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Exponential stability of numerical solutions to stochastic age-dependent population equations with Poisson jumps

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Abstract—The main aim of this paper is to investigate the exponential stability of the Euler method for a stochastic age-dependent population equations with Poisson random measures. It is proved that the Euler scheme is exponentially stable in mean square sense. An example is given for illustration.

Keywords—Stochastic age-dependent population equations, Poisson random measures, Numerical solutions, Exponential stability.

I. Introduction

STOCHASTIC partial functional differential equations are very important in stochastic models of biological, physical and economical systems, and the study of stochastic age-dependent population equations(SADPEs) has received a lot of attention. Pollard [1] studied the effects of adding stochastic terms to discrete-time age-dependent models that employ Leslie matrices. Zhang [2] investigated the existence, uniqueness and exponential stability for SADPEs. Li[3] and Pang[4]discussed the convergence and exponential stability of numerical solutions to SADPEs.

On the other hand, in the stochastic age-dependent population system, due to brusque variations from some rare events, the size of the population systems increases or decreases drastically, so Poisson jumps is embedded into the SADPEs. Recently, Li [5] and Wang [6] studied SADPEs with Poisson jump process and given some results about the numerical analysis. However, to the best of our knowledge, there is little work on the exponential stability of numerical solutions to SADPEs with Poisson random measures. Motivated by Pang [4], we will study the exponential stability of numerical solutions for the above systems. Although the way of analysis follows the ideas in [4,7,8,9], we need to develop several new techniques to deal with the Poisson random measure. Some known results in [4] are generalized and improved.

The paper is organized as follows. In Section 2, we present some basic preliminaries and define an Euler approximate solution to SADPEs with Poisson random measures. In section 3, we show that the Euler method applied to SADPEs with Poisson random measures is exponentially stable in mean square sense. In section 4, an example is provided to illustrate our theory.

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II. Preliminaries and the Euler approximation

Let $V = H^1([0,A]) \equiv \{\varphi|\varphi \in L^p([0,A]), \frac{\partial\varphi}{\partial x_i} \in L^p([0,A])$, where $\frac{\partial\varphi}{\partial x_i}$ is generalized partial derivatives. $\}$ V is a Sobolev space. $H = L^p([0,A]), (P \geq 2)$ such that $V \hookrightarrow H \equiv H' \hookrightarrow V'$. V' is the dual space of V. We denote by $||\cdot||, |\cdot|$, and $||\cdot||_*$ the norm in V, H and V', respectively; by $\langle\cdot,\cdot\rangle$ the duality product between V, V', and by $(\cdot,\cdot)$ the scalar product in H.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.Let W_t be a Wiener process defined on $(\Omega, \mathcal{F}, P)$ and taking its values in the separable Hilbert space K, with increment covariance W. For an operator $B \in \mathcal{L}(K,H)$ be the space of all bounded linear operators from K into H, we denote by $||B||_2$ the Hilbert-Schmidt norm, i.e. $||B||_2 = tr(BWB^T)$. Let $C = C([0,T];H)$ be the space of all right-continuous functions with left-hand limits from $[0,T]$ into H. The space $C = C([0,T];H)$ is assumed to be equipped with sup-norm $||\psi||_C = \sup_{0\leq t\leq T} |\psi(s)|$. $L_V^p = L^p([0,T];V)$ and $L_H^p = L^p([0,T];H)$.

Let $(U, \mathcal{B}(U))$ be a measurable space and $\pi(du)$ a σ- finite measure on it. Let $p = p(t), t \in D_p$ be a stationary $\mathcal{F}_t$-Poisson point process on R^n with characteristic measure π. Denote by $N(dt,du)$ the Poisson counting measure associated with p, i.e., $N(t,A) = \sum_{s\in D_p, s\leq t} I_A(p(s))$. Let $\tilde{N}(dt,du) := N(dt,du) - \pi(du)dt$ be the compensated Poisson random measure that is independent of W_t. We refer to Ikeda [10] for the details on Poisson point process.

