International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:9, No:6, 2015

Extending the Quantum Entropy to Multidimensional Signal Processing

Youssef Khmou, Said Safi, Miloud Frikel

Abstract—This paper treats different aspects of entropy measure in classical information theory and statistical quantum mechanics, it presents the possibility of extending the definition of Von Neumann entropy to image and array processing. In the first part, we generalize the quantum entropy using singular values of arbitrary rectangular matrices to measure the randomness and the quality of denoising operation, this new definition of entropy can be implemented to compare the performance analysis of filtering methods. In the second part, we apply the concept of pure state in quantum formalism to generalize the maximum entropy method for narrowband and farfield source localization problem. Several computer simulation results are illustrated to demonstrate the effectiveness of the proposed techniques.

Keywords—Von Neumann entropy, Filtering, array, DoA, Maximum Entropy Method.

I. Introduction

In communication theory, Shannon entropy [1] is statistical measure that characterizes the uncertainty of information source, it is logarithmic function of probability distribution of source, when the signal transmitted is deterministic, the entropy is null, in the other hand when the signal is random, entropy is maximal, so there is total lack of information. Shannon's entropy is applied in image processing [2] to measure the randomness in order to analyse the texture of images. In quantum information theory, the statistical measure of mixture of quantum systems uses Von Neumann entropy [3], which the extended concept of Shannon entropy to the quantum field, where it is based on eigenvalues of density operators.

In this paper, we propose an extension of Von Neumann entropy to several aspects of multidimensional signal processing, in the following section we brievely present a description of classical and quantum entropies, in the third section, we apply the quantum entropy in filtering process and in the last part we generalize the Maximum Entropy Method (MEM) for narrowband and farfield punctual source localization problem, using statistical processing of signals received by an array of sensors.

II. Classical and Quantum Entropies

A. Classical Entropy

We consider a continuous random process x, with probability density function $p(x)$, which takes values in an interval J, using natural logarithm, the corresponding entropy is defined by the following integral :

$$h_c(x) = -\int_J p(x)\ln p(x)dx \tag{1}$$

this definition requires the convention $0\ln 0 = 0$, for discrete process $x = \{x_1, ..., x_K\}$ with discrete probability distribution $p(x_i)$, the entropy is :

$$h_c(x) = -\sum_{i\geq 1} p_i \ln p_i \tag{2}$$

the function $h_c(x)$ represents a measure of uncertainty or lack of information, for example if we know the value of the second outcome $p(x_2) = 1$ and $p(x_i) = 0$ for $i \neq 2$ then $h_c(x) = -1\ln 1 = 0$, this result means that we have complete knowledge of the process x, in the other hand if all the probabilties are equal $p(x_i) = 1/K$, in this case we have $h_c(x) = lnK$ and the entropy is maximal, in fact this result can be obtained using Lagrange multipliers where we search for maximum of function $h_c(x)$ subject to the constraint that $\sum_i^K p(x_i) = 1$. This second example is rooted in statistical mechanics where the considered system (consisting of very large number of particles) undergoes a process and reaches the equilibrium when the entropy is maximal, the total energy of the system is equipartitionned over all the constituents. For random process x given by Gaussian distribution $\mathcal{N}(\mu, \sigma)$ the theoretical value [1] of the entropy is :

$$h(x) = -\int_{\mathbb{R}} p(x)\ln p(x)dx = \frac{1}{2}\left(1 + \ln\left(2\pi\sigma^2\right)\right) \tag{3}$$

$h_c(x)$ is only dependent on variance σ^2. Similarely, for N dimensional random process $X \in \mathbb{R}^N$, the corresponding probability density function [4] is:

$$p(X) = \frac{1}{\sqrt{2\pi^N|\Gamma|}} e^{-\frac{1}{2}(X-\mu)^T\Gamma^{-1}(X-\mu)} \tag{4}$$

with $|\Gamma|$ is the determinant of covariance matrix $\Gamma = \langle XX^T\rangle$ which is symmetric and positive-definite and mean vector $\mu \in \mathbb{R}^N$, the entropy is given by :

$$h_c(X) = \frac{1}{2}\left(N + \ln\left((2\pi)^N|\Gamma|\right)\right) \tag{5}$$

To calculate $h_c(x)$ or $h_c(X)$, of discrete vector or matrix respectively, requires the computation of normalized histogram, therefore it is necessary to choose an adequate number of bins. The above definitions are valid in signal and arry processing fields, in quantum information theory, the quantum entropy uses other mechanism that we present in the next part.

