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Extend three-wave method for the (3+1)-Dimensional Soliton Equation

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Abstract—In this paper, we study (3+1)-dimensional Soliton equation. We employ the Hirota's bilinear method to obtain the bilinear form of (3+1)-dimensional Soliton equation. Then by the idea of extended three-wave method, some exact soliton solutions including breather type solutions are presented.

Keywords—three-wave method; (3+1)-dimensional Soliton equation; Hirota's bilinear form.

I. Introduction

The study of exact solutions of nonlinear partial differential equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models. Recently, many kinds of powerful methods have been proposed to find exact solutions of nonlinear partial differential equations, e.g., the tanh-method [1], the homogeneous balance method [2], homotopy analysis method [3], [4], [5], [6], [7], [8], the $F-$expansion method [9], three-wave method [10], [11], [12], extended homoclinic test approach [13], [14], [15], the $(\frac{G'}{G})-$expansion method [16] and the exp-function method [17], [18], [19], [20], [21].
In this paper, by means of the Three-wave method, we will obtain some exact and new solutions for the (3+1)-dimensional soliton equation. In the following section we have a brief review on the Three-wave method and then we apply the method to find explicit formulas of solutions of the (3+1)-dimensional soliton equation in Section 3. The paper is concluded in Section 4.

II. Three-wave method

Dai et al. [22], suggested the three-wave method for nonlinear evolution equations. The basic idea of this method applies the Painlevé analysis to make a transformation as

$$u = T(f) \tag{1}$$

for some new and unknown function f.Then we use this transformation in a high dimensional nonlinear equation of the general form

$$F(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, \cdots) = 0, \tag{2}$$

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where $u = u(x, y, z, t)$ and F is a polynomial of u and its derivatives. By substituting (1) in (2), the first one converts into the Hirota's bilinear form, which it will solve by taking a special form for f and assuming that the obtained Hirota's bilinear form has three-wave solutions, we can specify the unknown function f. For more details see [22], [23].

III. (3+1)-dimensional Soliton equation

In this paper, we investigate explicit formula of solutions of the following (3+1)-dimensional Soliton equation given in [24]

$$3u_{xz} - (2u_t + u_{xxx} - 2uu_x)_y + 2(u_x\partial_x^{-1}u_y)_x = 0, \tag{3}$$

or equivalently

$$3u_{xxz} - (2u_xt + u_{xxxx} - 2u_xu_{xx})_y + 2(u_{xx}u_y)_x = 0. \tag{4}$$

To solve eq. (3) we introduce a new dependent variable w by

$$w = -3(\ln f)_x \tag{5}$$

where $f(x, y, z, t)$ is an unknown real function which will be determined. Substituting eq. (5) into eq. (4), we obtain the following Hirota's bilinear form

$$(3\,D_xD_z - D_yD_x^3 - 2\,D_yD_t)f \cdot f = 0, \tag{6}$$

where the D-operator is defined by

$$D_x^m D_y^k D_t^n f(x, y, t) \cdot g(x, y, t) =$$

$$(\tfrac{\partial}{\partial x_1} - \tfrac{\partial}{\partial x_2})^m(\tfrac{\partial}{\partial y_1} - \tfrac{\partial}{\partial y_2})^k(\tfrac{\partial}{\partial t_1} - \tfrac{\partial}{\partial t_2})^n \tag{7}$$

$$[f(x_1, y_1, t_1)g(x_2, y_2, t_2)],$$

and the right hand side is computed in

$$x_1 = x_2 = x,\ y_1 = y_2 = y,\ t_1 = t_2 = t.$$

Now we suppose the solution of eq. (6) as

$$f\,(x, y, z, t) = \mathrm{e}^{-\xi_1} + \delta_1\,\cos\,(\xi_2) + \delta_2\,\cosh\,(\xi_3) + \delta_3\,\mathrm{e}^{\xi_1} \tag{8}$$

