

Exponential stability of periodic solutions in inertial neural networks with unbounded delay

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Abstract—In this paper, the exponential stability of periodic solutions in inertial neural networks with unbounded delay are investigated. First, using variable substitution the system is transformed to first order differential equation. Second, by the fixed-point theorem and constructing suitable Lyapunov function, some sufficient conditions guaranteeing the existence and exponential stability of periodic solutions of the system are obtained. Finally, two examples are given to illustrate the effectiveness of the results.

Keywords—inertial neural networks, unbounded delay, fixed-point theorem, Lyapunov function, periodic solutions, exponential stability.

I. INTRODUCTION

CELLULAR neural networks (CNNs) [1,2] have been successfully applied in signal processing, pattern recognition, especially in static image treatment [3]. In delayed neural networks, a constant delay is only a special case. In most situations, delays are variable, and in fact unbounded. In some practical applications and hardware implementations of neural networks, the inevitable time delay may be unbounded, for example

$$\frac{dx_i(t)}{dt} = -\alpha_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau_{ij})) + \sum_{j=1}^n c_{ij} \int_{-\infty}^t K_{ij}(t-s) f_j(x_j(s)) ds + I_i(t), \quad (1)$$

for $i = 1, 2, \dots, n$,

Therefore, the studies of neural networks with time-varying delays and unbounded time delays are more important and necessary than those with constant delays, and the corresponding research can be seen in [4-14].

On the other hand, the inertia can be considered a useful tool that is added to help in the generation of chaos in neural systems. Babcock and Westervelt [15] combined inertia and driving to explore chaos in one- and two-neuron systems. Tani et al.[16-18] added inertia and a nonlinear oscillating resistance to neural equations as a way of chaotically searching for memories in neural networks. In [19], the authors considered the bifurcation and chaos in a single inertial neuron model with both time delay and inertial term. Liu et al.[20-23] investigated the Hopf bifurcation and dynamics of an inertial two-neuron system or in a single inertial neuron mode. In [24], the authors investigate the dynamical characteristics of a single inertial neuron model with time delay under periodic external. Ke and Miao[25-26] investigate stability of equilibrium point and periodic solutions for in inertial BAM neural networks with

time delay and unbounded delay, respectively.

In this paper, we will investigate the exponential stability of periodic solutions for the following inertial neural networks with unbounded delay.

$$\frac{d^2 x_i(t)}{dt^2} = -\beta_i \frac{dx_i(t)}{dt} - \alpha_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau_{ij})) + \sum_{j=1}^n c_{ij} \int_{-\infty}^t K_{ij}(t-s) f_j(x_j(s)) ds + I_i(t), \quad (2)$$

for $i = 1, 2, \dots, n$, where the second derivative are called an inertial term of system (2), $\alpha_i, \beta_i > 0$ are constants. $x_i(t)$ denotes the states variable of the i th neuron at the time t , a_{ij}, b_{ij} and c_{ij} are connection weights of the neural networks, f_j denotes the activation functions of j th neuron at the time t , τ_{ij} is time delay of j th neuron at the time t and satisfies $0 \leq \tau_{ij} \leq \tau$, $I_i(t)$ denotes the external inputs on the i th neuron at the time t , $K_{ij} : R^+ \rightarrow R^+$ are continuous functions, and satisfy $\int_0^{+\infty} e^{\eta s} K_{ij}(s) ds = \rho_{ij}(\eta)$, $i, j = 1, 2, \dots, n$, where $\rho_{ij}(\eta)$ are continuous functions in $[0, \delta)$ ($0 < \delta < 1$) and $\rho_{ij}(0) = 1$.

The initial values of system (2) are

$$x_i(s) = \varphi_i(s), \frac{dx_i(s)}{dt} = \psi_i(s), i = 1, 2, \dots, n, \quad (3)$$

where $-\tau \leq s \leq 0$, $\varphi_i(s), \psi_i(s)$ are bounded and continuous function.

Based on the standpoint of Mathematics and Physics, the system (2) is a two-order nonlinear dynamic system, and $\beta_i > 0$ is a damping coefficient. Then the neural networks of system (1) can be understood as a model which damping tends to infinity. But in some practical problems, we need to consider the existence and stability of periodic solution of the system when it has damping(or low damping). For example, pendulum equation with dissipation term

$$\frac{d^2 x(t)}{dt^2} = -\alpha \frac{dx(t)}{dt} - \beta x - \gamma \sin t,$$

and forced Duffing equation

$$\frac{d^2 x(t)}{dt^2} = -\alpha \frac{dx(t)}{dt} - x(\beta x + \gamma x^2) + \delta \cos vt,$$

which have applied background.

II. PRELIMINARIES

Throughout this paper, we make the following assumptions.

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- (H1) : The activation functions f_j satisfy Lipschitz condition, i.e., there exist constant $L_j > 0$, such that

$$|f_j(v_1) - f_j(v_2)| \leq L_j|v_1 - v_2|,$$

$j = 1, 2, \dots, n, v_1, v_2 \in R$, and there exist constant $\bar{f}_j > 0$, such that $0 \leq |f_j(x)| \leq \bar{f}_j, f_j(0) = 0, j = 1, 2, \dots, n, x \in R$.

- (H2) : $I_i(t)$ are ω - periodic continuously functions and satisfy $|I_i(t)| \leq \bar{I}_i, i = 1, 2, \dots, n, t \in R$.

Let variable transformation:

$$y_i(t) = \frac{dx_i(t)}{dt} + x_i(t), \quad i = 1, 2, \dots, n,$$

then of (2) and (3) can be rewritten as

$$\begin{cases} \frac{dx_i(t)}{dt} = -x_i(t) + y_i(t), \\ \frac{dy_i(t)}{dt} = -(1 + \alpha_i - \beta_i)x_i(t) - (\beta_i - 1)y_i(t) \\ + \sum_{j=1}^n a_{ij}f_j(x_j(t)) + \sum_{j=1}^n b_{ij}f_j(x_j(t - \tau_{ij})) \\ + \sum_{j=1}^n c_{ij} \int_{-\infty}^t K_{ij}(t - s)f_j(x_j(s))ds + I_i(t) \end{cases} \quad (4)$$

for $i = 1, 2, \dots, n$, and

$$\begin{cases} x_i(s) = \varphi_i(s), \quad \frac{dx_i(s)}{dt} = \psi_i(s), -\tau \leq s \leq 0, \\ y_i(s) = \varphi_i(s) + \psi_i(s) \doteq \bar{\varphi}_i(s), -\tau \leq s \leq 0, \end{cases} \quad (5)$$

for $i = 1, 2, \dots, n$.

Let $z_i(t) = (x_i(t), y_i(t))^T$,

$F_i(z_i(t)) =$

$$\begin{aligned} & [0, \sum_{j=1}^n a_{ij}f_j(x_j(t)) + \sum_{j=1}^n b_{ij}f_j(x_j(t - \tau_{ij})) \\ & + \sum_{j=1}^n c_{ij} \int_{-\infty}^t K_{ij}(t - s)f_j(x_j(s))ds + I_i(t)]^T, \end{aligned}$$

then system (4) can becomes

$$\frac{dz_i(t)}{dt} = A_i z_i(t) + F_i(z_i(t)), \quad (6)$$

where

$$A_i = \begin{pmatrix} -1 & 1 \\ \beta_i - \alpha_i - 1 & 1 - \beta_i \end{pmatrix},$$

for $i = 1, 2, \dots, n$.

Definition 1: Let

$$x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$$

be an ω - periodic solution of system (2) with initial value

$$\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \dots, \varphi_n^*(t))^T,$$

$$\psi^*(t) = (\psi_1^*(t), \psi_2^*(t), \dots, \psi_n^*(t))^T.$$

If there exist constants $\alpha > 0$ and $M > 0$, for every solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of system (2) with initial value

$$\varphi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t))^T,$$

$$\psi(t) = (\psi_1(t), \psi_2(t), \dots, \psi_n(t))^T,$$

such that

$$\sum_{i=1}^n |x_i(t) - x_i^*(t)|^2 \leq M e^{-\alpha t} \|\varphi - \varphi^*\|^2, t > 0,$$

then ω - periodic solution $x^*(t)$ is said to be exponentially stable, where

$$\|\varphi - \varphi^*\|^2 = \sup_{-\tau \leq t \leq 0} \sum_{i=1}^n |\varphi_i(t) - \varphi_i^*(t)|^2.$$

Definition 2: Let vector $u = (u_1, u_2, \dots, u_n)^T$ and matrix $B = (b_{ij})_{n \times n}$, we define norm

$$\|u\|^2 = \sum_{i=1}^n u_i^2, \quad \|B\|^2 = \sum_{i,j=1}^n b_{ij}^2.$$

Let $d_i = \sum_{j=1}^n \bar{f}_j[|a_{ij}| + |b_{ij}| + |c_{ij}|] + \bar{I}_i$,

$$e = \min_{1 \leq i \leq n} \{d_i\},$$

$$h_i = \max \left\{ \frac{2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)}{\sqrt{|\beta_i^2 - 4\alpha_i|}} + \frac{d_i}{\alpha_i}, \frac{2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)}{\sqrt{|\beta_i^2 - 4\alpha_i|}} + \varphi_i(0) \right\},$$

$$k_i = \psi_i(0) + \frac{1}{\beta_i} \left[\alpha_i h_i + \sum_{j=1}^n \bar{f}_j[|a_{ij}| + |b_{ij}| + |c_{ij}|] + \bar{I}_i \right],$$

$$h = \min_{1 \leq i \leq n} \{ \sqrt{3h_i^2 + 2k_i^2} \},$$

for $i = 1, 2, \dots, n$.

