $\mathcal{C}_{c,\delta}(b)$ if

$$\Re \left\{ 1 + \frac{1}{b} \frac{z(\mathcal{I}_c^\delta f(z))''}{\mathcal{I}_c^\delta f(z)} \right\} > 0, \tag{8}$$

$(c > 0, \delta \geq 0, b \in \mathbb{C} \setminus \{0\}, z \in \mathbb{U}).$

Note that

$$f \in \mathcal{C}_{c,\delta}(b) \Leftrightarrow z f' \in \mathcal{S}_{c,\delta}(b). \tag{9}$$

In particular, we have starlike and convex function classes, $\mathcal{S}_{c,0}(1) = \mathcal{S}^*$ and $\mathcal{C}_{c,0}(1) = \mathcal{C}$, respectively.

We denote by P a class of the analytic functions in \mathbb{U} with

$$p(0) = 1 \text{ and } \Re\{p(z)\} > 0.$$

To prove our results, we need the following Lemmas considered by Duren [8] Ravichandran et al. [9].

Lemma 1: [8] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$. Then

$$|c_n| \leq 2, \quad (n \geq 1).$$

Lemma 2: [9] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$. Then for any complex number γ

$$|c_2 - \gamma c_1^2| \leq 2 \max\{1, |2\gamma - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 3: [8] Let $p \in P$ with $p(z) = 1 + c_1 z + c_2 z^2 + \dots$. Then

$$\left| c_2 - \frac{1}{2} \lambda c_1^2 \right| \leq 2 + \frac{1}{2} (|\lambda - 1| - 1) |c_1|^2.$$

II. MAIN RESULTS

Theorem 1: Let $c, \delta \geq 0; b \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{S}_{c,\delta}(b)$, then

$$|a_2| \leq 2|b| \left(\frac{c+2}{c+1}\right)^\delta,$$

$$|a_3| \leq |b| \left(\frac{c+3}{c+1}\right)^\delta \max\{1, |1+2b|\},$$

and

$$\left| a_3 - \frac{1}{2} \left(\frac{(c+1)(c+3)}{(c+2)^2} \right)^\delta a_2^2 \right| \leq |b| \left(\frac{c+3}{c+1}\right)^\delta.$$

Proof. Denote

$$\mathcal{I}_c^\delta = z + A_2 z^2 + A_3 z^3 + \dots$$

Then by (3), we can write

$$A_2 = \left(\frac{c+1}{c+2}\right)^\delta a_2, \quad A_3 = \left(\frac{c+1}{c+3}\right)^\delta a_3. \quad (10)$$

$$= \left| a_3 - \frac{1}{2} \left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta a_2^2 \right|$$

$$= \left| \frac{b}{2} \left(\frac{c+3}{c+1}\right)^\delta (c_2 + bc_1^2) - \frac{b^2 c_1^2}{2} \left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta \left(\frac{c+2}{c+1}\right)^{2\delta} \right|$$

$$= \left| \frac{bc_2}{2} \left(\frac{c+3}{c+1}\right)^\delta \right|$$

$$\leq |b| \left(\frac{c+3}{c+1}\right)^\delta.$$

by the definition of the class $\mathcal{S}_{c,\delta}(b)$, there exists $p \in P$ such that:

$$1 + \frac{1}{b} \left(\frac{z(\mathcal{I}_c^\delta f(z))'}{\mathcal{I}_c^\delta f(z)} - 1 \right) = p(z),$$

$$\frac{z(\mathcal{I}_c^\delta f(z))'}{\mathcal{I}_c^\delta f(z)} = 1 - b + bp(z),$$

so that

$$\frac{z(1 + 2A_2z + 3A_3z^2 + \dots)}{z + A_2z^2 + A_3z^3 + \dots} = 1 - b + b(1 + c_1z + c_2z^2 + \dots),$$

which implies the equality

$$z + 2A_2z^2 + 3A_3z^3 + \dots$$

$$= z + (A_2 + bc_1)z^2 + (A_3 + bc_1A_2 + bc_2)z^3 + \dots$$

Equating the coefficients of both side, we have

$$A_2 = bc_1, \quad A_3 = \frac{b}{2}(c_2 + bc_1^2), \quad (11)$$

so that, on account of (10)

$$a_2 = b \left(\frac{c+2}{c+1}\right)^\delta c_1, \quad a_3 = \frac{b}{2} \left(\frac{c+3}{c+1}\right)^\delta (c_2 + bc_1^2). \quad (12)$$