We are concerned with stochastic age-dependent population equations driven by Poisson random measures:

$$\begin{cases} d_tP = [-\frac{\partial P}{\partial a} - u(t,a)P + f(t,P)]dt + g(t,P)dW_t \\ \qquad + \int_U h(u,P)\tilde{N}(dt,du), \quad in\ Q = (0,T)\times(0,A), \\ P(0,a) = P_0(a), \qquad\qquad in\ \ [0,A], \\ P(t,a) = \displaystyle\int_0^A \beta(t,a)P(t,a)da, \quad in\ \ [0,T], \end{cases} \tag{1}$$

where $T > 0, A > 0$. $P(t,a)$ denotes the population density of age a at time t, $\beta(t,a)$ denotes the fertility rate of females of age a at time t, $u(t,a)$ denotes the mortality rate of age a at time t. $f(t,P)$ denotes effects of external environment for population system, $g(t,P)$ is a diffusion coefficient, $h(u,P)$ is a jump coefficient. Let $f(t,.) : [0,T]\times L_H^2 \to H$ be a family of nonlinear operator, $\mathcal{F}_t$-measurable almost surely in t; $g(t,.) : [0,T] \times L_H^2 \to \mathcal{L}(K,H)$ is the family of nonlinear operator,

$\mathcal{F}_t$-measurable almost surely in t; $h(u,.) : U \times L_H^2 \to H$ is the family of nonlinear operator, $\mathcal{F}_t$-measurable almost surely in t.

Now,we define the Euler approximate solution. For system (1), the discrete implicit Euler approximation on $t \in \{0, h, 2h, \cdots\}$ is given by the iterative scheme

$$\begin{aligned} Q_t^{n+1} &= Q_t^n - \frac{\partial Q_t^{n+1}}{\partial a}h - u(t,a)Q_t^n h + f(t, Q_t^n)h \\ &\quad + g(t, Q_t^n)\Delta W_n + \int_U h(u, Q_t^n)\tilde{N}(h, du), \quad (2) \end{aligned}$$

with initial value $Q_t^0 = P(0, a)$, $Q^n(t, 0) = \int_0^A \beta(t, a)Q_t^n da$, $n \geq 1$. Here,Q_t^n is the approximation to $P(t_n, a)$, for $t_n = nh$, the time increment is $h = T/N$, for some large integer N such that $h << 1$. Brownian motion increment is $\Delta W_n = W(t_{n+1}) - W(t_n)$ and $\tilde{N}(h, du) = \tilde{N}(t_{n+1}, du) - \tilde{N}(t_n, du)$. We define the step functions:

$$Z_t = \sum_{n=0}^{N} Q_t^n I_{[nh,(n+1)h)}(t),$$

where I_G is the indicator function for the set G. Then we define the continuous Euler approximate solution

$$\begin{aligned} Q_t &= Q_0 - \int_0^t [\frac{\partial Q_s}{\partial a} + u(s,a)Z_s - f(s, Z_s)]ds \\ &\quad + \int_0^t g(s, Z_s)dW_s + \int_0^t \int_U h(u, Z_s)\tilde{N}(ds, du), (3) \end{aligned}$$

with $Q_0 = P(0, a), Q(t, 0) = \int_0^A \beta(t, a)Q_t da$, $Q_t = Q(t, a)$.

To state our main theorem, we shall impose the following conditions on the coefficients f, g and h.

(H_1) $f(t, 0) = 0$, $g(t, 0) = 0$, $h(t, 0) = 0$.

(H_2) $u(t, a), \beta(t, a)$ are continuous in $\bar{Q}$ such that

$$0 \leq u_0 \leq u(t, a) \leq \bar{\alpha} < \infty, \qquad 0 \leq \beta(t, a) \leq \bar{\beta} < \infty. \quad (4)$$

(H_3) There exists a positive constant K such that $x, y \in C$ and $u \in U$,

$$\begin{aligned} &|f(t,x) - f(t,y)|^2 \vee ||g(t,x) - g(t,y)||_2^2 \\ &\vee \int_U |h(x,u) - h(y,u)|^2\pi(du) \leq K||x - y|_C^2. \quad (5) \end{aligned}$$

(H_4) (Coercivity condition) there exists constants $\alpha > 0$, $\xi > 0$, $\lambda \in R$, and a nonnegative continuous function $\gamma(t)$, $t \in R^+$, such that

$$\begin{aligned} &2 < f(t,v), v > + ||g(t,v)||_2^2 + \int_U |h(u,v)|^2\pi(du) \\ &\qquad \leq -\alpha||v||^2 + \lambda|v|^2 + \gamma(t)e^{-\xi t}, \quad v \in V, \quad (6) \end{aligned}$$

where, for arbitrary $\delta > 0$, $\gamma(t)$ satisfies $\gamma(t) = o(e^{\delta t})$, as $t \to \infty$, i.e., $lim_{t\to\infty}\gamma(t)/e^{\delta t} = 0$.

Definition 2.1 For a given step size $h > 0$, a numerical method is said to be exponentially stable in mean square on Eq.(1) if for any $a \in [0, A]$ there is a pair of positive constants γ and M such that

$$E|Q_t|^2 \leq Me^{-\gamma t}, \quad \forall t \geq 0. \quad (7)$$

III. The main results

In this section, we shall provide some lemmas which are necessary for the proof of our result. Because Q_t interpolates the discrete numerical solution, we first study properties of Q_t.