Youssef Khmou and Said Safi are with Department of Mathematics and Informatics, polydisciplinary faculty, Sultan Moulay Slimane University, Beni Mellal, Morocco (e-mail: khmou.y@gmail.com, safi.said@gmail.com).
Miloud Frikel is with GREYC Lab UMR 6072 CNRS, Equipe Auto-matique, Avenue 6 juin, 14053, Caen, France. (e-mail: mfrikel@greyc.ensicaen.fr).

B. *Quantum Information Entropy*

The quantum theory [3] is a generalization of classical mechanics where the state of a system is described by space phase. If we conisder N dimensional Hilbert space, the state of system, in quantum information theory, is represented by an ensemble of operators. For example, a quantum source is described by state vector called ket $|\Psi_i\rangle \in \mathbb{C}^{N\times 1}$ that contains all the information of the system, as example the photon's polarization state. A density matrix of quantum system of mixed state is a statistical set of all possible quantum states with associated probilities λ_i, a density operator is given by spectral decomposition :

$$\rho = \sum_{i=1}^{N} \lambda_i \,|\Psi_i\rangle \langle\Psi_i| \tag{6}$$

where $|\Psi_i\rangle^+ = \langle\Psi_i|$ and $(.)^+$ is the conjugate transpose transformation, the states $|\Psi_i\rangle$ are orthogonal $\langle\Psi_i|\Psi_i\rangle = \delta_{ij}$ and the probabilties $\sum_i^N \lambda_i = 1$, the Von Neumann entropy [3] of mixed state is :

$$h_q(\rho) = -\sum_{i=1}^{N} \lambda_i \ln \lambda_i = -Tr(\rho \ln \rho) \tag{7}$$

The entropy of pure state is zero where the operator is idempotent $\rho^2 = \rho$, while that of mixed state is always greater than zero and becomes maximal $h_q(\rho) = \ln N$ when the state is totally mixed ($\rho = N^{-1} I_N$). Among the properties of $h_q(\rho)$ is that it is invariant under unitary transformation, for unitary matrix $U \in \mathbb{C}^{N\times N}$ such that $UU^+ = I_N$, we have $h_q(U\rho U^+) = h_q(\rho)$. From this definition, we conclude that the quantum entropy is dependent only on eigenvalues, $h_q(\rho)$ is defined for square operators. If we want to extend this concept into signal processing field, then the generalization of $h_q(\rho)$ in (7) consists of using normalized singular values instead of eigenvalues, with this proposition we can use the quantum entropy to measure the randomness of arbitrary rectangular matrix $X \in \mathbb{R}^{N\times K}$, this mechanism is explained in the following section.

III. Quantum Entropy and Filtering

In this part, we extend Von neuman entropy to image processing using singular values, and we demonstrate the advantage of this proposition. We begin by explainig the Singular Value Decomposition (SVD), let us consider $X \in \mathbb{R}^{N\times K}$, unitary matrices $U \in \mathbb{R}^{N\times N}$, $V \in \mathbb{R}^{K\times K}$ and diagonal matrix $\Sigma \in \mathbb{R}^{N\times K}$ verify $X = U\Sigma V^T$, the diagonal elements of Σ are the singular values $\Sigma_{ii} = s_i$ for $i = \{1, ..., N\}$, the "Quantum transformation" herein consists of normalization $\lambda_i = s_i / \sum_i^N s_i$, where λ_i are interpreted as probability coefficients such as $\sum_i^N \lambda_i = 1$, next the extended Von Neumann entropy is :