where

$$\xi_1 = a_1x + d_1t \tag{9}$$

and

$$\xi_i = a_ix + b_iy + c_iz + d_it, \quad i = 2, 3 \tag{10}$$

and a_1,d_1,δ_1,a_i,b_i,c_i,d_i and δ_i are some constants to be determined later. Substituting eq. (8) into eq. (6), and equating all coefficients of $\exp(\xi_1)$, $\exp(-\xi_1)$, $\sin\,(\xi_2)$,$\cos\,(\xi_2)$, $\sinh\,(\xi_3)$ and $\cosh\,(\xi_3)$ to zero, we get the following set of algebraic equations for a_1,d_1,δ_1,a_i,b_i,c_i,d_i,δ_i, $(i = 2, 3)$

$$
\begin{aligned}
&2\,d_3b_3 + {a_3}^3b_3 + 3\,{a_1}^2a_3b_3 - 3\,a_3c_3 = 0,\\
&-3\,{a_3}^2b_3a_1 + 3\,a_1c_3 - {a_1}^3b_3 - 2\,d_1b_3 = 0,\\
&-3\,{a_1}^2a_2b_2 + {a_2}^3b_2 + 3\,a_2c_2 - 2\,d_2b_2 = 0,\\
&{a_1}^3b_2 + 2\,d_1b_2 - 3\,{a_2}^2b_2a_1 - 3\,a_1c_2 = 0,\\
&-3\,{a_2}^2a_3b_3 - 3\,a_3c_3 - 3\,{a_3}^2a_2b_2 + {a_2}^3b_2\\
&+{a_3}^3b_3 - 2\,d_2b_2 + 2\,d_3b_3 + 3\,a_2c_2 = 0,\\
&-3\,{a_2}^2b_2a_3 + 2\,d_2b_3 + 2\,d_3b_2 + 3\,{a_3}^2b_3a_2\\
&-3\,a_2c_3 - 3\,a_3c_2 - {a_2}^3b_3 + {a_3}^3b_2 = 0,\\
&-2\,{\delta_1}^2d_2b_2 + 4\,{\delta_1}^2{a_2}^3b_2 - 3\,{\delta_2}^2a_3c_3\\
&+3\,{\delta_1}^2a_2c_2 + 4\,{\delta_2}^2{a_3}^3b_3 + 2\,{\delta_2}^2d_3b_3 = 0.
\end{aligned}
\tag{11}
$$

Solving the system of equations (11) with the aid of Maple, yields the following cases:

A. CaseI:

$$
a_2 = a_3 = d_2 = d_3 = 0, c_2 = \frac{b_2c_3}{b_3},\qquad d_1 = -\frac{a_1\left({a_1}^2b_3 - 3\,c_3\right)}{2\,b_3}
\tag{12}
$$

for some arbitrary complex constants a_1, b_2, b_3, c_3 and δ_i, $i = 1, 2, 3$. Substitute eq. (12) into eq. (5) with eq. (8), we obtain the solution as

$$
f\left(x, y, z, t\right) = \mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right) + \delta_3\mathrm{e}^{\xi_1} \tag{13}
$$

and

$$
u\left(x, y, z, t\right) = -3\,\frac{-a_1\mathrm{e}^{-\xi_1} + \delta_3a_1\mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right) + \delta_3\mathrm{e}^{\xi_1}} \tag{14}
$$

for

$$
\xi_1 = a_1x - \frac{a_1\left({a_1}^2b_3 - 3\,c_3\right)t}{2\,b_3},\qquad \xi_2 = b_2y + \frac{b_2c_3z}{b_3}\qquad,\qquad \xi_3 = b_3y + c_3z \tag{15}
$$