Lemma 3.1 Under conditions (H_1) - (H_3), we get

$$E(\sup_{0\leq t\leq T} |Q_t|^2) \leq C_1, \quad (8)$$

where $C_1 = 2E|Q_0|^2 e^{[2(A^2\bar{\beta}^2+u_0+2)+2(3+2K_1+C)K]T}$.

Proof: Applying Ito formula to $|Q_t|^2$ yields, it is easy to see from (3) that

$$\begin{aligned} &|Q_t|^2 \\ =\ &|Q_0|^2 + 2\int_0^t < -\frac{\partial Q_s}{\partial a} - u(s,a)Z_s, Q_s > ds \\ &+2\int_0^t (f(s,Z_s), Q_s)ds + \int_0^t ||g(s,Z_s)||_2^2 ds \\ &+2\int_0^t (Q_s, g(s,Z_s)dW_s) + \int_0^t \int_U |h(u,Z_s)|^2\pi(du)ds \\ &+\int_0^t \int_U [2(Q_s, h(u,Z_s)) + |h(u,Z_s)|^2]\tilde{N}(ds,du) \\ \leq\ &|Q_0|^2 - 2\int_0^t < \frac{\partial Q_s}{\partial a}, Q_s > ds - 2u_0\int_0^t (Z_s, Q_s)ds \\ &+2\int_0^t (f(s,Z_s), Q_s)ds + \int_0^t ||g(s,Z_s)||_2^2 ds \\ &+2\int_0^t (Q_s, g(s,Z_s)dW_s) + \int_0^t \int_U |h(u,Z_s)|^2\pi(du)ds \\ &+\int_0^t \int_U [2(Q_s, h(u,Z_s)) + |h(u,Z_s)|^2]\tilde{N}(ds,du). \end{aligned}$$

Note

$$\begin{aligned} &- < \frac{\partial Q_s}{\partial a}, Q_s > \\ =\ &-\int_0^A Q_s d_a(Q_s) = \frac{1}{2}(\int_0^A \beta(s,a)Q_s da)^2 \\ &\leq \frac{1}{2}(\int_0^A \beta^2(s,a)da \int_0^A Q_s^2 da \leq \frac{1}{2}A^2\bar{\beta}^2|Q_s|^2. \quad (9) \end{aligned}$$

Therefore, we get that

$$\begin{aligned} &|Q_t|^2 \\ \leq\ &|Q_0|^2 + A^2\bar{\beta}^2\int_0^t |Q_s|^2 ds + 2u_0\int_0^t |Z_s||Q_s|ds \\ &+2\int_0^t |f(s,Z_s)||Q_s|ds + \int_0^t ||g(s,Z_s)||_2^2 ds \\ &+2\int_0^t (Q_s, g(s,Z_s)dW_s) + \int_0^t \int_U |h(u,Z_s)|^2\pi(du)ds \\ &+\int_0^t \int_U [2(Q_s, h(u,Z_s)) + |h(u,Z_s)|^2]\tilde{N}(ds,du). \end{aligned}$$

Applying the inequality $2ab \leq a^2+b^2$, it follows that for any $t \in [0,T]$,

$$\begin{aligned} & E(\sup_{0\leq s\leq t}|Q_s|^2) \\ \leq\ & E|Q_0|^2 + (A^2\bar{\beta}^2+u_0+2)\int_0^t E\sup_{0\leq u\leq s}|Q_u|^2 ds \\ & +E\int_0^t |f(s,Z_s)|^2 ds + E\int_0^t\int_U |h(u,Z_s)|^2\pi(du)ds \\ +\ & 2E\sup_{0\leq s\leq t}\int_0^s (Q_u, g(u,Z_u)dW_u) + E\int_0^t ||g(s,Z_s)||_2^2 ds \\ +\ & E\sup_{0\leq s\leq t}\int_0^s\int_U [2(Q_v,h(u,Z_v)) + |h(u,Z_v)|^2]\tilde{N}(dv,du). \end{aligned} \quad (10)$$