$$h_q(X) = -\sum_{i}^{N} \lambda_i \ln \lambda_i \tag{8}$$

which is maximal if $\lambda_i = 1/N$. In image processing, comparing two samples is usually done using Root Mean Square Error (RMSE) because classical entropy is not effective in this case, to clarify this idea let us take a simple example of two matrices $X_1 = [1, 1; 3, 5]$ and $X_2 = [3, 3; 7, 9]$, classical entropy is the same for both samples, $h_c(X_1) = h_c(X_2) = 1.0397$ although $X_1 \neq X_2$, however if use the qantum extension in (8), we find that $h_q(X_1) = 0.206$ and $h_q(X_2) = 0.1648$, this example shows the advantage of using singular values. A second example that we present is focused on measuring the quality of filtering, for this purpose we consider a sample $X \in \mathbb{R}^{280\times 272}$ given in Fig. 1 with corresponding Von Neumann entropy $h_q = 3.1049$, we progressively add a zero mean Gaussian noise n where the noise standatd deviation σ is in the range $[0.01, 2]$ and the result $Y = X + n$ for final value $\sigma = 2$ represents an ergodic state as shown in Fig. 2, each time we denoise Y using Generalized Wiener filter [5].

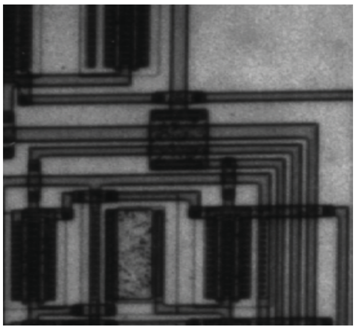

Fig. 1: Initial state.

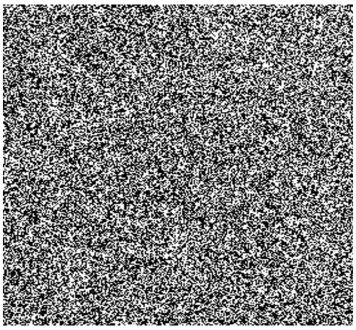

Fig. 2: Final state.

During the processing, we measure the quantum entropies of the additive noise $h_q(n)$, the noisy sample $h_q(Y)$ and the recovered sample $h_q(\hat{Y})$, the obtained results are given in Fig. 3, we realize that $h_q(n) \simeq 5.40$ is independent of noise power and tends to $\ln 280$, the function of noisy sample $h_q(Y)$ converges to $h_q(n)$ and $h_q(\hat{Y})$ diverges slowly

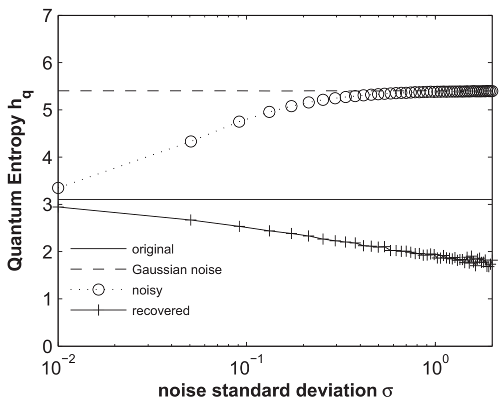

Fig. 3: Quantum entropy variations of four different matrices with respect to noise power.

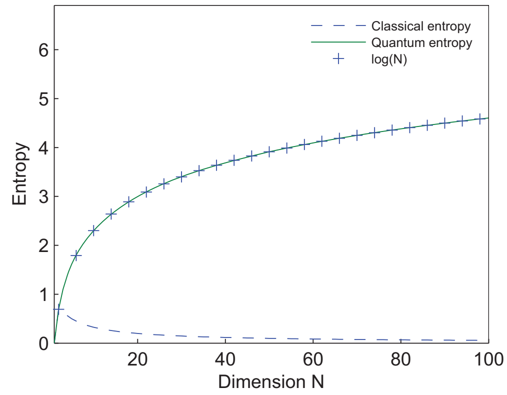

Fig. 4: Quantum and classical entropies of identity operator I_N with varying dimension N.

from original value $h_q(X)$, the gap between the entropy functions of original and recovered samples is interpreted as irreversibility of process. As conclusion of this section, Von Neumann entropy can be implemented to measure the quality of denoising process instead of RMSE.