Taking into account (12) and Lemma 1, we obtain

$$|a_2| \leq 2|b| \left(\frac{c+2}{c+1}\right)^\delta,$$

and Lemma 2

$$|a_3| = \left| \frac{b}{2} \left(\frac{c+3}{c+1}\right)^\delta (c_2 + bc_1^2) \right|$$

$$\leq |b| \left(\frac{c+3}{c+1}\right)^\delta \max\{1, |1 + 2b|\}.$$

Moreover, by Lemma 1

as asserted.

Now, we consider functional $|a_3 - \mu a_2^2|$ for complex μ .

Theorem 2: Let $c, \delta \geq 0; b \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{S}_{c,\delta}(b)$, then for $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq |b| \left(\frac{c+3}{c+1}\right)^\delta \max \left\{ 1, \left| 1 + 2b - 4\mu b \left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta \right| \right\}.$$

Moreover for each μ , there is a function in $\mathcal{S}_{c,\delta}(b)$ such that equality holds.

Proof. Taking into account (12) we have

$$a_3 - \mu a_2^2 = \frac{b}{2} \left(\frac{c+2}{c+1}\right)^\delta (c_2 + bc_1^2) - \mu b^2 c_1^2 \left(\frac{c+2}{c+1}\right)^{2\delta}$$

$$= \frac{b}{2} \left(\frac{c+2}{c+1}\right)^\delta (c_2 + \beta c_1^2), \quad (13)$$

where

$$\beta = -b + 2\mu b \left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta.$$

Then, with the aid of Lemma 2, we obtain

$$|a_3 - \mu a_2^2|$$

$$\leq |b| \left(\frac{c+3}{c+1}\right)^\delta \max \left\{ 1, \left| 1 + 2b - 4\mu b \left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta \right| \right\}. \quad (14)$$

as asserted. An examination of the proof shows that equality is attained for the first case when $c_1 = 0$ and $c_2 = 2$ and the corresponding $f \in \mathcal{S}_{c,\delta}(b)$ is given by

$$\frac{z(\mathcal{I}_c^\delta f(z))'}{\mathcal{I}_c^\delta f(z)} = \frac{1 + (2b - 1)z^2}{1 - z^2}, \quad (15)$$

and likewise for the second case when $c_1 = c_2 = 2$ the corresponding $f \in \mathcal{S}_{c,\delta}(b)$ is given by

$$\frac{z(\mathcal{I}_c^\delta f(z))'}{\mathcal{I}_c^\delta f(z)} = \frac{1 + (2b - 1)z}{1 - z}, \quad (16)$$

respectively.

Taking $\delta = 0$ and $b = 1$ in Theorem 2, we have :

Corollary 1: [10] If $f \in S^*$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \max\{1, |4\mu - 3|\}.$$

Moreover for each μ , there is a function in S^* such that equality holds.

We next consider the case when μ and b are real. Then we have

Theorem 3: Let $c, \delta \geq 0; b > 0$. If $f \in \mathcal{S}_{c,\delta}(b)$, then for $\mu \in \mathbb{R}$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} b\left(\frac{c+3}{c+1}\right)^\delta \left[1 + 2b - 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta\right] & \text{if } \mu \leq \frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta \\ b\left(\frac{c+3}{c+1}\right)^\delta & \text{if } \frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta \leq \mu \leq \frac{1+b}{2b}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta \\ b\left(\frac{c+3}{c+1}\right)^\delta \left[-1 - 2b + 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta\right] & \text{if } \mu \geq \frac{1+b}{2b}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta \end{cases}$$

Moreover for each μ , there is a function in $\mathcal{S}_{c,\delta}(b)$ such that equality holds.