Applying Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned} & E\sup_{0\leq s\leq t}\int_0^s (Q_u, g(u,Z_u)dW(u)) \\ \leq\ & 3E[\sup_{0\leq s\leq t}|Q_s|(\int_0^t ||g(s,Z_s)||_2^2 ds)^{\frac{1}{2}}] \\ \leq\ & \frac{1}{6}E\sup_{0\leq s\leq t}|Q_s|^2 + K_1 E\int_0^t ||g(s,Z_s)||_2^2 ds, \end{aligned} \quad (11)$$

and

$$\begin{aligned} & E\sup_{0\leq s\leq t}\int_0^s\int_U [2(Q_v,h(u,Z_v)) + |h(u,Z_v)|^2]\tilde{N}(dv,du) \\ \leq\ & CE[([M,M]_t^{\frac{1}{2}}), \end{aligned} \quad (12)$$

where $M_t = \int_0^t\int_U [2(Q_s,h(u,Z_s)) + |h(u,Z_s)|^2]\tilde{N}(ds,du)$. By the definition of quadratic variation,

$$\begin{aligned} & [M,M]_t^{\frac{1}{2}} \\ =\ & \{\sum_{s\in D_p, s\leq t}(2(Q_s,h(u,Z_s)) + |h(u,Z_s)|^2)^2\}^{\frac{1}{2}} \\ \leq\ & C(\sum_{s\in D_p, s\leq t}|Q_s|^2|h(u,Z_s)|^2)^{\frac{1}{2}} \\ & +C(\sum_{s\in D_p, s\leq t}|h(u,Z_s)|^4)^{\frac{1}{2}} \\ \leq\ & C\sup_{0\leq t\leq t_1}|Q_s|(\sum_{s\in D_p, s\leq t}|h(u,Z_s)|^2)^{\frac{1}{2}} \\ & +C(\sum_{s\in D_p, s\leq t_1}|h(u,Z_s)|^2) \\ \leq\ & \frac{1}{6}\sup_{0\leq s\leq t}|Q_s|^2 + C(\sum_{s\in D_p, s\leq t}|h(u,Z_s)|^2). \end{aligned} \quad (13)$$

So we have

$$\begin{aligned} & E\sup_{0\leq s\leq t}\int_0^s\int_U [2(Q_v,h(u,Z_v)) + |h(u,Z_v)|^2]\tilde{N}(dv,du) \\ \leq\ & \frac{1}{6}E\sup_{0\leq s\leq t}|Q_s|^2 + CE(\sum_{s\in D_p, s\leq t}|h(u,Z_s)|^2) \\ \leq\ & \frac{1}{6}E\sup_{0\leq s\leq t}|Q_s|^2 + CE\int_0^t\int_U |h(u,Z_s)|^2\pi(du)dt. \end{aligned} \quad (14)$$

Inserting (11) and (14) into (10) gives

$$\begin{aligned} & E(\sup_{0\leq s\leq t}|Q_s|^2) \\ \leq\ & 2E|Q_0|^2 + 2(A^2\bar{\beta}^2+u_0+2)\int_0^t E\sup_{0\leq u\leq s}|Q_u|^2 ds \\ & +2E\int_0^t |f(s,Z_s)|^2 ds + 2(1+2K_1)E\int_0^t ||g(s,Z_s)||_2^2 ds \\ & +2(1+C)E\int_0^t\int_U |h(u,Z_s)|^2\pi(du)ds. \end{aligned} \quad (15)$$

we then compute, by (H_1) and (H_3), that

$$\begin{aligned} & E(\sup_{0\leq s\leq t}|Q_s|^2) \\ \leq\ & 2E|Q_0|^2 + 2(A^2\bar{\beta}^2+u_0+2)\int_0^t E\sup_{0\leq u\leq s}|Q_u|^2 ds \\ & +2(3+2K_1+C)KE\int_0^t ||Z_s||_c^2 ds \\ \leq\ & 2E|Q_0|^2 + [2(A^2\bar{\beta}^2+u_0+2) \\ & +2(3+2K_1+C)K]\int_0^t E\sup_{0\leq u\leq s}|Q_u|^2 ds. \end{aligned} \quad (16)$$

The well-known Gronwall inequality implies

$$E(\sup_{0\leq s\leq t}|Q_s|^2) \leq 2E|Q_0|^2 e^{[2(A^2\bar{\beta}^2+u_0+2)+2(3+2K_1+C)K]T}.$$

Lemma3.2 Under conditions (H_1)-(H_3) and $E|\frac{\partial Q_s}{\partial a}|^2 ds < \gamma E|Q_s|^2$, for each $t\in[0,T]$,

$$\sup_{0\leq t\leq T}E|Q_t - Z_t|^2 \leq C_2 h(\sup_{0\leq t\leq T}E|Q_t|^2), \quad (17)$$

where $C_2 = 5(\gamma+u_0^2)T + 5(T+2)K$.
Proof: For any $t\in[0,T]$, choose a n such that $t\in[nh,(n+1)h)$. Then

$$\begin{aligned} & Q_t - Z_t \\ =\ & -\int_{nh}^t \frac{\partial Q_s}{\partial a}ds - \int_{nh}^t u(s,a)Z_s ds + \int_{nh}^t f(s,Z_s)ds \\ & +\int_{nh}^t g(s,Z_s)dW_s + \int_{nh}^t\int_U h(u,Z_s)\tilde{N}(ds,du). \end{aligned}$$