Depending on the nature of input matrix X, interpreting the result of quantum entropy $h_q(X)$ can have an ambiguity relativeley to the classsical entropy $h_c(X)$, we treat two examples to clarify this difference.

Let us consider an identity operator of dimension N such as $[I_N]_{ij} = \delta_{ij}$, this operator has two values $Card(I_N) = \{0, 1\}$, we compute the two associated entropies to show the difference of interpretation, the normalized eigenvalues of I_N are $\lambda_i = 1/N$ and the quantum entropy is maximal :

$$h_q(I_N) = -\sum_{i=1}^{N} \lambda_i \log \lambda_i = \log N \tag{9}$$

In the other hand, the probabilites of inputs 0 and 1, in terms of classical entropy, are :

$$\begin{cases} p(1) = \dfrac{1}{N} \\ \\ p(0) = \dfrac{N-1}{N} \end{cases} \tag{10}$$

the classical entropy has the following expression :

$$h_c(I_N) = -\sum_{i=1}^{2} p(i) \log p(i) = \log N - \frac{(N-1)}{N} \log(N-1) \tag{11}$$

Fig. 4 represents the variations of two metrics $h_c(I_N)$ and $h_q(I_N)$ with respect to the variation of dimension N.

We can remark that at $N = 2$, the two functions are equal $h_c(I_2) = h_q(I_2) = \log 2$ and the interpretation of the state of matrix is ambiguous because when the dimension N tends to infinity, the quantum entropy indicates that I_N represents totally mixed state in terms of quantum mechanics. Contrary to the classical entropy, the result of $h_c(I_N)$ is valid interpretation because when the dimension is larger, the randomness of I_N is diminished which is expressed by the behavior of $h_c(I_N)$ that tends to zero.

A counter example that we present is a class of constant square matrices $X \in \mathbb{R}^{N \times N}$ defined as :

$$X = \begin{pmatrix} a & \dots & a \\ \vdots & \ddots & \vdots \\ a & \dots & a \end{pmatrix} \tag{12}$$

where a is constant, in this cas we demonstrate that the interpretation of the entropy result for both $h_c(X)$ and $h_q(X)$ are in agreement. For classical entropy, the input matrix X is uniform as $p(a) = 1$ and the entropy is zero $h_c(X) = -p(a) \log p(a) = 0$. For quantum entropy, we use the normalized eigenvalues, the spectrum of constant matrix X is given by :

$$\sigma_X = [Na, 0_{1\times N-1}] \tag{13}$$

normalizing the spectrum σ_X yields to the spectrum $[1, 0_{1\times N-1}]$ that corresponds to null quantum entropy $h_q(X) = -1 \log 1 - (N-1) 0 \log 0 = 0$, we remark in this example that the results are in agreement.

In the next section we present another viewpoint of maximum entropy principle in array processing, a field that combines statistical signal processing, interferometry and wave propagation.

IV. Array Processing and Maximum Entropy Principle

A. Signal Model

The aim of array processing is to characterize radiating sources by analyzing the wavefield intercepted by array of N antennas [6], it is mainly based on statistics of received data subject to the geometry of the antenna and the nature of emitting sources. In this part, we only focus on studying a single parameter which is the angular positions of punctual sources located in large distance relatively to the array as illustrated in Fig. 5.

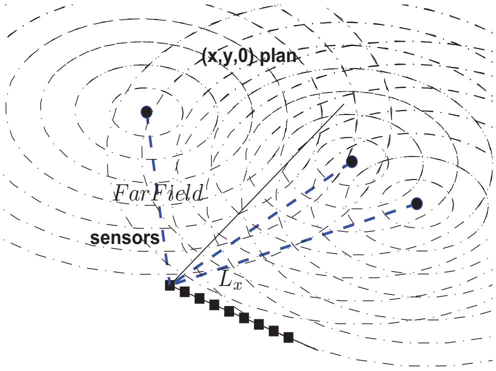

Fig. 5: Two dimensional propagation model, three punctual sources imping on linear array of sensors in horizontal plane.