Lemma 3: For system (2), under hypotheses (H1) – (H2), if $\beta_i^2 - 4\alpha_i \neq 0, \psi_i(0) > 0, \varphi_i(0) \geq \frac{e}{\alpha_i} > 0, i = 1, 2, \dots, n$, then $x(t)$ and $\frac{dx_i(t)}{dt}$ is bounded, and

$$|x_i(t)| \leq h_i, \quad \left| \frac{dx_i(t)}{dt} \right| \leq k_i \quad i = 1, 2, \dots, n.$$

Proof. From (2), we obtain

$$\frac{d^2 x_i(t)}{dt^2} + \beta_i \frac{dx_i(t)}{dt} + \alpha_i x_i(t)$$

$$\leq \sum_{j=1}^n \bar{f}_j[|a_{ij}| + |b_{ij}| + |c_{ij}|] + \bar{I}_i, \quad i = 1, 2, \dots, n.$$

We consider the following differential equations

$$\frac{d^2 x_i(t)}{dt^2} + \beta_i \frac{dx_i(t)}{dt} + \alpha_i x_i(t) = e, \quad i = 1, 2, \dots, n. \quad (7)$$

If $\beta_i^2 - 4\alpha_i > 0$, We obtain general solution of (7)

$$x_i(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \frac{e}{\alpha_i}, \quad i = 1, 2, \dots, n,$$

where

$$\lambda_1 = \frac{1}{2}[-\beta_i + \sqrt{\beta_i^2 - 4\alpha_i}] < 0,$$

$$\lambda_2 = \frac{1}{2}[-\beta_i - \sqrt{\beta_i^2 - 4\alpha_i}] < 0,$$

C_1, C_2 are arbitrary constant.

For initial values $x_i(0) = \varphi_i(0), \frac{dx_i(0)}{dt} = \psi_i(0)$, if $\psi_i(0) > 0, \varphi_i(0) \geq \frac{e}{\alpha_i} > 0$, we can obtain,

$$C_1 = \frac{1}{\sqrt{\beta_i^2 - 4\alpha_i}} [\psi_i(0) - \lambda_2(\varphi_i(0) - e/\alpha_i)] > 0,$$

$$C_2 = \frac{1}{\sqrt{\beta_i^2 - 4\alpha_i}} [\lambda_1(\varphi_i(0) - e/\alpha_i) - \psi_i(0)] < 0.$$

We have

$$\begin{aligned} |x_i(t)| & \leq \frac{1}{\sqrt{\beta_i^2 - 4\alpha_i}} [\psi_i(0) - \lambda_2(\varphi_i(0) - e/\alpha_i) \\ & - \lambda_1(\varphi_i(0) - e/\alpha_i) + \psi_i(0)] + \frac{e}{\alpha_i} \\ & \leq \frac{2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)}{\sqrt{\beta_i^2 - 4\alpha_i}} + \frac{d_i}{\alpha_i} \leq h_i, \quad i = 1, 2, \dots, n. \end{aligned}$$

If $\beta_i^2 - 4\alpha_i < 0$, We obtain general solution of (7)

$$\begin{aligned} x_i(t) & = e^{-\beta_i/2t} [C_1 \cos \frac{\sqrt{4\alpha_i - \beta_i^2}}{2} t \\ & + C_2 \sin \frac{\sqrt{4\alpha_i - \beta_i^2}}{2} t + \frac{e}{\alpha_i}], \quad i = 1, 2, \dots, n, \end{aligned}$$

where C_1, C_2 are arbitrary constant.

For initial values $x_i(0) = \varphi_i(0)$, $\frac{dx_i(0)}{dt} = \psi_i(0)$, if $\psi_i(0) > 0$, $\varphi_i(0) \geq \frac{e}{\alpha_i} > 0$, we can obtain,

$$C_1 = \varphi_i(0) - e/\alpha_i > 0,$$

$$C_2 = \frac{1}{\sqrt{|\beta_i^2 - 4\alpha_i|}} [2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)] > 0.$$

We have

$$|x_i(t)| \leq \varphi_i(0) - e/\alpha_i + \frac{1}{\sqrt{|\beta_i^2 - 4\alpha_i|}} [2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)] + \frac{e}{\alpha_i} \\ \leq \frac{2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)}{\sqrt{|\beta_i^2 - 4\alpha_i|}} + \varphi_i(0) \leq h_i, \\ i = 1, 2, \dots, n.$$

On the other hand, from (2), we have

$$\frac{d}{dt} [e^{\beta_i t} \frac{dx_i(t)}{dt}] = e^{\beta_i t} [-\alpha_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t - \tau_{ij})) + \sum_{j=1}^n c_{ij} \int_{-\infty}^t K_{ij}(t-s) f_j(x_j(s)) ds + I_i(t)], \\ \text{for } i = 1, 2, \dots, n. \text{ We obtain} \\ \frac{dx_i(t)}{dt} = e^{-\beta_i t} \psi_i(0) + \int_0^t e^{-\beta_i(t-u)} [-\alpha_i x_i(u) + \sum_{j=1}^n a_{ij} f_j(x_j(u)) + \sum_{j=1}^n b_{ij} f_j(x_j(u - \tau_{ij})) + \sum_{j=1}^n c_{ij} \int_{-\infty}^u K_{ij}(u-s) f_j(x_j(s)) ds + I_i(u)] du, \\ i = 1, 2, \dots, n.$$

Since $|x_i(t)| \leq h_i$, from the above equation, we obtain

$$|\frac{dx_i(t)}{dt}| \leq \psi_i(0) + \int_0^t e^{-\beta_i(t-u)} [\alpha_i h_i + \sum_{j=1}^n \bar{f}_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) + \bar{I}_i] du \\ \leq \psi_i(0) + \frac{1}{\beta_i} [\alpha_i h_i + \sum_{j=1}^n \bar{f}_j(|a_{ij}| + |b_{ij}| + |c_{ij}|) + \bar{I}_i] \\ = k_i, \quad i = 1, 2, \dots, n.$$

Proof of completion of Lemma 3.

Lemma 4: For matrix

$$A_i = \begin{pmatrix} -1 & 1 \\ \beta_i - \alpha_i - 1 & 1 - \beta_i \end{pmatrix}, \quad i = 1, 2, \dots, n.$$

we have

$$\|exp(-A_i t)\| \leq M_i e^{\mu_i t}, \quad i = 1, 2, \dots, n, \quad t \geq 0,$$

$$\|(exp(-A_i \omega) - E)^{-1}\| \leq N_i, \quad i = 1, 2, \dots, n,$$

where, 1) if $\beta_i^2 - 4\alpha_i > 0$ ($i = 1, 2, \dots, n$), then

$$\mu_i = \frac{1}{2} [\beta_i + \sqrt{|\beta_i^2 - 4\alpha_i|}], \\ M_i = \left[\frac{2(\beta_i - \alpha_i)^2 + 2(\beta_i - 2)^2}{|\beta_i^2 - 4\alpha_i|} \right]^{1/2},$$

$$N_i = \left[|\beta_i^2 - 4\alpha_i|^{3/2} |e^{\beta_i \omega} - e^{\frac{\beta_i \omega}{2}}| \left(e^{\frac{\sqrt{\beta_i^2 - 4\alpha_i} \omega}{2}} + e^{-\frac{\sqrt{\beta_i^2 - 4\alpha_i} \omega}{2}} \right) + 1 \right]^{-1}$$

$$\cdot \{ [(\alpha_i - \beta_i)^2 + (2 - \beta_i)^2] e^{\beta_i \omega} (e^{\sqrt{\beta_i^2 - 4\alpha_i} \omega} + e^{-\sqrt{\beta_i^2 - 4\alpha_i} \omega}) + 2(\beta_i^2 - 4\alpha_i) \}^{1/2},$$

2) if $\beta_i^2 - 4\alpha_i < 0$ ($i = 1, 2, \dots, n$), then

$$\mu_i = \frac{1}{2} \beta_i, \\ M_i = 2 \left[\frac{2\alpha_i - 2\beta_i + 3 + (|1 - \delta_i| + |\alpha_i + 1 - \beta_i|)^2}{|\beta_i^2 - 4\alpha_i|} \right]^{1/2}, \\ N_i = \frac{1}{2e^{\beta_i \omega/2} \left[\frac{\delta_i [2\delta_i^2 (1 - e^{-\beta_i \omega/2})^2 + (\delta_i - 1) \sin 2\delta_i \omega]}{\delta_i} \right]^{1/2}}$$

$$\cdot \{ 2\delta_i - 1 + [2(\frac{\beta_i}{2} - 1)^2 + (1 + \alpha_i - \beta_i)^2] \sin^2 \delta_i \omega \\ + 3\delta_i^2 \cos^2 \delta_i \omega + 2(\delta_i e^{-\beta_i \omega/2} + 1)^2 \\ - 2\delta_i (\cos \delta_i \omega + 1)^2 \}^{1/2}, \\ \delta_i = \frac{\sqrt{|\beta_i^2 - 4\alpha_i|}}{2}.$$