Using the basic inequality and (H_2), we have

$$\begin{aligned} & |Q_t - Z_t|^2 \\ \leq\ & 5h\int_{nh}^t |\frac{\partial Q_s}{\partial a}|^2 ds + 5hu_0^2\int_{nh}^t |Z_s|^2 ds \\ & +5h\int_{nh}^t |f(s,Z_s)|^2 ds + 5|\int_{nh}^t g(s,Z_s)dW_s|^2 \\ & +5|\int_{nh}^t\int_U h(u,Z_s)\tilde{N}(ds,du)|^2. \end{aligned}$$

By (H_3) and martingale isometries, it follows

$$\begin{aligned}
&\sup_{0\le t\le T} E|Q_t - Z_t|^2 \\
\le\ & 5hE\int_0^T |\frac{\partial Q_s}{\partial a}|^2 ds + 5hu_0^2 E\int_0^T |Z_s|^2 ds \\
&+5hE\int_0^T |f(s,Z_s)|^2 ds \\
&+5\sup_{0\le t\le T} E\max_{n=0,1,2,\cdot,N-1} |\int_{nh}^t g(s,Z_s)dW_s|^2 \\
&+5\sup_{0\le t\le T} E\max_{n=0,1,2,\cdot,N-1} |\int_{nh}^t\int_U h(u,Z_s)\tilde{N}(ds,du)|^2 \\
\le\ & 5h\gamma\int_0^T E|Q_s|^2 ds + 5hu_0^2 T\sup_{0\le s\le T} E|Q_s|^2 \\
&+5hE\int_0^T |f(s,Z_s)|^2 ds \\
&+5\max_{n=0,1,2,\cdot,N-1} E\int_{nh}^{(n+1)h} |g(s,Z_s)|^2 ds \\
&+5\max_{n=0,1,2,\cdot,N-1} E\int_{nh}^{(n+1)h}\int_U |h(u,Z_s)|^2\pi(du)ds \\
\le\ & 5(\gamma+u_0^2)hT\sup_{0\le s\le T} E|Q_s|^2 + 5hKE\int_0^T ||Z_s||_C^2 ds \\
&+10K\max_{n=0,1,2,\cdot,N-1} E\int_{nh}^{(n+1)h} ||Z_s||_C^2 ds \\
\le\ & 5(\gamma+u_0^2)hT\sup_{0\le s\le T} E|Q_s|^2 \\
&+5(T+2)Kh\sup_{0\le s\le T} E|Q_s|^2 \\
\le\ & C_2 h\sup_{0\le s\le T} E|Q_s|^2.
\end{aligned}$$

The proof is completed.

Lemma 3.3 Under conditions (H_1)-(H_3), then the Euler approximate solution (3) will converge to the exact solution of Eq.(1), i.e.,

$$\sup_{0\le t\le T} E|P_t - Q_t|^2 \le C_3 h\sup_{0\le t\le T} E|Q_t|^2, \quad \forall\ \ T>0,$$

where $P_t = P(t,a)$ and $C_3 = [4(3+C+2K_2)C_2KT + 4u_0C_2T]e^{2(A^2\bar{\beta}^2+3u_0+1)+4(3+C+2K_2)K}$.

Proof: Combing (1) with (3) has

$$\begin{aligned}
&P_t - Q_t \\
=\ & -\int_0^t \frac{\partial(P_s-Q_s)}{\partial a}ds - \int_0^t u(s,a)(P_s-Q_s)ds \\
&+\int_0^t [f(s,P_s)-f(s,Z_s)]ds \\
&+\int_0^t [g(s,P_s)-g(s,Z_s)]dW_s \\
&+\int_0^t\int_U [h(u,P_s)-h(u,Z_s)]\tilde{N}(ds,du). \qquad (18)
\end{aligned}$$