After a period T of data acquisition, if there were P sources that were impinging, such that $P < N$, then after downconversion into digital form, the received signals at instant t are given by:

$$x(t) = A(\theta)s(t) + n(t) \tag{14}$$

where $x(t) \in \mathbb{C}^{N\times1}$, $A(\theta) \in \mathbb{C}^{N\times P}$ is the steering matrix that depends on the geometry of the array, $s(t) \in \mathbb{C}^{P\times1}$ is the vector of complex envelopes of radiating sources and $n(t) \in \mathbb{C}^{N\times1}$ is zero mean and complex random process due to thermal noise and interefences between the waves during the propagation. A simple configuration of the antenna is the Uniform Linear Array (ULA) where the steering matrix has Vandermonde structure $A(\theta) = [a(\theta_1), ..., a(\theta_P)]$ with vector $a(\theta_i) = \left[1, e^{-j\mu_i}, ..., e^{-j(N-1)\mu_i}\right]^T$, the spatial frequencies are $\mu_i = 2\pi d \sin(\theta_i)\lambda^{-1}$ with d being the inter-element distance between sensors and θ_i is the i^{th} Direction of Arrival (DoA). λ is the wavelength of incoming waves where we suppose that all sources have the same carrier frequency $f = c\lambda^{-1}$ with c is the speed of phase.

In (9) we suppose that the signal sources $s(t)$ are slowly varying envelopes relatively to the carrier waves, and the distance d must be less or equal to half the wavelength since the spatial frequency is bounded by $-\pi \le \mu_i \le \pi$. The analysis of random matrix $X \in \mathbb{C}^{N\times K}$ is usually done using second order statistics $\langle x(t)x^+(t)\rangle$, indeed, the theoretical expression of covariance matrix is :

$$\Gamma = \langle x(t)x^+(t)\rangle = \lim_{K\to+\infty}\frac{1}{K}\sum_{t=1}^{K} X(t)X^+(t) = A\Gamma_{ss}A^+ + \Gamma_n \tag{15}$$

Only an estimate of Γ is obtained because of finite number of samples K, $\Gamma_{ss} = \langle ss^+\rangle$ is the correlation matrix of symbols $s(t)$ and $\Gamma_n = \sigma^2 I_N$ is covariance matrix of noise $n(t)$ which is uncorrelated between the sensors and statistically independent of $s(t)$ with same variance σ^2.

Let us consider an N dimensional Hilbert space, based on the fact that data is a combination of signal plus noise, the spectral decomposition of Γ contains two subspaces (signal and noise), the decomposition is given by :

$$\Gamma = \sum_{i=1}^{N}\lambda_i \left|u_i\right\rangle\left\langle u_i\right| = \sum_{i=1}^{P}\lambda_i \left|u_i\right\rangle\left\langle u_i\right| + \sum_{j=P+1}^{N}\lambda_j \left|u_j\right\rangle\left\langle u_j\right| \tag{16}$$

λ_i is the eigenvalue that corresponds to the i^{th} eigenvector $|u_i\rangle \in \mathbb{C}^{N\times1}$, in compact form we have $\Gamma = U\Lambda U^+$ where U is unitary matrix and Λ is diagonal matrix. The eigenspace is written as $U = [U_s, U_n]$ with $U_s \in \mathbb{C}^{N\times P}$ is the signal subspace and $U_n \in \mathbb{C}^{N\times N-P}$ is noise subspace. Following this partition, the projectors into both subspaces are $P_s = U_sU_s^+$ and $P_s = U_nU_n^+$ where they form a complete base $P_s + P_n = I_N$.