Proof. We consider the following linear differential equations

$$Z_i'(t) = -A_i Z_i(t). \quad (8)$$

By calculation, we can obtain the eigenvalue of matrix $-A_i$

$$\lambda_1 = \frac{1}{2} [\beta_i + \sqrt{\beta_i^2 - 4\alpha_i}], \\ \lambda_2 = \frac{1}{2} [\beta_i - \sqrt{\beta_i^2 - 4\alpha_i}].$$

Corresponding eigenvector of the λ_1 and λ_2 , respectively

$$V_1 = (1, 1 - \lambda_1)^T, \quad V_2 = (1, 1 - \lambda_2)^T.$$

If $\beta_i^2 - 4\alpha_i > 0$, we obtain the fundamental solution matrix of system (8) is

$$\phi_i(t) = \begin{bmatrix} e^{\lambda_1 t} & e^{\lambda_2 t} \\ (1 - \lambda_1)e^{\lambda_1 t} & (1 - \lambda_2)e^{\lambda_2 t} \end{bmatrix}.$$

By calculation, we obtain

$$\phi_i^{-1}(0) = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} 1 - \lambda_2 & -1 \\ \lambda_1 - 1 & 1 \end{bmatrix}.$$

Since $exp(-A_i t) = \phi_i(t) \phi_i^{-1}(0)$, we have

$$exp(-A_i t) = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} (1 - \lambda_2)e^{\lambda_1 t} + (\lambda_1 - 1)e^{\lambda_2 t} & -e^{\lambda_1 t} \\ (1 - \lambda_1)(1 - \lambda_2)(e^{\lambda_1 t} - e^{\lambda_2 t}) & (1 - \lambda_2)e^{\lambda_2 t} + (\lambda_1 - 1)e^{\lambda_1 t} \end{bmatrix}.$$

$$\|exp(-A_i t)\| \leq \frac{1}{|\lambda_1 - \lambda_2|} \{ [(1 - \lambda_2)e^{\lambda_1 t} + (\lambda_1 - 1)e^{\lambda_2 t}]^2 + [(1 - \lambda_1)(1 - \lambda_2)(e^{\lambda_1 t} - e^{\lambda_2 t})]^2 + [(1 - \lambda_2)e^{\lambda_2 t} + (\lambda_1 - 1)e^{\lambda_1 t}]^2 \}^{1/2} \\ = \frac{1}{|\lambda_1 - \lambda_2|} \{ [(\lambda_1 - 1)^2 + (\lambda_2 - 1)^2 + (\lambda_1 - 1)^2(\lambda_2 - 1)^2 + 1] (e^{2\lambda_1 t} + e^{2\lambda_2 t}) + [4(\lambda_1 - 1)(1 - \lambda_2) - 2(\lambda_1 - 1)^2(\lambda_2 - 1)^2 - 2] e^{(\lambda_1 + \lambda_2)t} \}^{1/2} \\ = \frac{1}{|\lambda_1 - \lambda_2|} \{ [(\beta_i - \alpha_i)^2 + (\beta_i - 2)^2] (e^{2\lambda_1 t} + e^{2\lambda_2 t}) - 2[(1 - \lambda_2)(\lambda_1 - 1) - 1]^2 e^{(\lambda_1 + \lambda_2)t} \}^{1/2} \\ \leq \frac{\sqrt{(\beta_i - \alpha_i)^2 + (\beta_i - 2)^2}}{\sqrt{|\beta_i^2 - 4\alpha_i|}} \sqrt{e^{2\lambda_1 t} + e^{2\lambda_2 t}}.$$

Then we have

$$\|exp(-A_i t)\| \leq M_i e^{\mu_i t}, \quad i = 1, 2, \dots, n, \quad t \geq 0.$$

Since

$$exp(-A_i \omega) - E = \frac{1}{\lambda_1 - \lambda_2}$$

$$\begin{bmatrix} (1 - \lambda_2)e^{\lambda_1 \omega} + (\lambda_1 - 1)e^{\lambda_2 \omega} - \lambda_1 - \lambda_2 & -e^{\lambda_1 \omega} \\ (1 - \lambda_1)(1 - \lambda_2)(e^{\lambda_1 \omega} - e^{\lambda_2 \omega}) & (1 - \lambda_2)e^{\lambda_2 \omega} + (\lambda_1 - 1)e^{\lambda_1 \omega} - \lambda_1 - \lambda_2 \end{bmatrix}.$$

We can obtain

$$[exp(-A_i \omega) - E]^{-1} = \frac{1}{(\lambda_2 - \lambda_1)^3 [e^{\beta_i \omega} - e^{\lambda_1 \omega} - e^{\lambda_2 \omega} + 1]} \begin{bmatrix} (1 - \lambda_2)e^{\lambda_2 \omega} + (\lambda_1 - 1)e^{\lambda_1 \omega} - \lambda_1 - \lambda_2 & -e^{\lambda_1 \omega} \\ (1 - \lambda_1)(1 - \lambda_2)(e^{\lambda_2 \omega} - e^{\lambda_1 \omega}) & (1 - \lambda_2)e^{\lambda_1 \omega} + (\lambda_1 - 1)e^{\lambda_2 \omega} - \lambda_1 - \lambda_2 \end{bmatrix}.$$

we can obtain

$$\begin{aligned} & \|(\exp(-A_i\omega) - E)^{-1}\| \\ & \leq \frac{1}{(\lambda_2 - \lambda_1)^3 |e^{\beta_i\omega} - e^{\lambda_1\omega} - e^{\lambda_2\omega} + 1|} \{[(1 - \lambda_1)^2 \\ & + (1 - \lambda_2)^2 + (1 - \lambda_1)^2(1 - \lambda_2)^2 + 1] \\ & \cdot (e^{2\lambda_1\omega} + e^{2\lambda_2\omega}) - 2[(1 - \lambda_2)(\lambda_1 - 1) - 1]^2 e^{(\lambda_1 + \lambda_2)\omega} \\ & - 2(\lambda_1 - \lambda_2)^2 (e^{\lambda_1\omega} + e^{\lambda_2\omega}) + 2(\lambda_1 - \lambda_2)^2\}^{1/2} \\ & \leq \frac{1}{(\lambda_2 - \lambda_1)^3 |e^{\beta_i\omega} - e^{\lambda_1\omega} - e^{\lambda_2\omega} + 1|} \{[(\alpha_i - \beta_i)^2 \\ & + (2 - \beta_i)^2] (e^{(\beta_i + \sqrt{\beta_i^2 - 4\alpha_i})\omega} + e^{(\beta_i - \sqrt{\beta_i^2 - 4\alpha_i})\omega}) \\ & + 2(\beta_i^2 - 4\alpha_i)\}^{1/2} = N_i. \end{aligned}$$

If $\beta_i^2 - 4\alpha_i < 0$, let $\delta_i = \frac{1}{2}\sqrt{4\alpha_i - \beta_i^2}$, we obtain the fundamental solution matrix of system (8) is

$$\phi_i(t) = e^{\beta_i t/2} \begin{bmatrix} \cos\delta_i t & \\ (1 - \frac{\beta_i}{2})\cos\delta_i t + \delta_i \sin\delta_i t & \\ \sin\delta_i t & \\ (1 - \frac{\beta_i}{2})\sin\delta_i t - \delta_i \cos\delta_i t & \end{bmatrix}.$$

By calculation, we obtain

$$\phi_i^{-1}(0) = \frac{1}{\delta_i} \begin{bmatrix} \delta_i & 0 \\ \lambda_1 - \frac{\beta_i}{2} & -1 \end{bmatrix}.$$

Since $\exp(-A_i t) = \phi_i(t)\phi_i^{-1}(0)$, we have $\exp(-A_i t) = \frac{1}{\delta_i} e^{\frac{\beta_i t}{2}}$

$$\begin{bmatrix} \delta_i \cos\delta_i t + (1 - \frac{\beta_i}{2})\sin\delta_i t & \\ (\delta_i - 1)\cos\delta_i t + (\alpha_i + 1 - \beta_i)\sin\delta_i t & \\ -\sin\delta_i t & \\ (\frac{\beta_i}{2} - 1)\sin\delta_i t + \delta_i \cos\delta_i t & \end{bmatrix}.$$

$$\begin{aligned} \| \exp(-A_i t) \| & \leq \frac{e^{\beta_i t/2}}{\delta_i} \{2\delta_i^2 \cos^2 \delta_i t \\ & + [1 + 2(1 - \frac{\beta_i}{2})^2] \sin^2 \delta_i t \\ & + [(\delta_i - 1)\cos\delta_i t + (\alpha_i + 1 - \beta_i)\sin\delta_i t]^2\}^{1/2} \\ & \leq \frac{e^{\beta_i t/2}}{\delta_i} \{2\alpha_i - 2\beta_i + 3 + (|1 - \delta_i| + |\alpha_i + 1 - \beta_i|)^2\}^{1/2}. \end{aligned}$$

Then we have

$$\| \exp(-A_i t) \| \leq M_i e^{\mu_i t}, \quad i = 1, 2, \dots, n, \quad t \geq 0.$$