Then applying Ito formula to $|P_t - Q_t|^2$, we have

$$\begin{aligned}
&|P_t-Q_t|^2 \\
=\ & -2\int_0^t <\frac{\partial(P_s-Q_s)}{\partial a}, P_s-Q_s> ds \\
&-2\int_0^t (u(s,a)(P_s-Z_s), P_s-Q_s)ds \\
&+2\int_0^t (P_s-Q_s, f(s,P_s)-f(s,Z_s))ds \\
&+\int_0^t ||g(s,P_s)-g(s,Z_s)||_2^2 ds \\
&+2\int_0^s (P_u-Q_u,(g(u,P_u)-g(u,Z_u))dW_u) \\
&+\int_0^t\int_U |h(u,P_s)-h(u,Z_s)|^2\pi(du)ds \\
&+\int_0^t\int_U [2(P_s-Q_s, h(s,P_s)-h(s,Z_s)) \\
&+|h(s,P_s)-h(s,Z_s)|^2]\tilde{N}(ds,du). \qquad (19)
\end{aligned}$$

Hence, for any $t\in[0,T]$,

$$\begin{aligned}
&E\sup_{0\le s\le t} |P_s-Q_s|^2 \\
\le\ & E\int_0^t A^2\bar{\beta}^2|P_s-Q_s|^2 ds \\
&+u_0E\int_0^t (3|P_s-Q_s|^2+2|Q_s-Z_s|^2)ds \\
&+E\int_0^t |P_s-Q_s|^2 ds + E\int_0^t |f(s,P_s)-f(s,Z_s)|^2 ds \\
&+E\int_0^t ||g(s,P_s)-g(s,Z_s)||_2^2 ds \\
&+2E\sup_{0\le s\le t}\int_0^s (P_u-Q_u,(g(u,P_u)-g(u,Z_u))dW_u) \\
&+E\int_0^t\int_U |h(u,P_s)-h(u,Z_s)|^2\pi(du)ds \\
&+E\sup_{0\le s\le t}\int_0^s\int_U [2(P_v-Q_v, h(v,P_v)-h(v,Z_v)) \\
&+|h(v,P_v)-h(v,Z_v)|^2]\tilde{N}(dv,du). \qquad (20)
\end{aligned}$$

By (H3) and Lemma 3.2, it follows that

$$\begin{aligned}
&E\int_0^t |f(s,P_s)-f(s,Z_s)|^2 ds \\
\le\ & K\int_0^t E||P_s-Z_s||_C^2 ds \\
\le\ & 2K\int_0^t E||P_s-Q_s||_C^2 ds + 2K\int_0^t E||Q_s-Z_s||_C^2 ds \\
\le\ & 2K\int_0^t E||P_s-Q_s||_C^2 ds + 2KTC_2h(\sup_{0\le t\le T} E|Q_t|^2). \qquad (21)
\end{aligned}$$

Similarly, we have

$$E\int_0^t ||g(s,P_s)-g(s,Z_s)||_2^2 ds$$

$$\leq 2K\int_0^t E||P_s - Q_s||_C^2 ds + 2KTC_2 h(\sup_{0\leq t\leq T} E|Q_t|^2). \quad (22)$$

and

$$E\int_0^t \int_U |h(u,P_s) - h(u,Z_s)|^2 \pi(du)ds$$
$$\leq 2K\int_0^t E||P_s - Q_s||_C^2 ds + 2KTC_2 h(\sup_{0\leq t\leq T} E|Q_t|^2). \quad (23)$$

Applying the Burkholder-Davis-Gundy inequality, (H3) and lemma 3.2, we have

$$\begin{aligned}
& E\sup_{0\leq s\leq t}\int_0^s (P_u - Q_u, (g(u,P_u) - g(u,Z_u))dW_u) \\
\leq\ & CE[\sup_{0\leq s\leq t}|P_s - Q_s|(\int_0^t ||g(s,P_s) - g(s,Z_s)||_2^2 ds)^{\frac{1}{2}}] \\
\leq\ & \frac{1}{8}E\sup_{0\leq s\leq t}|P_s - Q_s|^2 \\
& +K_2 E\int_0^t ||g(s,P_s) - g(s,Z_s)||_2^2 ds \\
\leq\ & \frac{1}{8}E\sup_{0\leq s\leq t}|P_s - Q_s|^2 + 2K_2K\int_0^t E||P_s - Q_s||_C^2 ds \\
& +2K_2KTC_2 h(\sup_{0\leq t\leq T} E|Q_t|^2). \qquad (24)
\end{aligned}$$

and

$$\begin{aligned}
& E\sup_{0\leq s\leq t}\int_0^s\int_U [2(P_v - Q_v, h(v,P_v) - h(v,Z_v)) \\
& +|h(v,P_v) - h(v,Z_v)|^2]\tilde{N}(dv,du) \\
\leq\ & \frac{1}{4}E\sup_{0\leq s\leq t}|P_s - Q_s|^2 \\
& +CE(\sum_{s\in D_p, s\leq t}|h(u,P_s) - h(u,Z_s)|^2) \\
\leq\ & \frac{1}{4}E\sup_{0\leq s\leq t}|P_s - Q_s|^2 \\
& +CE\int_0^t\int_U |h(u,P_s) - h(u,Z_s)|^2\pi(du)dt \\
\leq\ & \frac{1}{4}E\sup_{0\leq s\leq t}|P_s - Q_s|^2 + 2CK\int_0^t E||P_s - Q_s||_C^2 ds \\
& +2CKTC_2 h(\sup_{0\leq t\leq T} E|Q_t|^2). \qquad (25)
\end{aligned}$$