Proof. By (14), we obtain

$$a_3 - \mu a_2^2 = \frac{b}{2}\left(\frac{c+3}{c+1}\right)^\delta \left[c_2 - \frac{c_1^2}{2} + \frac{c_1^2}{2}\left(1 + 2b - 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta\right) \right]. \tag{17}$$

First, let $\mu \leq \frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta$. in this case, by (17), Lemma 1 and Lemma 3 give

$$|a_3 - \mu a_2^2| \leq \frac{b}{2}\left(\frac{c+3}{c+1}\right)^\delta \left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(1 + 2b - 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta\right) \right] \leq b\left(\frac{c+3}{c+1}\right)^\delta \left[1 + 2b - 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta \right].$$

Now let $\frac{1}{2}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta \leq \mu \leq \frac{1+b}{2b}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta$. Then, using the above calculations, we get

$$|a_3 - \mu a_2^2| \leq b\left(\frac{c+3}{c+1}\right)^\delta.$$

Finally, if $\mu \geq \frac{1+b}{2b}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta$, then we obtain

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{b}{2}\left(\frac{c+3}{c+1}\right)^\delta \left[2 - \frac{|c_1|^2}{2} + \frac{|c_1|^2}{2}\left(-1 - 2b + 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta\right) \right] \\ &\leq \frac{b}{2}\left(\frac{c+3}{c+1}\right)^\delta \left[2 + \frac{|c_1|^2}{2}\left(-2 - 2b + 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta\right) \right] \\ &\leq b\left(\frac{c+3}{c+1}\right)^\delta \left[-1 - 2b + 4\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta \right]. \end{aligned}$$

Equality is attained for the second case on choosing $c_1 = 0, c_2 = 2$ in (15) and in (16) $c_1 = c_2 = 2; c_1 = 2i, c_2 = -2$ for the first and third case, respectively. Thus the proof is complete.

Using the relation (9), we easily obtain bounds of coefficients and a solution of the Fekete-Szegő problem in $\mathcal{C}_{c,\delta}$.

Theorem 4: Let $c, \delta \geq 0; b \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{C}_{c,\delta}(b)$, then

$$|a_2| \leq |b|\left(\frac{c+2}{c+1}\right)^\delta,$$

$$|a_3| \leq \frac{|b|}{3}\left(\frac{c+3}{c+1}\right)^\delta \max\{1, |1 + 2b|\},$$

and

$$\left| a_3 - \frac{2}{3}\left(\frac{(c+1)(c+3)}{(c+2)^2}\right)^\delta a_2^2 \right| \leq \frac{|b|}{3}\left(\frac{c+3}{c+1}\right)^\delta.$$

Reasoning in the same line as in proof of Theorem 2 obtain :

Theorem 5: Let $c, \delta \geq 0; b \in \mathbb{C} \setminus \{0\}$. If $f \in \mathcal{C}_{c,\delta}(b)$, then for $\mu \in \mathbb{C}$, we have

$$|a_3 - \mu a_2^2| \leq \frac{|b|}{3}\left(\frac{c+3}{c+1}\right)^\delta \max\left\{1, \left|1 + 2b - 3\mu b\left(\frac{(c+2)^2}{(c+1)(c+3)}\right)^\delta\right|\right\}.$$

Moreover for each μ , there is a function in $\mathcal{C}_{c,\delta}(b)$ such that equality holds.

By taking $\delta = 0$ and $b = 1$ in Theorem 5, we have

Corollary 2: [10] If $f \in \mathcal{C}^*$, then for $\mu \in \mathbb{C}$ we have

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{3}, |\mu - 1|\right\}.$$

Moreover for each μ , there is a function in \mathcal{C}^* such that equality holds.

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