

# Existence and globally exponential stability of equilibrium for BAM neural networks with mixed delays and impulses

Xiaomei Wang, Shouming Zhong

**Abstract**—In this paper, a class of generalized bi-directional associative memory (BAM) neural networks with mixed delays is investigated. On the basis of Lyapunov stability theory and contraction mapping theorem, some new sufficient conditions are established for the existence and uniqueness and globally exponential stability of equilibrium, which generalize and improve the previously known results. One example is given to show the feasibility and effectiveness of our results.

**Keywords**—Bi-directional associative memory (BAM) neural networks, mixed delays, Lyapunov stability theory, contraction mapping theorem, existence, equilibrium, globally exponential stability.

## I. INTRODUCTION

**R**ECENTLY, a class of two-layer heteroassociative networks called bi-directional associative memory (BAM) networks [1-4] with or without axonal signal transmission delays has been proposed and used in many fields, such as pattern recognition and automatic control. They were first proposed by Kosko [1-3], Cao [5-8,16-17,19-20], Liao and Yu [10], Gopalsamy and He [11], Liang [12], Wei and Ruan [13], Jin [14] and Xia and Cao [15]. Though the non-impulsive systems have been well studied in theory and in practice (for example see [1-20] and references cited therein), the theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulse, but also represents a more natural framework for mathematical modeling of many real-world phenomena, such as population dynamic and the neural networks. In recent years, the impulsive differential equations have been extensively studied (see the monographs and the works [21-25]). This class of neural networks has been showed to be a useful network model for applications in pattern recognition, solving optimization problems and automatic control engineering. Hence, they have been the object of intensive analysis by numerous authors in recent years [16-17,19-20,24-25]. In particular, there are extensive results on the problem of the stability and other dynamical behaviors of impulsive. In the present paper, we investigate the following more general BAM neural networks with impulses

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$$\left\{ \begin{array}{l} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^p c_{ji} f_j(y_j(t - \sigma_{ji})) \\ \quad + \sum_{j=1}^p \int_0^{\tau} p_{ji} f_j(y_j(t - s)) ds + r_i, \quad t \geq 0, t \neq t_k, \\ \Delta x_i(t) = (\alpha_{ik} - 1)x_i(t), \\ \quad i = 1, 2, \dots, n, k = 1, 2, \dots, t = t_k, \\ y'_j(t) = -b_j y_j(t) + \sum_{i=1}^n d_{ij} g_i(x_i(t - \delta_{ij})) \\ \quad + \sum_{i=1}^n \int_0^{\tau} q_{ij} g_i(x_i(t - s)) ds + s_j, \quad t \geq 0, t \neq t_k, \\ \Delta y_j(t) = (\beta_{jk} - 1)y_j(t), \\ \quad j = 1, 2, \dots, p, k = 1, 2, \dots, t = t_k. \end{array} \right. \quad (1)$$

where  $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$  and  $\Delta y_j(t_k) = y_j(t_k^+) - y_j(t_k^-)$  are the impulses at moments  $t_k$  then there are  $x_i(t_k^-) = x_i(t_k)$  and  $y_j(t_k^-) = y_j(t_k)$   $t_1 < t_2 < \dots$  is a strictly increasing sequence such that  $\lim_{k \rightarrow \infty} t_k = +\infty$ .  $n$  and  $p$  correspond to the number of neurons in X-layer and Y-layer.  $x_i$  and  $y_j$  are the activations of the  $i$ th neurons and the  $j$ th neurons, respectively.  $c_{ji}, d_{ij}, p_{ji}, q_{ij}$  are the connection weight.  $\sigma_{ji} > 0, \delta_{ij} > 0$  are the transmission delay,  $\tau$  is distributed time-varying delay  $r_i$  and  $s_j$  denote the external inputs  $a_i > 0$  and  $b_j > 0$  represent the rate with which the  $i$ th neuron and  $j$ th neuron will rest its potential to the resting state in isolation when disconnected from the network and external inputs, respectively. Then system (1) is supplemented with initial values given by  $x_i(t) = \varphi_i(t), t \in [-\sigma, 0], \sigma = \max\{\max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}} \sigma_{ji}, \tau\}$ ,  $i = 1, 2, \dots, n, y_j(t) = \psi_j(t), t \in [-\delta, 0], \delta = \max\{\max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq p}} \delta_{ij}, \tau\}, j = 1, 2, \dots, p$  where  $\varphi_i(t)$  and  $\psi_i(t)$  denote real-valued continuous functions defined on  $[-\sigma, 0]$  and  $[-\delta, 0]$ .  $z(t) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_p(t))^T \in R^n$  in which  $x_i(\cdot), \dots, y_j(\cdot)$  are piecewise continuous on  $(0, \beta)$  for some  $\beta > 0$  such that  $z(t_k^+)$  and  $z(t_k^-)$  exist and  $z(\cdot)$  is differentiable on intervals of the form  $(t_{k-1}, t_k) \subset (0, \beta)$  and satisfies(1); We assume that  $z(t)$  is left continuous with  $z(t_k^-) = z(t_k)$ , and  $\alpha_{ik}, \beta_{jk}, r_i, s_j$  are real numbers. we assume that:

(H1)  $a_i, b_j \in (0, \infty), c_{ji}, d_{ij}, p_{ji}, q_{ij}, r_i, s_j \in R, \sigma_{ji}, \delta_{ij} \in (0, \infty), i = 1, \dots, n, j = 1, \dots, p$ .

(H2)  $f_j$  and  $g_i$  are Lipschitz-continuous on  $R$  with Lipschitz constant  $L_j^f, (j = 1, \dots, p)$  and  $L_i^g (i = 1, \dots, n)$ , that is,

$$|f_j(x) - f_j(y)| \leq L_j^f |x - y|, \forall x, y \in R.$$

$$|g_i(x) - g_i(y)| \leq L_i^g |x - y|, \forall x, y \in R.$$

The aim of this paper is to derive some criteria for the existence and globally exponential stability of a unique equilibrium of system (1). The rest of this paper is organized as follows. In Section II, we shall establish some sufficient conditions for the existence and uniqueness of equilibrium. In Section III, we will investigate the globally exponential stability of the unique equilibrium. Finally, one example is presented to illustrate that our results are feasible and more general.

## II. EXISTENCE AND UNIQUENESS OF EQUILIBRIUM

When neural networks are used for the solution of optimization problems, one of the fundamental issues in the design of a network is concerned with the existence and uniqueness and globally exponentially stable equilibrium state of network without requiring the boundedness, differentiability or monotonicity, we establish a easily verifiable sufficient conditions for the existence of a unique equilibrium state in this section. An equilibrium solution of (1) is a constant vector

$$z^* = (x_1^*, \dots, x_n^*, y_1^*, \dots, y_p^*)^T \in R^{n+p},$$

which satisfies the system

$$\begin{cases} a_i x_i^* = \sum_{j=1}^p c_{ji} f_j(y_j^*) + \tau \sum_{j=1}^p p_{ji} f_j(y_j^*) + r_i, i = 1, \dots, n \\ b_j y_j^* = \sum_{i=1}^n d_{ij} g_i(x_i^*) + \tau \sum_{i=1}^n q_{ij} g_i(x_i^*) + s_j, j = 1, \dots, p. \end{cases} \quad (2)$$

Or

$$\begin{cases} x_i^* = \sum_{j=1}^p \frac{1}{a_i} (c_{ji} + \tau p_{ji}) f_j(y_j^*) + \frac{r_i}{a_i}, i = 1, 2, \dots, n \\ y_j^* = \sum_{i=1}^n \frac{1}{b_j} (d_{ij} + \tau q_{ij}) g_i(x_i^*) + \frac{s_j}{b_j}, j = 1, 2, \dots, p. \end{cases} \quad (3)$$

when the impulsive jumps as assumed to satisfy

$$\alpha_{ik}(x_i^*) = \beta_{jk}(y_j^*) = 0.$$

We denote the spectral radius of the matrix  $F$  by  $\rho(F)$ .

**Lemma 1.** ([15]). Let  $N$  be a positive integer and  $B$  be an Banach space. If the mapping  $\phi^N : B \rightarrow B$  is a contraction mapping, then  $\phi : B \rightarrow B$  has unique fixed point in  $B$ , where  $\phi^N = \phi(\phi^{N-1})$ .

**Theorem 1.** In addition to (H1)-(H2), assume further that  $\rho(K) < 1$ , where  $K = (K_{ij})_{(n+p) \times (n+p)}$ ,  $M_{ji} = \frac{1}{a_i} (|c_{ji}| + |\tau p_{ji}|)$ , and  $N_{ij} = \frac{1}{b_j} (|d_{ij}| + |\tau q_{ij}|)$ ,

$$k_{ij} = \begin{cases} M_{j-n,i} L_{j-n}^f, & 1 \leq i \leq n, n+1 \leq j \leq n+p, \\ N_{j,i-p} L_i^g, & p+1 \leq i \leq n+p, 1 \leq j \leq p, \\ 0, & 1 \leq i, j \leq n, n+1 \leq i, j \leq n+p. \end{cases}$$

Then, there exists unique equilibrium of system (1).