Substituting (21)-(25) into (20), we obtain that

$$\begin{aligned}
& E[\sup_{0\leq s\leq t}|P_s - Q_s|^2] \\
\leq\ & [2(A^2\bar{\beta}^2 + 3u_0 + 1) + 4(3 + C + 2K_2)K] \\
& \int_0^t E\sup_{0\leq u\leq s}|P_u - Q_u|^2 ds + [4(3 + C + 2K_2)C_2KT \\
& +4u_0C_2T]h(\sup_{0\leq t\leq T} E|Q_t|^2). \qquad (26)
\end{aligned}$$

By the Gronwall inequality, we have

$$E[\sup_{0\leq s\leq t}|P_s - Q_s|^2] \leq M_1 h(\sup_{0\leq t\leq T} E|Q_t|^2)e^{M_2T}. \quad (27)$$

where $M_1 = 4(3 + C + 2K_2)C_2KT + 4u_0C_2T$, $M_2 = 2(A^2\bar{\beta}^2 + 3u_0 + 1) + 4(3 + C + 2K_2)K$.

Lemma 3.4 Under condition (H4), the exact solutions of Eq.(1) is exponentially stable in mean square. That is, there exist two positive constants C, τ such that

$$E|P_t|^2 \leq C(E|P_0|^2 + \int_0^t \gamma(s)e^{-\delta s})e^{-\tau t} \quad (28)$$

Proof: we can choose $\delta > 0$ small enough such that $\xi - \delta > 0$. Then Ito formula implies

$$\begin{aligned}
& e^{(\xi-\delta)t}|P_t|^2 - |P_0|^2 \\
=\ & (\xi-\delta)\int_0^t e^{(\xi-\delta)s}|P_s|^2 ds \\
& +2\int_0^t e^{(\xi-\delta)s} < -\frac{\partial P_s}{\partial a} - u(s,a)P_s, P_s > ds \\
& +2\int_0^t e^{(\xi-\delta)s}(f(s,P_s),P_s)ds + \int_0^t e^{(\xi-\delta)s}||g(s,P_s)||_2^2 ds \\
& +2\int_0^t e^{(\xi-\delta)s}(P_s, g(s,P_s)dW_s) \\
& +\int_0^t\int_U e^{(\xi-\delta)s}|h(u,Z_s)|^2\pi(du)ds \\
& +\int_0^t\int_U e^{(\xi-\delta)s}[2(Q_s,h(u,Z_s)) + |h(u,Z_s)|^2]\tilde{N}(ds,du). \qquad (29)
\end{aligned}$$

Since $\int_0^t e^{(\xi-\delta)s}(P_s, g(s,P_s)dW_s)$ and $\int_0^t\int_U e^{(\xi-\delta)s}[2(Q_s,h(u,Z_s)) + |h(u,Z_s)|^2]\tilde{N}(ds,du)$ are martingale, they follow that $E\int_0^t e^{(\xi-\delta)s}(P_s, g(s,P_s)dW_s) = 0$ and

$$E\int_0^t\int_U e^{(\xi-\delta)s}[2(Q_s,h(u,Z_s)) + |h(u,Z_s)|^2]\tilde{N}(ds,du) = 0.$$

Therefore

$$\begin{aligned}
& e^{(\xi-\delta)t}E|P_t|^2 \\
\leq\ & E|P_0|^2 + (\xi-\delta)\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds \\
& -2E\int_0^t e^{(\xi-\delta)s} < -\frac{\partial P_s}{\partial a}, P_s > ds \\
& -2u_0E\int_0^t e^{(\xi-\delta)s}(P_s,P_s)ds \\
& +2E\int_0^t e^{(\xi-\delta)s}(f(s,P_s),P_s)ds \\
& +E\int_0^t e^{(\xi-\delta)s}||g(s,P_s)||_2^2 ds \\
& +E\int_0^t\int_U e^{(\xi-\delta)s}|h(u,Z_s)|^2\pi(du)ds \\
\leq\ & E|P_0|^2 + (\xi-\delta)\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds \\
& +E\int_0^t e^{(\xi-\delta)s}(\int_0^A \beta(s,a)P_s da)^2 ds \\
& -2u_0\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds \\
& +2E\int_0^t e^{(\xi-\delta)s}(f(s,P_s),P_s)ds
\end{aligned}$$