As mentionned earlier, we only search for angular positions of punctual emitters, thus we have to use the expression of steering matrix in (9) and choose a range of scan $\Omega = [\theta_{min}, \theta_{max}]$, if we consider a steering vector $a(\theta)$ with $\theta \in \Omega$, we have :

$$f(\theta) = a^+(\theta)P_na(\theta) = \begin{cases} 0 & \text{If } \theta \text{ is DoA} \\ \neq 0 & \text{Otherwise} \end{cases} \tag{17}$$

After assembling all the values for θ, we obtain an angular spectrum $f(\theta) = \left(a^+(\theta)P_na(\theta)\right)^{-1}$ whose indexes of highest peaks represent the estimated DoAs. The subspace U_n is extracted from U based on the clustering of eigenvalues λ_i into two sets where noise eigenvalues $\lambda_j \simeq \sigma^2$, this mechanism is known as Multiple Signal Classification (MUSIC) [6]. Among other techniques of localization is the Maximum Enropy Method which we generalize in the next section.

B. Generalized Maximum Entropy Method

In narrowband localization problem, the Maximum Entropy Method (MEM) [7] gives accurate results when the carried signals $s(t)$ are ergodic and complex random processes, which is the model of cosmic sources, this technique is based on an optimization problem where we search for a vector which minimizes the output power of the array subject to the constraint that at least one element of solution vector equals one :

$$\text{Min}\{a^+\Gamma a\} \text{ Subject to } a^+e_i = 1 \tag{18}$$

e_i is the i^{th} column of identity matrix, the Lagrangian of this problem is given by:

$$L(a, \lambda) = a^+\Gamma a - \lambda(1 - a^+e_i) \tag{19}$$

A minimum of the above function is reached when the first order derivative is zero $\partial L(a, \lambda)/\partial a = 0$, the covariance matrix Γ is Hermitian ($\Gamma^+ = \Gamma$) which yields to $2\Gamma a - \lambda e_i = 0$, using the constraint, the solution is given by :

$$a = \frac{\Gamma^{-1}e_i}{{e_i}^T\Gamma^{-1}e_i} = \alpha\Gamma^{-1}e_i \tag{20}$$

where α is a constant, the MEM matrix is given by the relation $P_i = aa^+ = \Gamma^{-1}e_ie_i^T\Gamma^{-1}$ which is dyadic operator

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:9, No:6, 2015

constructed from i^{th} column of inverse of covariance matrix. If the localization problem is two dimensional (azimuth θ and elevation φ), P_i is of rank one and may present some deviations [7] in locating highest peaks, therefore we propose a Generalized MEM (G-MEM) which is full rank operator.

The idea is to sum over all possible operators P_i for $i = \{1, ..., N\}$ after transforming them into projectors, P_i contains a single non zero eigenvalue β as $Tr(P_i) = \beta$, using the definition of density matrix of pure state in quantum mechanics, we have :

$$\rho_i = \frac{P_i}{Tr(P_i)} \tag{21}$$

ρ_i is projector such that $\forall n \in \mathbb{N}$, $\rho_i^n = \rho_i$, the Von Neumann entropy becomes $h(\rho_i) = -\sum_{i=1}^{N} \lambda_i \ln \lambda_i = -1 \ln 1 = 0$, finally the generalized operator is given by :

$$Q = \sum_{i}^{N} \rho_i = \sum_{i}^{N} \frac{P_i}{Tr(P_i)} \tag{22}$$

$rank\{Q\} = N$, the corresponding eigenvalues are given in ascending order $\{\lambda_1 \simeq \lambda_2 \simeq ... \simeq \lambda_P < \lambda_{P+1} \leq ... \leq \lambda_N\}$, the advantage of the generalized operator's spectrum is that signal eigenvalues are attenuated or "annihilated". To evaluate the performance of G-MEM we present in the next section a comparative study.