Since

$$\exp(-A_i\omega) - E = \frac{e^{\beta_i\omega/2}}{\delta_i}$$

$$\begin{bmatrix} \delta_i \cos\delta_i\omega + (1 - \frac{\beta_i}{2})\sin\delta_i\omega - \delta_i e^{-\beta_i\omega/2} & \\ (\delta_i - 1)\cos\delta_i\omega + (\alpha_i + 1 - \beta_i)\sin\delta_i\omega & \\ -\sin\delta_i\omega & \\ (\frac{\beta_i}{2} - 1)\sin\delta_i\omega + \delta_i \cos\delta_i\omega - \delta_i e^{-\beta_i\omega/2} & \end{bmatrix}.$$

We can obtain

$$\begin{aligned} & [\exp(-A_i\omega) - E]^{-1} \\ & = \frac{e^{\beta_i\omega/2}}{\delta_i [\delta_i^2 (1 - 2\cos\delta_i\omega e^{-\beta_i\omega/2} + e^{-\beta_i\omega}) + (\delta_i - 1)\sin\delta_i\omega \cos\delta_i\omega]} \\ & \begin{bmatrix} (\frac{\beta_i}{2} - 1)\sin\delta_i\omega + \delta_i \cos\delta_i\omega - \delta_i e^{-\beta_i\omega/2} & \\ (1 - \delta_i)\cos\delta_i\omega - (\alpha_i + 1 - \beta_i)\sin\delta_i\omega & \\ \sin\delta_i\omega & \\ \delta_i \cos\delta_i\omega + (1 - \frac{\beta_i}{2})\sin\delta_i\omega - \delta_i e^{-\beta_i\omega/2} & \end{bmatrix}. \end{aligned}$$

By calculation, we can obtain

$$\|(\exp(-A_i\omega) - E)^{-1}\|$$

$$\begin{aligned} & \leq \frac{e^{\beta_i\omega/2}}{\delta_i [\delta_i^2 (1 - 2\cos\delta_i\omega e^{-\beta_i\omega/2} + e^{-\beta_i\omega}) + (\delta_i - 1)\sin\delta_i\omega \cos\delta_i\omega]} \\ & \cdot \{2(\frac{\beta_i}{2} - 1)^2 \sin^2 \delta_i\omega + 2\delta_i^2 \cos^2 \delta_i\omega \\ & + 2\delta_i^2 e^{-\beta_i\omega} - 4\delta_i^2 \cos\delta_i\omega e^{-\beta_i\omega/2} + \sin^2 \delta_i\omega \\ & + [(1 - \delta_i)\cos\delta_i\omega - (\alpha_i + 1 - \beta_i)\sin\delta_i\omega]^2\}^{1/2} \\ & = \frac{e^{\beta_i\omega/2}}{\delta_i [\delta_i^2 (1 - 2\cos\delta_i\omega e^{-\beta_i\omega/2} + e^{-\beta_i\omega}) + (\delta_i - 1)\sin\delta_i\omega \cos\delta_i\omega]} \\ & \cdot \{[1 + 2(\frac{\beta_i}{2} - 1)^2 + (1 + \alpha_i - \beta_i)^2] \sin^2 \delta_i\omega + [1 + 3\delta_i^2 - \\ & 2\delta_i] \cos^2 \delta_i\omega + 2\delta_i^2 e^{-\beta_i\omega} - 4\delta_i (\cos\delta_i\omega - e^{-\beta_i\omega/2}) - 2(1 - \delta_i)(1 + \\ & \alpha_i - \beta_i) \sin\delta_i\omega \cos\delta_i\omega\}^{1/2} \\ & \leq \frac{2e^{\beta_i\omega/2}}{\delta_i [2\delta_i^2 (1 - e^{-\beta_i\omega/2})^2 + (\delta_i - 1)\sin 2\delta_i\omega]} \\ & \cdot \{2\delta_i - 1 + [2(\frac{\beta_i}{2} - 1)^2 + (1 + \alpha_i - \beta_i)^2] \sin^2 \delta_i\omega \\ & + 3\delta_i^2 \cos^2 \delta_i\omega + 2(\delta_i e^{-\beta_i\omega/2} + 1)^2 \\ & - 2\delta_i (\cos\delta_i\omega + 1)^2\}^{1/2} = N_i, \end{aligned}$$

where $\delta_i = \frac{1}{2}\sqrt{4\alpha_i - \beta_i^2}$.

III. MAIN RESULTS

In this section, we will derive some sufficient conditions which ensure the existence and the exponential stability of periodic solution for system (2).

Let

$$X = \{x = (x_1(t), x_2(t), \dots, x_n(t))^T \in C(R, R^n) : x(t + \omega) = x(t), t \in R\}.$$

$$Y = \{y = (y_1(t), y_2(t), \dots, y_n(t))^T \in C(R, R^n) : y(t + \omega) = y(t), t \in R\}.$$

Defined

$$\|z_i\| = \max_{t \in [0, \omega]} \{|z_i(t)|\}, \quad \|z\| = \max_{1 \leq i \leq n} \{\|z_i\|\}.$$

Define the cone P in X, Y by

$$\begin{aligned} P = \{z = (x^T, y^T)^T, x \in X, y \in Y : \\ |x_i(t)| \leq h_i, |y_i(t)| \leq h_i + k_i, \\ \|z_i(t)\| < h, t \in [0, \omega], i = 1, 2, \dots, n\}, \end{aligned}$$

where $h = \max_{1 \leq i \leq n} \{\sqrt{3h_i^2 + 2k_i^2}\}$.

Theorem 5: Under the hypotheses (H1) – (H2), and all of the conditions in Lemma 3 are satisfied, if

$$\|(\exp(-A_i\omega) - E)^{-1}\| \omega M_i e^{\mu_i \omega} [\sum_{j=1}^n L_i(|a_{ij}|$$

$$+ |b_{ij}| + |c_{ij}|) + \frac{\bar{L}_i}{h}] < 1, \quad i = 1, 2, \dots, n,$$

then system (2) has a unique ω -periodic solution.

Proof. Let the map T be defined by

$$(Tz)(t) = ((Tz)_1, (Tz)_2, \dots, (Tz)_n)^T,$$

$$z \in P, t \in R, \text{ where, for } i = 1, 2, \dots, n,$$

$$(Tz)_i(t) = [\exp(-A_i\omega) - E]^{-1}$$

$$\times \int_t^{t+\omega} \exp(-A_i(s-t)) F_i(z_i(s)) ds. \quad (9)$$

Since $F_i(z_i(s + \omega)) = F_i(z_i(s))$, $|x_i(t)| \leq h_i < h, i = 1, 2, \dots, n$, for any $z \in P$, from (9), we obtain

$$\begin{aligned} (Tz)_i(t + \omega) & = [\exp(-A_i\omega) - E]^{-1} \\ & \times \int_{t+\omega}^{t+2\omega} \exp(-A_i(s-t-\omega)) F_i(z_i(s)) ds \\ & = [\exp(-A_i\omega) - E]^{-1} \int_t^{t+\omega} \exp(-A_i(s-t)) \\ & \times F_i(z_i(s)) ds = (Tz)_i(t). \end{aligned}$$

By Lemma 4, we have

$$\begin{aligned} \|(Tz)_i\| & = \max_{0 \leq t \leq \omega} \{|(Tz)_i(t)|\} \\ & = \max_{0 \leq t \leq \omega} \|[\exp(-A_i\omega) - E]^{-1} \\ & \times \int_t^{t+\omega} \exp(-A_i(s-t)) F_i(z_i(s)) ds\| \end{aligned}$$

$$\begin{aligned} &\leq \max_{0 \leq t \leq \omega} \|\exp(-A_i \omega) - E\|^{-1} \\ &\quad \times \int_t^{t+\omega} M_i e^{\mu_i(s-t)} \|F_i(z_i(s))\| ds \\ &\leq \max_{0 \leq t \leq \omega} \|\exp(-A_i \omega) - E\|^{-1} \\ &\quad \times M_i e^{\mu_i \omega} \int_0^\omega \|F_i(z_i(s))\| ds \\ &\leq \|\exp(-A_i \omega) - E\|^{-1} \|M_i e^{\mu_i \omega} \\ &\quad \times \int_0^\omega [\sum_{j=1}^n |a_{ij} f_j(x_j(u))| + \sum_{j=1}^n |b_{ij} f_j(x_j(u - \tau_{ij}))| \\ &\quad + \sum_{j=1}^n |c_{ij} \int_{-\infty}^u K_{ij}(u-s) f_j(x_j(s))| ds \\ &\quad + I_i(u)] du \\ &\leq \|\exp(-A_i \omega) - E\|^{-1} \|M_i e^{\mu_i \omega} \\ &\quad \times \int_0^\omega [\sum_{j=1}^n L_j (|a_{ij}| |x_j(u)| + |b_{ij}| |x_j(u - \tau_{ij})| \\ &\quad + |c_{ij}| \int_{-\infty}^u K_{ij}(u-s) |x_j(s)| ds) + I_i(u)] du \\ &\leq \|\exp(-A_i \omega) - E\|^{-1} \|M_i e^{\mu_i \omega} h \omega [\sum_{j=1}^n L_j (|a_{ij}| \\ &\quad + |b_{ij}| + |c_{ij}|) + \frac{I_i}{h}] < h. \end{aligned}$$

Hence we obtain $Tz \in P$.