If $\delta_3 > 0$, then we obtain the exact breather cross-kink solution

$$
u\left(x, y, z, t\right) = \frac{6\,a_1\sqrt{\delta_3}\sinh(\xi_1 - \theta)}{2\,\sqrt{\delta_3}\cosh(\xi_1 - \theta) + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right)}
$$

for

$$
\theta = \frac{1}{2}\,\ln(\delta_3).
$$

If $\delta_3 < 0$, then we obtain the exact breather cross-kink solution

$$
u\left(x, y, z, t\right) = \frac{6\,a_1\sqrt{-\delta_3}\cosh(\xi_1 - \theta)}{2\,\sqrt{-\delta_3}\sinh(\xi_1 - \theta) + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right)}
$$

for

$$
\theta = \frac{1}{2}\,\ln(-\delta_3).
$$

B. CaseII:

$$
a_2 = b_3 = c_3 = d_2 = 0, c_2 = \frac{b_2\left(2\,d_1 + {a_1}^3\right)}{3\,a_1},\qquad d_3 = \frac{a_3\left(2\,d_1 + {a_1}^3 - {a_3}^2a_1\right)}{2\,a_1}
\tag{16}
$$

for some arbitrary complex constants a_1, a_3, b_2, d_1,δ_i, $i = 1, 2, 3$. Substitute eq. (16) into eq. (5) with eq. (8), we obtain the solution as follows

$$
f\left(x, y, z, t\right) = \mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right) + \delta_3\mathrm{e}^{\xi_1} \tag{17}
$$

and

$$
u\left(x, y, z, t\right) = \frac{-3\left(-a_1\mathrm{e}^{-\xi_1} + \delta_2\sinh\left(\xi_3\right)a_3 + \delta_3a_1\mathrm{e}^{\xi_1}\right)}{\mathrm{e}^{-\xi_1} + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right) + \delta_3\mathrm{e}^{\xi_1}} \tag{18}
$$

for

$$
\xi_1 = a_1x + d_1t,\qquad \xi_2 = b_2y + \frac{b_2\left(2\,d_1 + {a_1}^3\right)z}{3\,a_1},\qquad \xi_3 = a_3x + \frac{a_3\left(2\,d_1 + {a_1}^3 - {a_3}^2a_1\right)t}{2\,a_1},
\tag{19}
$$

If $\delta_3 > 0$, then we obtain the exact breather cross-kink solution

$$
u\left(x, y, z, t\right) = \frac{6\,a_1\sqrt{\delta_3}\sinh(\xi_1 - \theta) - 3\,\delta_2\sinh\left(\xi_3\right)a_3}{2\,\sqrt{\delta_3}\cosh(\xi_1 - \theta) + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right)}
$$

for

$$
\theta = \frac{1}{2}\,\ln(\delta_3).
$$

If $\delta_3 < 0$, then we obtain the exact breather cross-kink solution

$$
u\left(x, y, z, t\right) = \frac{6\,a_1\sqrt{-\delta_3}\cosh(\xi_1 - \theta) - 3\,\delta_2\sinh\left(\xi_3\right)a_3}{2\,\sqrt{-\delta_3}\sinh(\xi_1 - \theta) + \delta_1\cos\left(\xi_2\right) + \delta_2\cosh\left(\xi_3\right)}
$$

for

$$
\theta = \frac{1}{2}\,\ln(-\delta_3).
$$

C. CaseIII:

$$a_1 = d_1 = 0, c_3 = \frac{{a_2}^3 b_3 - 2\,a_3 c_2 + {a_3}^2 b_3 a_2}{2\,a_2},$$

$$b_2 = -\frac{a_2 b_3}{a_3}, d_3 = \frac{a_3\left(3\,{a_2}^3 b_3 - 6\,a_3 c_2 + {a_3}^2 b_3 a_2\right)}{4\,a_2 b_3},$$

$$d_2 = \frac{{a_2}^3 b_3 - 3\,a_3 c_2}{2\,b_3}, \delta_1 = \frac{\delta_2 {a_3}^2}{{a_2}^2}, \tag{20}$$

for some arbitrary complex constants $a_2, d_2, \delta_1, \delta_2$. Substitute eq. (20) into eq. (5) with eq. (8), we obtain the solution as follows

$$f(x,y,z,t) = 1 + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \tag{21}$$

and

$$u(x,y,z,t) = \frac{3\,\delta_1 \sin(\xi_2)\,a_2 - 3\,\delta_2 \sinh(\xi_3)\,a_3}{1 + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3}, \tag{22}$$