*Proof:* In order to show that (3) has a unique solution. Now consider a mapping:  $\phi : \phi(z) = (\phi_1(z), \dots, \phi_n(z), \phi_{n+1}(z), \dots, \phi_{n+p}(z))^T, R^{n+p} \rightarrow R^{n+p}$

$$\phi(z) = \phi(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_p)^T,$$

$$z = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_p)^T$$

$$\bar{z} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n, \bar{y}_1, \bar{y}_2, \dots, \bar{y}_p)^T. \quad (4)$$

We have

$$|\phi(z) - \phi(\bar{z})| = (|(\phi(z) - \phi(\bar{z}))_1|, \dots, |(\phi(z) - \phi(\bar{z}))_n|, |(\phi(z) - \phi(\bar{z}))_{n+1}|, \dots, |(\phi(z) - \phi(\bar{z}))_{n+p}|)^T$$

$$\leq \left[ \sum_{j=1}^p (M_{j1} |f_j(y_j) - f_j(\bar{y}_j)|), \right.$$

$$\dots, \sum_{j=1}^p (M_{jn} |f_j(y_j) - f_j(\bar{y}_j)|),$$

$$\dots, \sum_{i=1}^n (N_{i1} |g_i(x_i) - g_i(\bar{x}_i)|),$$

$$\dots, \sum_{i=1}^n (N_{ip} |g_i(x_i) - g_i(\bar{x}_i)|) \Big]^T$$

$$\leq \left[ \sum_{j=1}^p (M_{j1} L_j^f |y_j - \bar{y}_j|), \right.$$

$$\dots, \sum_{j=1}^p (M_{jn} L_j^f |y_j - \bar{y}_j|) \dots, \sum_{i=1}^n (N_{i1} L_i^g |x_i - \bar{x}_i|),$$

$$\dots, \sum_{i=1}^n (N_{ip} L_i^g |x_i - \bar{x}_i|) \Big]^T$$

$$= \begin{pmatrix} 0, & \dots, & 0, & M_{11} L_1^f, & \dots, & M_{1p} L_p^f \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & \dots, & 0, & M_{n1} L_1^f & \dots & M_{np} L_p^f \\ N_{11} L_1^g & \dots & N_{1n} L_n^g & 0, & \dots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ N_{p1} L_1^g & \dots & N_{pn} L_n^g & 0, & \dots, & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} |x_1 - \bar{x}_1| \\ \vdots \\ |x_n - \bar{x}_n| \\ |y_1 - \bar{y}_1| \\ \vdots \\ |y_p - \bar{y}_p| \end{pmatrix}$$

$$= K (|x_1 - \bar{x}_1|, \dots, |x_n - \bar{x}_n|, \dots, |y_1 - \bar{y}_1|, \dots, |y_p - \bar{y}_p|)^T$$

$$= K (|(z - \bar{z})_1|, \dots, |(z - \bar{z})_n|,$$

$$\dots, |(z - \bar{z})_{n+1}|, \dots, |(z - \bar{z})_{n+p}|)^T. \quad (5)$$

Let  $m$  be a positive integer. Then from (5), we get

$$(|(\phi^m(z) - \phi^m(\bar{z}))_1|, \dots, |(\phi^m(z) - \phi^m(\bar{z}))_n|, |(\phi^m(z) - \phi^m(\bar{z}))_{n+1}|, \dots, |(\phi^m(z) - \phi^m(\bar{z}))_{n+p}|)^T$$

$$\leq K (|\phi^{m-1}(z) - \phi^{m-1}(\bar{z})|_1,$$

$$\dots, |\phi^{m-1}(z) - \phi^{m-1}(\bar{z})|_n, |(\phi^{m-1}(z) - \phi^{m-1}(\bar{z}))_{n+1}|,$$

$$\dots, |(\phi^{m-1}(z) - \phi^{m-1}(\bar{z}))_{n+p}|)^T$$

$$\leq K^m (|(z - \bar{z})_1|, \dots, |(z - \bar{z})_n|,$$

$$\dots, |(z - \bar{z})_{n+1}|, \dots, |(z - \bar{z})_{n+p}|)^T. \quad (6)$$

From the assumption  $\rho(K) < 1$ , we obtain

$$\lim_{m \rightarrow +\infty} K^m = 0,$$

which implies that there exists a positive integer  $N$  and positive constant  $\eta < 1$  such that

$$K^N = (h_{kl})_{(n+p) \times (n+p)} \text{ and } \sum_{l=1}^{n+p} h_{kl} \leq \eta, k = 1, \dots, n+p. (7)$$