Next we will prove T is contraction mapping. For any $z, \bar{z} \in P$, we have

$$\begin{aligned} &\|(Tz) - (T\bar{z})\| \\ &= \max_{1 \leq i \leq n} \max_{0 \leq t \leq \omega} \{ \|(Tz)_i(t) - (T\bar{z})_i(t)\| \} \\ &= \max_{1 \leq i \leq n} \max_{0 \leq t \leq \omega} \|\exp(-A_i \omega) - E\|^{-1} \\ &\quad \times \int_t^{t+\omega} \exp(-A_i(s-t)) (F_i(z_i(s)) - F_i(\bar{z}_i(s))) ds \\ &\leq \max_{1 \leq i \leq n} \|\exp(-A_i \omega) - E\|^{-1} \|M_i e^{\mu_i \omega} \\ &\quad \times \int_0^\omega [\sum_{j=1}^n L_j (|a_{ij}| |x_j(u) - \bar{x}_j(u)| \\ &\quad + |b_{ij}| |x_j(u - \tau_{ij}) - \bar{x}_j(u - \tau_{ij})| \\ &\quad + |c_{ij}| \int_{-\infty}^u K_{ij}(u-s) |x_j(s) - \bar{x}_j(s)| ds) du \\ &\leq \max_{1 \leq i \leq n} \|\exp(-A_i \omega) - E\|^{-1} \|M_i e^{\mu_i \omega} h \omega \\ &\quad \times \sum_{j=1}^n L_j (|a_{ij}| + |b_{ij}| + |c_{ij}|) \|z - \bar{z}\| < \|z - \bar{z}\|. \end{aligned}$$

Thus, T is contraction mapping, by fixed-point theorem, it follows that there exist uniqueness an fixed point $z^*(t)$ satisfying $(Tz^*)(t) = z^*(t)$.

Now we will show that $z^*(t)$ is the ω - periodic solution of (1). From (9), we obtain

$$\begin{aligned} &\frac{d(Tz)_i(t)}{dt} = A_i(Tz)_i(t) \\ &\quad + [\exp(-A_i \omega) - E]^{-1} \exp(-A_i \omega) F_i(z_i(t)) \\ &\quad - [\exp(-A_i \omega) - E]^{-1} F_i(z_i(t)) \\ &= A_i(Tz)_i(t) + [\exp(-A_i \omega) - E]^{-1} \\ &\quad \times [\exp(-A_i \omega) - E] F_i(z_i(t)) \\ &= A_i(Tz)_i(t) + F_i(z_i(t)). \end{aligned}$$

We have

$$\frac{dz_i^*(t)}{dt} = A_i z_i^*(t) + F_i(z_i^*(t)),$$

and $z_i^*(t + \omega) = z_i^*(t)$. From the above we can see, system (2) has a unique ω - periodic solution.

Theorem 6: All the conditions of Theorem 5 are satisfied, if

$$\begin{aligned} &-2 + |\beta_i - \alpha_i| + \sum_{j=1}^n L_j (|a_{ij}| + |b_{ji}| \\ &\quad + |c_{ji}| \int_0^{+\infty} K_{ji}(s) ds) < 0, \end{aligned}$$

$$-2\beta_i + 2 + |\beta_i - \alpha_i| + \sum_{j=1}^n L_j (|a_{ij}| + |b_{ij}| + |c_{ij}|) < 0,$$

for $i = 1, 2, \dots, n$, then system (2) has a unique ω - periodic solution which exponentially stable.

Proof. By using Theorem 5, system (2) has a unique ω - periodic solution. In the following we will prove the unique positive ω - periodic solution is exponentially stable.

Let $x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T$ be an ω - periodic solution of system (2) with initial value

$$x_i^*(s) = \varphi_i^*(s), \quad \frac{dx_i^*(s)}{dt} = \psi_i^*(s),$$

$i = 1, 2, \dots, n, \quad -\tau \leq s \leq 0$.

$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be an any solution of system (2) with initial value

$$x_i(s) = \varphi_i(s), \quad \frac{dx_i(s)}{dt} = \psi_i(s),$$

$i = 1, 2, \dots, n, \quad -\tau \leq s \leq 0$.

Let $y_i^*(t) = x_i^*(t) + \frac{dx_i^*(t)}{dt}$,

$$u_i(t) = x_i(t) - x_i^*(t), \quad v_i(t) = y_i(t) - y_i^*(t),$$

$$\bar{f}_i(u_i(t)) = f_i(x_i(t)) - f_i(x_i^*(t)),$$

for $i = 1, 2, \dots, n$.

From (4), we can obtain

$$\begin{cases} \frac{du_i(t)}{dt} = -u_i(t) + v_i(t), \\ \frac{dv_i(t)}{dt} = -(1 + \alpha_i - \beta_i)u_i(t) \\ \quad - (\beta_i - 1)v_i(t) + \sum_{j=1}^n a_{ij} \bar{f}_j(u_j(t)) \\ \quad + \sum_{j=1}^n b_{ij} \bar{f}_j(u_j(t - \tau_{ij})) \\ \quad + \sum_{j=1}^n c_{ij} \int_{-\infty}^t K_{ij}(t-s) \bar{f}_j(u_j(s)) ds \end{cases} \quad (10)$$

for $i = 1, 2, \dots, n$.

From (10), we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (u_i^2(t) + v_i^2(t)) = -u_i^2(t) + u_i(t)v_i(t) \\ &\quad - (1 + \alpha_i - \beta_i)u_i(t)v_i(t) - (\beta_i - 1)v_i^2(t) \\ &\quad + \sum_{j=1}^n a_{ij} v_i(t) \bar{f}_j(u_j(t)) + \sum_{j=1}^n b_{ij} v_i(t) \bar{f}_j(u_j(t - \tau_{ij})) \\ &\quad + \sum_{j=1}^n c_{ij} \int_{-\infty}^t K_{ij}(t-s) v_i(t) \bar{f}_j(u_j(s)) ds \\ &\leq (\frac{|\beta_i - \alpha_i|}{2} - 1) u_i^2(t) + (1 - \beta_i + \frac{|\beta_i - \alpha_i|}{2}) v_i^2(t) \\ &\quad + \sum_{j=1}^n |a_{ij}| L_j |v_i(t)| |u_j(t)| \\ &\quad + \sum_{j=1}^n |b_{ij}| L_j |v_i(t)| |u_j(t - \tau_{ij})| \\ &\quad + \sum_{j=1}^n |c_{ij}| \int_{-\infty}^t K_{ij}(t-s) |v_i(t)| |u_j(s)| ds \\ &\leq (\frac{|\beta_i - \alpha_i|}{2} - 1) u_i^2(t) + (1 - \beta_i + \frac{|\beta_i - \alpha_i|}{2}) v_i^2(t) \\ &\quad + \sum_{j=1}^n \frac{|a_{ij}|}{2} L_j [v_i^2(t) + u_j^2(t)] \\ &\quad + \sum_{j=1}^n \frac{|b_{ij}|}{2} L_j [v_i^2(t) + u_j^2(t - \tau_{ij})] \\ &\quad + \sum_{j=1}^n \frac{|c_{ij}|}{2} L_j \int_{-\infty}^t K_{ij}(t-s) [v_i^2(t) + u_j^2(s)] ds. \end{aligned} \quad (11)$$

We consider the Lyapunov functional:

$$V(t) = \sum_{i=1}^n \{ \frac{u_i^2(t) + v_i^2(t)}{2} \} e^{\epsilon t}$$

$$\begin{aligned}
 & + \sum_{j=1}^n \frac{|b_{ij}|}{2} L_j \int_{t-\tau_{ji}}^t e^{\varepsilon(\eta+\tau_{ji})} u_j^2(\eta) d\eta \\
 & + \sum_{j=1}^n \frac{|c_{ij}|}{2} L_j \int_0^{+\infty} K_{ij}(s) \left[\int_{t-s}^t e^{\varepsilon(\eta+s)} z_j^2(\eta) d\eta \right] ds \} \quad (12)
 \end{aligned}$$

$\varepsilon > 0$ is a small number.

Calculating the upper right Dini-derivative $D^+V(t)$ of $V(t)$ along the solution of (10), using (11) we have