$$
\begin{aligned}
&+E\int_0^t e^{(\xi-\delta)s}||g(s,P_s)||_2^2 ds\\
&+E\int_0^t\int_U e^{(\xi-\delta)s}|h(u,Z_s)|^2\pi(du)ds\\
\leq\ & E|P_0|^2+(\xi-\delta)\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds\\
&+\int_0^t e^{(\xi-\delta)s}(\int_0^A \beta^2(s,a)P_s da E\int_0^A P_s^2 da)ds\\
&-2u_0\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds\\
&+2E\int_0^t e^{(\xi-\delta)s}(f(s,P_s),P_s)ds\\
&+E\int_0^t e^{(\xi-\delta)s}||g(s,P_s)||_2^2 ds\\
&+E\int_0^t\int_U e^{(\xi-\delta)s}|h(u,Z_s)|^2\pi(du)ds. \qquad (30)
\end{aligned}
$$

By condition (H4), we have

$$
\begin{aligned}
& e^{(\xi-\delta)t}E|P_t|^2\\
\leq\ & E|P_0|^2+(\xi-\delta+\lambda)\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds\\
&-\alpha E\int_0^t e^{(\xi-\delta)s}E||P_s||^2 ds\\
&+(A\bar\beta^2-2u_0)\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds\\
&+\int_0^t \gamma(s)e^{-\delta s}ds\\
\leq\ & E|P_0|^2+(\xi-\delta-v)\int_0^t e^{(\xi-\delta)s}E|P_s|^2 ds\\
&+\int_0^t \gamma(s)e^{-\delta s}ds. \qquad (31)
\end{aligned}
$$

where $v=\alpha/m^2-\lambda-A\bar\beta^2+2u_0$.
Following the proof of [2], we have that the exact solutions of Eq.(1) is exponentially stable in mean square. The proof is completed.

Now we can state our main result of this paper.
Theorem 3.1 Under conditions (H1)-(H4) and $\alpha/m^2-\lambda-A\bar\beta^2+2u_0>0$ hold, then the Euler method applied to Eq.(1) is exponentially stable in mean square.
Proof: The proof is basically similar to those of Theorem 2.2 in Pang [4], we thus omit it here.
Remark 3.1 When $h=0$, the Eq.(1) becomes the usual SADPEs which was studied by Zhang[2] and Pang[4]. Hence, Theorem 3.1 in this paper is a generalization of Theorem 2.2 of [4].

IV. An Example

Consider a stochastic age-dependent population equations with jumps of the form

$$
\begin{cases}
d_tP=[-\frac{\partial P}{\partial a}-\frac{1}{1-a}P-tP]dt\\
\quad+\int_{|u|\leq 1}uP\tilde N(dt,du), \quad in\ Q=(0,T)\times(0,1),\\
P(0,a)=1-a, \qquad in\ \ [0,1],\\
P(t,a)=\int_0^1\frac{1}{1-a}P(t,a)da, \qquad in\ \ [0,T],
\end{cases} \qquad (32)
$$

Where $\tilde N(dt,du)$ is a compensated Poisson random measure on $[0,\infty)\times[-1,1]$. Let $H=L^2([0,1])$, and $V=W_0^1([0,1])$ (a Sobolev space with elements satisfying the boundary conditions above), $M=R$, $u(t,a)=\beta(t,a)=\frac{1}{1-a}$, $f(t,P)=-tP$, $h(u,P)=uP$, $P(0,a)=1-a$.

Clearly, $u(t,a)$ and $\beta(t,a)$ satisfy condition (H_2), the coefficients f and h satisfy conditions (H1) and (H3). on the other hand, it is easy to deduce for arbitrary $x\in V$ that

$$
2<f(t,x),x>+\int_U |h(u,x)|^2\pi(du)\leq -\varepsilon||x|^2+\lambda|x|^2,
$$

where $\varepsilon=2t>0$ is small enough and $\lambda=\int_{-1}^1 u^2\pi(du)$. Therefore, it follows that condition (H4) is satisfied. Consequently, the approximate solution to Eq.(32) will be exponentially stable for any $(t,a)\in(0,T)\times(0,1)$ in the sense of Theorem 3.1, provided $E|\frac{\partial Q_s}{\partial a}|^2ds<\gamma E|Q_s|^2$.

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