C. Results and Discussion

In this section, we conduct several computer simulations to evaluate the resolution power of the operator Q, for all tests we consider a uniform array of $N = 20$ identical and isotropic sensors separated by half the wavelength, Rayleigh limit angular resolution of this array is $\theta_{HPBW} \simeq 6°$, we suppose that $P = 2$ far field and narrowband punctual sources are impinging from azimuths $\theta_1 = 13°$ and $\theta_2 = 15°$ in the same horizontal plan with the array, which means that the elevation angles are $\varphi_1 = \varphi_2 = \pi/2$. The sources are equipowered $\sigma_1 = \sigma_2 = 1$ watt, and transmit random BPSK symbols. The number of samples is set to $K = 60$ snapshots. In the first test, we compute the average of $L = 100$ Monte Carlo trials of G-MEM operator where $SNR = 5dB$, the result is given in Fig. 6.
We remark that the localization function is of high resolution as given in the magnified plot where the two sources are separated, with no sidelobes in other directions. To evaluate G-MEM comparatively to other spectra, we choose five different subspace operators which are MUSIC projector, Orthonormal Propagator [8], Ermolaev-Gershman [9] and Minimum Norm [6]-[9] operators, for each value of SNR we calculate the average of $L = 100$ trials of RMSE between true and estimated DoAs, the results are given in Fig. 7.

In the range $[-5dB, 5dB]$, MUSIC projector gives most accurate results, the performance of Minimum Norm and Ermolaev-Gershman is almost the same, G-MEM function comes in the fourth position and Orthogonal Propagator has the height incertainty. Staring from $10dB$, the quadratic errors

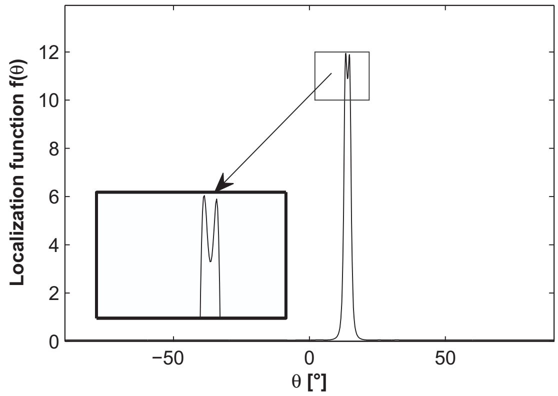

Fig. 6: Average of 100 realizations of G-MEM localization function with $N = 20$, $K = 60$, $\theta = [13°, 15°]$, $d = \lambda/2$ and $SNR = 5dB$.

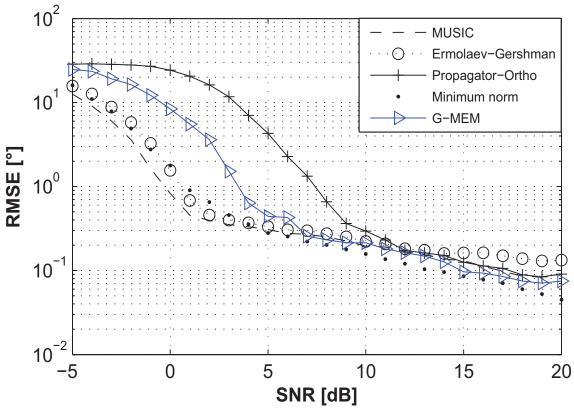

Fig. 7: Average of 100 realizations of RMSE for each DoA operator with $N = 20$, $K = 60$, $\theta = [13°, 15°]$ and $d = \lambda/2$.

of all operators are almost equivalent in this configuration.

In the last test, we measure the resolution of the G-MEM localization function comparatively to MEM operator in the presence of spatially correlated noise, we choose the noise spectral matrix Γ_b by the following [10] :

$$\Gamma_b(x, y) = \begin{cases} \sigma^2 \rho^{|x-y|} e^{j\pi(x-y)/2} \\ 0 \text{ if } |x - y| > k \end{cases} \tag{23}$$

where k is the spatial correlation length and σ^2 is the noise power. Using the same configuration and anntennas-sources as in the previous test, Fig. 8 represents the results of G-MEM and MEM localization functions where the index of MEM operator is 1 and the parameters of correlated noise are $\sigma^2 = 1$, $\rho = 0.4$ and $k = 6$.

Fig. 9 represents the absolute value of noise spectral matrix Γ_b.

The effet of spatially non uniform noise is present in both angular functions at $-30°$ approximately and the DoAs are detected with different magnitudes.