$$\begin{aligned}
 D^+V(t) & = \sum_{i=1}^n \left\{ \varepsilon \frac{u_i^2(t)+v_i^2(t)}{2} e^{\varepsilon t} \right. \\
 & + \frac{1}{2} \frac{d}{dt} (u_i^2(t) + v_i^2(t)) e^{\varepsilon t} \\
 & + \sum_{j=1}^n \frac{|b_{ij}|}{2} L_j [u_j(t) e^{\varepsilon(t+\tau_{ji})} - u_j(t - \tau_{ji}) e^{\varepsilon t}] \\
 & + \sum_{j=1}^n \frac{|c_{ij}|}{2} L_j \int_0^{+\infty} K_{ij}(s) [e^{\varepsilon(t+s)} u_j^2(t) \\
 & - e^{\varepsilon t} u_j^2(t-s)] ds \} \\
 & \leq e^{\varepsilon t} \sum_{i=1}^n \left\{ \varepsilon \frac{u_i^2(t)+v_i^2(t)}{2} + \left(\frac{|\beta_i - \alpha_i|}{2} - 1 \right) u_i^2(t) \right. \\
 & + (1 - \beta_i + \frac{|\beta_i - \alpha_i|}{2}) v_i^2(t) \\
 & + \sum_{j=1}^n \frac{|a_{ij}|}{2} L_j [v_i^2(t) + u_j^2(t)] \\
 & + \sum_{j=1}^n \frac{|b_{ij}|}{2} L_j [v_i^2(t) + u_j^2(t - \tau_{ij})] \\
 & + \sum_{j=1}^n \frac{|c_{ij}|}{2} L_j \int_{-\infty}^t K_{ij}(t-s) [v_i^2(t) + u_j^2(s)] ds \\
 & + \sum_{j=1}^n \frac{|b_{ij}|}{2} L_j [u_j(t) e^{\varepsilon \tau} - z_j(t - \tau_{ji})] \\
 & + \sum_{j=1}^n \frac{|c_{ij}|}{2} L_j \int_0^{+\infty} K_{ij}(s) [e^{\varepsilon s} u_j^2(t) \\
 & - u_j^2(t-s)] ds \} \\
 & \leq \frac{e^{\varepsilon t}}{2} \sum_{i=1}^n \left\{ [\varepsilon - 2 + |\beta_i - \alpha_i| + \sum_{j=1}^n L_i (|a_{ji}| + |b_{ji}|) e^{\varepsilon \tau} \right. \\
 & + |c_{ji}| \int_0^{+\infty} K_{ji}(s) e^{\varepsilon s} ds] u_i^2(t) \\
 & + [\varepsilon - 2\beta_i + 2 + |\beta_i - \alpha_i| \\
 & + \sum_{j=1}^n L_j (|a_{ij}| + |b_{ij}| + |c_{ij}|)] v_i^2(t) \}. \quad (13)
 \end{aligned}$$

From condition of Theorem 6, we can choose a small $\varepsilon > 0$ such that

$$\begin{aligned}
 \varepsilon - 2 + |\beta_i - \alpha_i| + \sum_{j=1}^n L_i (|a_{ji}| + |b_{ji}|) e^{\varepsilon \tau} \\
 + |c_{ji}| \int_0^{+\infty} K_{ji}(s) e^{\varepsilon s} ds \leq 0,
 \end{aligned}$$

$$\varepsilon - 2\beta_i + 2 + |\beta_i - \alpha_i| + \sum_{j=1}^n L_j (|a_{ij}| + |b_{ij}| + |c_{ij}|) \leq 0,$$

for $i = 1, 2, \dots, n$.

From (13), we get $D^+V(t) \leq 0$, and so $V(t) \leq V(0)$, for all $t \geq 0$.

From (12), we have

$$\begin{aligned}
 V(t) & \geq \sum_{i=1}^n \frac{u_i^2(t)+v_i^2(t)}{2} e^{\varepsilon t} \\
 & = \sum_{i=1}^n \frac{e^{\varepsilon t}}{2} [(x_i - x_i^*)^2 + (y_i - y_i^*)^2]. \quad (14)
 \end{aligned}$$

$$\begin{aligned}
 V(0) & = \sum_{i=1}^n \left\{ \frac{u_i^2(0)+v_i^2(0)}{2} \right. \\
 & + \sum_{j=1}^n \frac{|b_{ij}|}{2} L_j \int_{-\tau_{ij}}^0 u_j^2(s) e^{\varepsilon(\eta+\tau_{ij})} d\eta \\
 & + \sum_{j=1}^n \frac{|c_{ij}|}{2} L_j \int_0^{+\infty} K_{ij}(s) \left[\int_{-s}^0 e^{\varepsilon(\eta+s)} u_j^2(\eta) d\eta \right] ds \} \\
 & = \sum_{i=1}^n \left\{ \frac{(\varphi_i(0) - \varphi_i^*(0))^2}{2} + \frac{(\bar{\psi}_i(0) - \bar{\psi}_i^*(0))^2}{2} \right. \\
 & + \sum_{j=1}^n \frac{|b_{ij}|}{2} L_j \int_{-\tau_{ij}}^0 (\varphi_j(\eta) - \varphi_j^*(\eta))^2 e^{\varepsilon(s+\tau_{ij})} d\eta \\
 & + \sum_{j=1}^n \frac{|c_{ij}|}{2} L_j \int_0^{+\infty} K_{ij}(s) \\
 & \quad \times \left[\int_{-s}^0 e^{\varepsilon(\eta+s)} (\varphi_j(\eta) - \varphi_j^*(\eta))^2 d\eta \right] ds \} \\
 & \leq \frac{\|\varphi - \varphi^*\|^2}{2} + \frac{\|\bar{\psi} - \bar{\psi}^*\|^2}{2} \\
 & + \tau \sum_{j=1}^n \max_{1 \leq i \leq n} \left\{ \frac{|b_{ij}|}{2} L_i \right\} e^{\varepsilon \tau} \|\varphi - \varphi^*\|^2 \\
 & + \sum_{j=1}^n \max_{1 \leq i \leq n} \left\{ \frac{|c_{ij}|}{2} L_i \right\} \int_0^{+\infty} K_{ji}(s) \\
 & \quad \times \left[\int_{-s}^0 e^{\varepsilon(\eta+s)} d\eta \right] ds \|\varphi - \varphi^*\|^2 \\
 & \leq \frac{1}{2} [1 + \tau \sum_{j=1}^n \max_{1 \leq i \leq n} \{|b_{ij}| L_j\} e^{\varepsilon \tau} \\
 & + \sum_{j=1}^n \max_{1 \leq i \leq n} \{ |c_{ji}| L_i \frac{1}{\varepsilon} \rho_{ji}(\varepsilon) \}] \|\varphi - \varphi^*\|^2 + \frac{1}{2} \|\bar{\psi} - \bar{\psi}^*\|^2. \quad (15)
 \end{aligned}$$

Since $V(0) \geq V(t)$, from (14) and (15), we obtain

$$\begin{aligned}
 \sum_{i=1}^n \frac{e^{\varepsilon t}}{2} [(x_i - x_i^*)^2 + (y_i - y_i^*)^2] \\
 \leq \frac{1}{2} [1 + \tau \sum_{j=1}^n \max_{1 \leq i \leq n} \{|b_{ij}| L_j\} e^{\varepsilon \tau} \\
 + \sum_{j=1}^n \max_{1 \leq i \leq n} \{ |c_{ji}| L_i \frac{1}{\varepsilon} \rho_{ji}(\varepsilon) \}] \|\varphi - \varphi^*\|^2 + \frac{1}{2} \|\bar{\psi} - \bar{\psi}^*\|^2. \quad (16)
 \end{aligned}$$

By multiplying both sides of (16) with $2e^{-\varepsilon t}$, we get

$$\sum_{i=1}^n [(x_i - x_i^*)^2 + (y_i - y_i^*)^2] \leq M e^{-\varepsilon t} \|\varphi - \varphi^*\|^2. \quad (17)$$

for all $t \geq 0$, where

$$\begin{aligned}
 M & = \{1 + \tau \sum_{j=1}^n \max_{1 \leq i \leq n} \{|b_{ij}| L_j\} e^{2\varepsilon \tau} \\
 & + \sum_{j=1}^n \max_{1 \leq i \leq n} \{ |c_{ji}| L_i \frac{1}{\varepsilon} \rho_{ji}(\varepsilon) \} + \frac{\|\bar{\psi} - \bar{\psi}^*\|^2}{\|\varphi - \varphi^*\|^2} \}.
 \end{aligned}$$

From (17), we obtain

$$\sum_{i=1}^n (x_i - x_i^*)^2 \leq M e^{-\varepsilon t} \|\varphi - \varphi^*\|^2, \quad t > 0.$$

By Definition 1, system (2) has one ω - periodic solution which exponentially stable.

IV. NUMERICAL SIMULATION

In this Section, we give two examples for $\beta_i^2 - 4\alpha_i > 0$ and $\beta_i^2 - 4\alpha_i < 0$, respectively.

Example 4.1. We consider the following inertial neural networks with unbounded delay ($n = 2$)

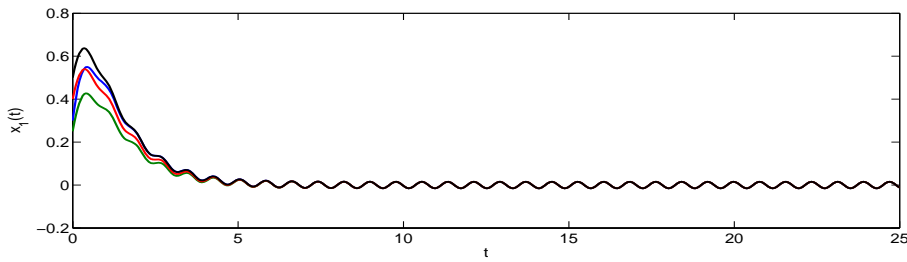


Fig.1. Transient response of state variables $x_1(t)$ of Example 4.1

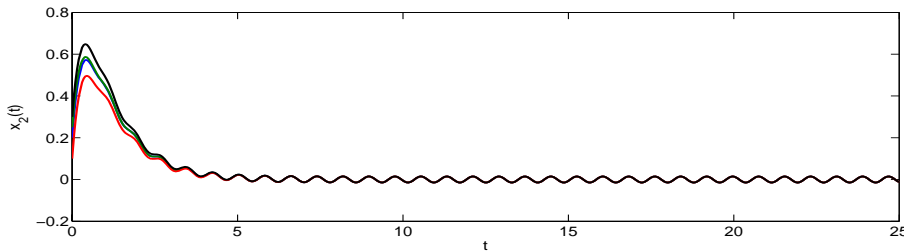


Fig.2. Transient response of state variables $x_2(t)$ of Example 4.1

$$\begin{aligned} \frac{d^2 x_i(t)}{dt^2} &= -\beta_i \frac{dx_i(t)}{dt} - \alpha_i x_i(t) \\ &+ \sum_{j=1}^2 a_{ij} f_j(x_j(t)) + \sum_{j=1}^2 b_{ij} f_j(x_j(t - \tau_{ij})) \\ &+ \sum_{j=1}^2 c_{ij} \int_{-\infty}^t K_{ij}(t-s) f_j(x_j(s)) ds + I_i(t), \end{aligned} \quad (18)$$