for

$$\xi_2 = a_2 x - \frac{a_2 b_3 y}{a_3} + c_2 z + \frac{\left({a_2}^3 b_3 - 3\,a_3 c_2\right) t}{2\,b_3},$$

$$\xi_3 = a_3 x + b_3 y + \frac{\left({a_2}^3 b_3 - 2\,a_3 c_2 + {a_3}^2 b_3 a_2\right) z}{2\,a_2} + \frac{a_3\left(3\,{a_2}^3 b_3 - 6\,a_3 c_2 + {a_3}^2 b_3 a_2\right) t}{4\,a_2 b_3},$$

and

$$\delta_1 = \frac{\delta_2 {a_3}^2}{{a_2}^2}$$

D. CaseIV:

$$a_1 = -a_3, a_2 = i a_3, c_2 = \frac{b_2 c_3}{b_3}, d_1 = \frac{a_3\left(4\,{a_3}^2 b_3 - 3\,c_3\right)}{2\,b_3},$$

$$d_2 = -\frac{i\left(4\,{a_3}^2 b_3 - 3\,c_3\right) a_3}{2\,b_3}, d_3 = -\frac{a_3\left(4\,{a_3}^2 b_3 - 3\,c_3\right)}{2\,b_3}, \tag{23}$$

for some arbitrary complex constants $a_3, b_2, b_3, c_3, \delta_i$, $i = 1, 2, 3$. Substitute eq. (23) into eq. (5) with eq. (8), we obtain the solution as follows

$$f(x,y,z,t) = \mathrm{e}^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1} \tag{24}$$

and

$$u(x,y,z,t) = -3\,\frac{a_3 \mathrm{e}^{-\xi_1} + i\delta_1 \sin(\xi_2)\,a_3 - \delta_2 \sinh(\xi_3)\,a_3 - \delta_3 a_3 \mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3) + \delta_3 \mathrm{e}^{\xi_1}} \tag{25}$$

for

$$\xi_1 = -a_3 x + \frac{a_3\left(4\,{a_3}^2 b_3 - 3\,c_3\right) t}{2\,b_3}$$

$$\xi_2 = -i a_3 x - b_2 y - \frac{b_2 c_3 z}{b_3} + \frac{i\left(4\,{a_3}^2 b_3 - 3\,c_3\right) a_3 t}{2\,b_3},$$

$$\xi_3 = -a_3 x - b_3 y - c_3 z + \frac{a_3\left(4\,{a_3}^2 b_3 - 3\,c_3\right) t}{2\,b_3}.$$

If $\delta_3 > 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,z,t) = \frac{-6\,a_3\sqrt{\delta_3}\sinh(\xi_1 - \theta) - 3i\delta_1 \sin(\xi_2)\,a_3 + 3\delta_2 \sinh(\xi_3)\,a_3}{2\sqrt{\delta_3}\cosh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)}$$

for

$$\theta = \frac{1}{2}\ln(\delta_3).$$

If $\delta_3 < 0$, then we obtain the exact breather cross-kink solution

$$u(x,y,z,t) = \frac{-6\,a_3\sqrt{-\delta_3}\cosh(\xi_1 - \theta) - 3i\delta_1 \sin(\xi_2)\,a_3 + 3\delta_2 \sinh(\xi_3)\,a_3}{2\sqrt{-\delta_3}\sinh(\xi_1 - \theta) + \delta_1 \cos(\xi_2) + \delta_2 \cosh(\xi_3)}$$

for

$$\theta = \frac{1}{2}\ln(-\delta_3).$$

IV. Conclusion

In this paper, using the three-wave solution method we obtained some explicit formulas of solutions for the (3+1)-dimensional Soliton equation. Three-wave solution method with the aid of a symbolic computation software like Maple or Mathematica is an easy and straightforward method which can be apply to other nonlinear partial differential equations. It must be noted that, all obtained solutions have checked in the (3+1)-dimensional Soliton equation. All solutions satisfy in the equations.

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