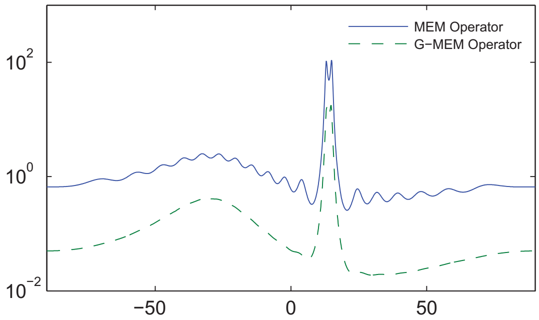

Fig. 8: G-MEM and MEM localization functions in the presence of colored noise with $N = 20$, $K = 60$, $\theta = [13°, 15°]$, $d = \lambda/2$, $\sigma^2 = 1$, $\rho = 0.4$ and $k = 6$

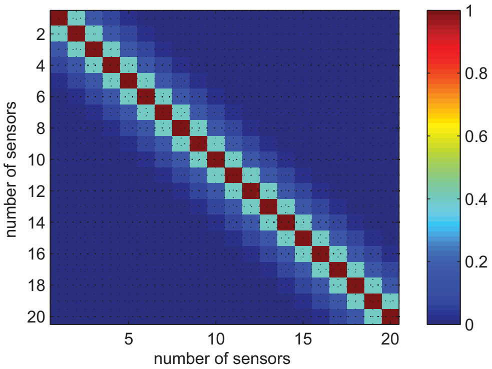

Fig. 9: Absolute value of operator Γ_b configured with parameters $N = 20$, $\sigma^2 = 1$, $\rho = 0.4$ and $k = 6$.

V. Conclusion

The quantum entropy is a generalization of Shannon information entropy, in this paper, we have proposed the application of quantum version of entropy in image and array processing fields. To measure the quality of filtering operation, we have extended Von Neumann entropy using singular values. Concerning the narrowband source localization problem, we have generalized the maximum entropy method into a full rank attenuator operator of signal subspace generated by radiating sources. Monte Carlo simulation results proved the accuracy of the proposed methods.

References

[1] Shannon, C.E., "A mathematical theory of communication," Bell System Technical Journal, The , vol.27, no.3, pp.379,423, July 1948.
[2] Yue Wu, Yicong Zhou, George Saveriades, Sos Agaian, Joseph P. Noonan, Premkumar Natarajan, Local Shannon entropy measure with statistical tests for image randomness, Information Sciences, Volume 222, 10 February 2013, Pages 323-342.
[3] John von Neumann, Mathematische Grundlagen der Quantenmechanik, Springer, 1995.
[4] Peter J. Schreier and Louis L. Scharf, Statistical Signal Processing of Complex-Valued Data, Cambridge University Press, 2010.
[5] Pratt, W., "Generalized Wiener Filtering Computation Techniques," Computers, IEEE Transactions on , vol.C-21, no.7, pp.636,641, July 1972.
[6] Zhizhang Chen, Gopal Gokeda and Yiqiang Yu, Introduction to Direction-Of-Arrival Estimation, Artech House, 2010.
[7] Youssef Khmou, Said Safi and Miloud Frikel, Generalized Maximum Entropy Method for Cosmic Source Localization, International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering, 2014.
[8] Sylvie Marcos, Alain Marsal, Messaoud Benidir, The propagator method for source bearing estimation, Signal Processing, Volume 42, Issue 2, March 1995, Pages 121-138.
[9] Ermolaev, V.T.; Gershman, A.B., "Fast algorithm for minimum-norm direction-of-arrival estimation," Signal Processing, IEEE Transactions on , vol.42, no.9, pp.2389,2394, Sep 1994.
[10] Bouri, M. , Bourennane, S.. "High Resolution Methods Based On Rank Revealing Triangular Factorizations ". World Academy of Science, Engineering and Technology, International Science Index 3, International Journal of Mathematical, Computational, Statistical, Natural and Physical Engineering (2007), 1(3), 190 - 193.