for $i = 1, 2$, where $\alpha_1 = 2, \alpha_2 = 3, \beta_1 = 3, \beta_2 = 4$,

$$a_{11} = \frac{1}{32}, \quad a_{12} = \frac{1}{64}, \quad a_{21} = -\frac{1}{32}, \quad a_{22} = -\frac{1}{64},$$

$$b_{11} = -\frac{1}{32}, \quad b_{12} = \frac{1}{64}, \quad b_{21} = \frac{1}{32}, \quad b_{22} = -\frac{1}{64},$$

$$c_{11} = \frac{1}{64}, \quad c_{12} = \frac{1}{32}, \quad c_{21} = -\frac{1}{64}, \quad c_{22} = \frac{1}{32},$$

$$f_i(x) = \frac{1}{8} \sin(8x), \quad K_{ij}(x) = e^{-x}, \quad I_i(t) = \frac{1}{4} \cos(8t),$$

for $i = 1, 2$.

Obviously,

$$|f_i(x) - f_i(y)| \leq |x - y|, \quad i = 1, 2,$$

$$\int_0^{+\infty} K_{ij}(s) ds = \int_0^{+\infty} e^{-s} ds = 1, \quad i, j = 1, 2,$$

$$\bar{I}_i = 1/4, \quad i = 1, 2.$$

We select $L_i = 1, \bar{f}_i = 1/8, \bar{I}_i = 1/4 (i = 1, 2), \omega = \frac{\pi}{4}$, hypotheses (H1) – (H2) are hold.

For numerical simulation, let $\tau_{11} = 0.2, \tau_{12} = 0.3, \tau_{21} = 0.1, \tau_{22} = 0.4$, the following four cases are given:

case 1 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.3, 0.2, 1.5, 2, 3];$$

case 2 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.25, 0.25, 1.1, 2.2];$$

case 3 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.4, 0.1, 1.15, 2.24];$$

case 4 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.5, 0.3, 1.28, 2.35].$$

Figs. 1-2 depict the time responses of state variables of $x_1(t)$ and $x_2(t)$ of system in example 4.1, respectively.

On the other hand, by calculation, we have the following results

$$\beta_1^2 - 4\alpha_1 = 1 > 0, \quad \beta_2^2 - 4\alpha_2 = 4 > 0, \quad \psi_i(0) > 0, \quad i = 1, 2.$$

$$d_1 = \sum_{j=1}^2 \bar{f}_j [|a_{1j}| + |b_{1j}| + |c_{1j}|] + \bar{I}_1 = \frac{137}{512},$$

$$d_2 = \sum_{j=1}^2 \bar{f}_j [|a_{2j}| + |b_{2j}| + |c_{2j}|] + \bar{I}_2 = \frac{137}{512},$$

$$e = \min_{1 \leq i \leq 2} \{d_i\} = \frac{137}{512},$$

$$\varphi_1(0) > \frac{e}{\alpha_1} = \frac{137}{1024} \approx 0.1338,$$

$$\varphi_2(0) > \frac{e}{\alpha_2} = \frac{137}{1536} \approx 0.089.$$

Since $\beta_i^2 - 4\alpha_i > 0 (i = 1, 2)$, By Lemma 4 we have

$$\mu_1 = \frac{1}{2} [\beta_1 + \sqrt{|\beta_1^2 - 4\alpha_1|}] = 2,$$

$$\mu_2 = \frac{1}{2} [\beta_2 + \sqrt{|\beta_2^2 - 4\alpha_2|}] = 3,$$

$$M_1 = \left[\frac{2(\beta_1 - \alpha_1)^2 + 2(\beta_1 - 2)^2}{|\beta_1^2 - 4\alpha_1|} \right]^{\frac{1}{2}} = 2,$$

$$M_2 = \left[\frac{2(\beta_2 - \alpha_2)^2 + 2(\beta_2 - 2)^2}{|\beta_2^2 - 4\alpha_2|} \right]^{\frac{1}{2}} = \frac{\sqrt{10}}{2}.$$

$$N_1 = \frac{1}{|\beta_1^2 - 4\alpha_1|^{\frac{3}{2}} |e^{\beta_1 \omega} - e^{-\frac{\beta_1 \omega}{2}} (e^{\frac{\sqrt{\beta_1^2 - 4\alpha_1} \omega}{2}} + e^{-\frac{\sqrt{\beta_1^2 - 4\alpha_1} \omega}{2}}) + 1|}$$

$$\cdot \{[(\alpha_1 - \beta_1)^2 + (2 - \beta_1)^2]e^{\beta_1\omega} (e^{\sqrt{\beta_1^2 - 4\alpha_1}\omega} + e^{-\sqrt{\beta_1^2 - 4\alpha_1}\omega}) + 2(\beta_1^2 - 4\alpha_1)\}^{1/2} < \frac{7.64}{21},$$

$$N_2 = \frac{1}{|\beta_2^2 - 4\alpha_2|^{\frac{3}{2}} |e^{\beta_2\omega} - e^{-\frac{\beta_2\omega}{2}} (e^{\frac{\sqrt{\beta_2^2 - 4\alpha_2}\omega}{2}} + e^{-\frac{\sqrt{\beta_2^2 - 4\alpha_2}\omega}{2}})| + 1}$$

$$\cdot \{[(\alpha_2 - \beta_2)^2 + (2 - \beta_2)^2]e^{\beta_2\omega} (e^{\sqrt{\beta_2^2 - 4\alpha_2}\omega} + e^{-\sqrt{\beta_2^2 - 4\alpha_2}\omega}) + 2(\beta_2^2 - 4\alpha_2)\}^{1/2} < \frac{4.2}{11},$$

$$h_i = \frac{2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)}{\sqrt{\beta_i^2 - 4\alpha_i}} + \frac{d_i}{\alpha_i} > 2, i = 1, 2,$$

$$k_i = \psi_i(0) + \frac{1}{\beta_i}[\alpha_i h_i + \sum_{j=1}^2 \bar{f}_j (|a_{ij}| + |b_{ij}| + |c_{ij}|) + \bar{I}_i] > 1, i = 1, 2.$$

$$h = \max_{1 \leq i \leq 2} \{\sqrt{3h_i^2 + 2k_i^2}\} > \sqrt{14}.$$

$$\|(\exp(-A_1 \frac{\pi}{4}) - E)^{-1}\| \leq N_1 < \frac{7.641}{21}.$$

$$\|(\exp(-A_2 \frac{\pi}{4}) - E)^{-1}\| \leq N_2 < \frac{4.24}{11}.$$

We have

$$\|(\exp(-A_i \omega) - E)^{-1}\| \omega M_i e^{\mu_i \omega}$$

$$\times [\sum_{j=1}^2 L_j (|a_{ij}| + |b_{ij}| + |c_{ij}|) + \frac{\bar{I}_i}{h}] < 1, i = 1, 2.$$

$$-2 + |\beta_1 - \alpha_1| + \sum_{j=1}^2 L_j (|a_{j1}| + |b_{j1}| + |c_{j1}| \int_0^{+\infty} K_{j1}(s) ds) = -\frac{27}{32} < 0,$$

$$-2\beta_1 + 2 + |\beta_1 - \alpha_1| + \sum_{j=1}^2 (|a_{1j}| + |b_{1j}| + |c_{1j}|) = -\frac{183}{64} < 0,$$

$$-2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 L_j (|a_{j2}| + |b_{j2}| + |c_{j2}| \int_0^{+\infty} K_{j2}(s) ds) = -\frac{7}{8} < 0,$$

$$-2\beta_2 + 2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 (|a_{2j}| + |b_{2j}| + |c_{2j}|) = -\frac{311}{64} < 0.$$

Then, the conditions of Theorem 5 and Theorem 6 hold, the system (18) has one $\pi/4$ - periodic solution, and all other solutions of system exponentially converge to it as $t \rightarrow +\infty$. Evidently, this consequence is coincident with the results of numerical simulation.

Example 4.2. For system (18), we let

$$\alpha_1 = \frac{5}{4}, \alpha_2 = \frac{5}{2}, \beta_1 = 2, \beta_2 = 3,$$

$$f_i(x) = \frac{1}{128} \sin(8x), I_i(t) = \frac{1}{64} \cos(8t), i = 1, 2,$$

the other parameters are the same as that in Example 4.1. Obviously,

$$|f_i(x) - f_i(y)| \leq \frac{1}{16} |x - y|, \bar{I}_i = \frac{1}{64}, i = 1, 2.$$

We select $L_i = \frac{1}{16}, \bar{f}_i = 1/128, \bar{I}_i = \frac{1}{64} (i = 1, 2), \omega = \frac{\pi}{4}$, hypotheses (H1) - (H2) are hold.

For numerical simulation, let $\tau_{11} = 0.02, \tau_{12} = 0.03, \tau_{21} = 0.1, \tau_{22} = 0.2$, the following four cases are given:

case 1 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.3; 0.7; 1.2; 2.1];$$

case 2 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.1; 0.5; 1.1; 2.2];$$

case 3 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.4, 0.1, 1.15, 2.24];$$

case 4 with the initial state

$$[\varphi_1(0), \varphi_2(0), \psi_1(0), \psi_2(0)] = [0.5; 0.3; 1.2; 2.3].$$

Figs. 3-4 depict the time responses of state variables of $x_1(t)$ and $x_2(t)$ of system in example 4.2, respectively.

On the other hand, by calculation, we have the following results

$$\beta_1^2 - 4\alpha_1 = -1 < 0, \beta_2^2 - 4\alpha_2 = -1 < 0, \psi_i(0) > 0, i = 1, 2.$$

$$d_1 = \sum_{j=1}^2 \bar{f}_j (|a_{1j}| + |b_{1j}| + |c_{1j}|) + \bar{I}_1 = \frac{137}{64 \times 128},$$

$$d_2 = \sum_{j=1}^2 \bar{f}_j (|a_{2j}| + |b_{2j}| + |c_{2j}|) + \bar{I}_2 = \frac{137}{64 \times 128},$$

$$e = \min_{1 \leq i \leq 2} \{d_i\} = \frac{137}{64 \times 128},$$

$$\varphi_1(0) > \frac{e}{\alpha_1} \approx 0.01338,$$

$$\varphi_2(0) > \frac{e}{\alpha_2} \approx 0.0069.$$

Since $\beta_i^2 - 4\alpha_i < 0 (i = 1, 2)$, By Lemma 4 we have

$$\mu_1 = \frac{1}{2}\beta_1 = 1,$$

$$\mu_2 = \frac{1}{2}\beta_2 = \frac{3}{2},$$

$$M_1 = 2 \left[\frac{2\alpha_1 - 2\beta_1 + 3 + (|1 - \delta_1| + |\alpha_1 + 1 - \beta_1|)^2}{|\beta_1^2 - 4\alpha_1|} \right]^{1/2}$$

$$= \frac{\sqrt{33}}{2},$$

$$M_2 = 2 \left[\frac{2\alpha_2 - 2\beta_2 + 3 + (|1 - \delta_2| + |\alpha_2 + 1 - \beta_2|)^2}{|\beta_2^2 - 4\alpha_2|} \right]^{1/2}$$

$$= 2\sqrt{6},$$

$$N_1 = \frac{2e^{\beta_1\omega/2}}{\delta_1 |2\delta_1^2(1 - e^{-\beta_1\omega/2})^2 + (\delta_1 - 1)\sin 2\delta_1\omega|}$$

$$\cdot \{2\delta_1 - 1 + [2(\frac{\beta_1}{2} - 1)^2 + (1 + \alpha_1 - \beta_1)^2] \sin^2 \delta_1\omega$$

$$+ 3\delta_1^2 \cos^2 \delta_1\omega + 2(\delta_1 e^{-\beta_1\omega/2} + 1)^2$$

$$- 2\delta_1(\cos \delta_1\omega + 1)\}^{1/2} < 6.5,$$

$$N_2 = \frac{2e^{\beta_2\omega/2}}{\delta_2 |2\delta_2^2(1 - e^{-\beta_2\omega/2})^2 + (\delta_2 - 1)\sin 2\delta_2\omega|}$$

$$\cdot \{2\delta_2 - 1 + [2(\frac{\beta_2}{2} - 1)^2 + (1 + \alpha_2 - \beta_2)^2] \sin^2 \delta_2\omega$$

$$+ 3\delta_2^2 \cos^2 \delta_2\omega + 2(\delta_2 e^{-\beta_2\omega/2} + 1)^2$$

$$- 2\delta_2(\cos \delta_2\omega + 1)\}^{1/2} < 4.64,$$

$$h_i = \frac{2\psi_i(0) + \beta_i(\varphi_i(0) - e/\alpha_i)}{\sqrt{\beta_i^2 - 4\alpha_i}} + \frac{d_i}{\alpha_i} > 2, i = 1, 2,$$

$$k_i = \psi_i(0) + \frac{1}{\beta_i}[\alpha_i h_i + \sum_{j=1}^2 \bar{f}_j (|a_{ij}| + |b_{ij}| + |c_{ij}|) + \bar{I}_i] > 1, i = 1, 2.$$

$$h = \max_{1 \leq i \leq 2} \{\sqrt{3h_i^2 + 2k_i^2}\} > \sqrt{14}.$$

$$\|(\exp(-A_1 \frac{\pi}{4}) - E)^{-1}\| \leq N_1 < 6.5.$$

$$\|(\exp(-A_2 \frac{\pi}{4}) - E)^{-1}\| \leq N_2 < 4.64.$$

We have

$$\|(\exp(-A_i \omega) - E)^{-1}\| \omega M_i e^{\mu_i \omega}$$

$$\times [\sum_{j=1}^2 L_j (|a_{ij}| + |b_{ij}| + |c_{ij}|) + \frac{\bar{I}_i}{h}] < 1, i = 1, 2.$$

$$-2 + |\beta_1 - \alpha_1| + \sum_{j=1}^2 L_j (|a_{j1}| + |b_{j1}| + |c_{j1}| \int_0^{+\infty} K_{j1}(s) ds) = -\frac{635}{512} < 0,$$

$$-2\beta_1 + 2 + |\beta_1 - \alpha_1| + \sum_{j=1}^2 (|a_{1j}| + |b_{1j}| + |c_{1j}|) = -\frac{35}{32} < 0,$$

$$-2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 L_j (|a_{j2}| + |b_{j2}| + |c_{j2}| \int_0^{+\infty} K_{j2}(s) ds) = -\frac{191}{128} < 0,$$

$$-2\beta_2 + 2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 (|a_{2j}| + |b_{2j}| + |c_{2j}|) = -\frac{215}{64} < 0.$$

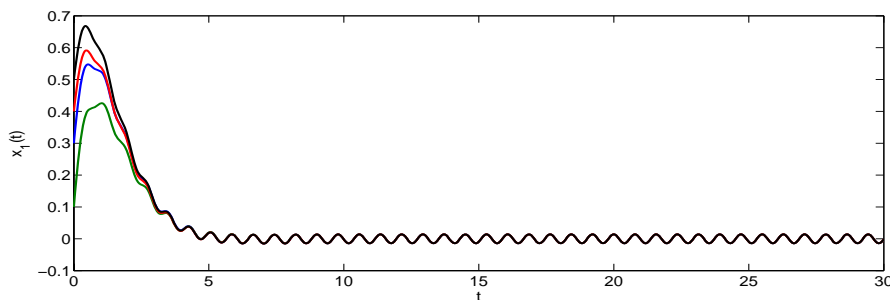
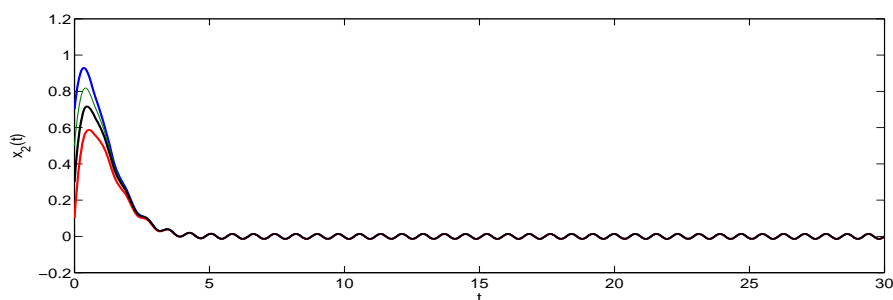
$$-2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 L_j (|a_{j2}| + |b_{j2}| + |c_{j2}| \int_0^{+\infty} K_{j2}(s) ds) = -\frac{191}{128} < 0,$$

$$-2\beta_2 + 2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 (|a_{2j}| + |b_{2j}| + |c_{2j}|) = -\frac{215}{64} < 0.$$

$$-2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 L_j (|a_{j2}| + |b_{j2}| + |c_{j2}| \int_0^{+\infty} K_{j2}(s) ds) = -\frac{191}{128} < 0,$$

$$-2\beta_2 + 2 + |\beta_2 - \alpha_2| + \sum_{j=1}^2 (|a_{2j}| + |b_{2j}| + |c_{2j}|) = -\frac{215}{64} < 0.$$

Then, the conditions of Theorem 5 and Theorem 6 hold,

Fig.3. Transient response of state variables $x_1(t)$ of Example 4.2Fig.4. Transient response of state variables $x_2(t)$ of Example 4.2

the system (18) has one $\pi/4$ - periodic solution, and all other solutions of system exponentially converge to it as $t \rightarrow +\infty$. Evidently, this consequence is coincident with the results of numerical simulation.

V. CONCLUSIONS

Since the periodic solutions for system is very important in theories and applications. In this paper, we give theorems to ensure the existence and the exponential stability of the periodic solution for inertial neural networks with unbounded delay. Novel existence and stability conditions are stated in simple algebraic forms and their verification and applications are straightforward and convenient.

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