In view of (6) and (7), we have

$$\left| (\phi^N(z) - \phi^N(\bar{z}))_k \right| \leq \sum_{l=1}^{n+p} h_{kl} \max_{1 \leq i \leq n+p} |z_i - \bar{z}_i| \leq \eta \|z - \bar{z}\| (8)$$

for all  $k = 1, 2, \dots, n+p$ . Therefore, it follows from (8) that

$$\left| (\phi^N(z) - \phi^N(\bar{z})) \right| \leq \eta \|z - \bar{z}\|_{B^*}. (9)$$

This implies that the mapping

$$\phi^N : R^{n+p} \rightarrow R^{n+p}$$

is a contraction mapping. By Lemma 1,  $\phi$  has unique fixed point  $z^* \in R^{n+p}$  such that  $\phi(z^*) = z^*$ . Thus, system (1) has unique equilibrium. This completes the proof of Theorem 1.

### III. EXPONENTIAL STABILITY OF EQUILIBRIUM

In this section, we will show that the conditions in Theorem 2 also guarantee globally exponentially stability the unique equilibrium of the impulsive system (1). For convenience, we introduce some definitions.

*Definition 1.* Let functions

$$\varphi(s) = (\varphi_1(s), \dots, \varphi_n(s)), s \in [-\sigma, 0], R^{n+p} \rightarrow R^{n+p}$$

and

$$\psi(s) = (\psi_1(s), \dots, \psi_n(s)), s \in [-\delta, 0], R^{n+p} \rightarrow R^{n+p}$$

The function  $\phi(s) = (\varphi(s), \psi(s))$  is said to be a continuous function if the following two conditions are satisfied:

(a)  $\phi$  is piecewise continuous with first kind discontinuity at the points  $\{t_k\}$ , and  $\phi$  is left-continuous at each discontinuity points.

(b)

$$\varphi_i(t^+) = \alpha_{ik} \varphi_i(t), t \in t_k \cap [-\sigma, 0],$$

$$\psi_j(t^+) = \beta_{jk} \psi_j(t), t \in t_k \cap [-\delta, 0],$$

$$i = 1, 2, \dots, n, k = 1, 2, \dots$$

We define the norms  $\|(z(t))\|$  as

$$\|(z(t))\| = \sum_{i=1}^n |x_i(t)| + \sum_{j=1}^p |y_j(t)|,$$

and the norm  $\|\phi\|$  by

$$\|\phi\| = \sup \left\{ \sum_{i=1}^n |\varphi_i(s)| + \sum_{j=1}^p |\psi_j(s)| \right\}.$$

*Definition 2.* The unique equilibrium  $z^* = (x_1^*, \dots, x_n^*, y_1^*, \dots, y_p^*)^T$  of (1) is said to be globally exponentially stable,

if there exist constant  $\varepsilon > 0$  and  $M(\varepsilon) > 0$  such that

$$\|z - z^*\| \leq M(\varepsilon) e^{-\varepsilon t} \|\phi - z^*\|, \forall t \geq 0.$$

*Definition 3.* The Dini right upper derivative of a continuous function, which is defined by

$$D^+ f(t) = \lim_{h \rightarrow 0^+} \sup_{0 \leq \Delta t \leq h} \left\{ \frac{f(t + \Delta t) - f(t)}{\Delta t} \right\},$$

and

$$D^+ |f(t)| = \begin{cases} f'(t), & \text{if } f(t) > 0 \text{ and } f'(t) > 0 \\ -f'(t), & \text{if } f(t) < 0 \text{ and } f'(t) < 0. \\ 0 & \text{if } f(t) = 0 \text{ and } f'(t) = 0 \end{cases}$$

*Theorem 2.* Under assumptions (H1) and (H2), system (1) is globally exponentially stable if the following conditions are satisfied:

(1)  $|\alpha_{ik}| \leq 1, |\beta_{jk}| \leq 1, i = 1, 2, \dots, n, j = 1, 2, \dots, p, k = 1, 2, \dots$ .

(2) There exist positive constants  $\lambda_i, \mu_j$  such that

$$\begin{cases} \sum_{j=1}^p \mu_j L_i^g(d_{ij} + q_{ij}\tau) - \lambda_i a_i < 0, j = 1, 2, \dots, p. \\ \sum_{i=1}^n \lambda_i L_j^f(c_{ji} + p_{ji}\tau) - \mu_j b_j < 0, i = 1, 2, \dots, n. \end{cases}$$

*Proof.* From condition (2) of Theorem 2 for continuation, there is a small positive constant  $\varepsilon$  such that

$$\begin{cases} \sum_{j=1}^p \mu_j L_i^g(d_{ij} e^{\varepsilon \delta_{ij}} + q_{ij} \int_0^\tau e^{\varepsilon s} ds) + \lambda_i (\varepsilon - a_i) < 0 \\ \sum_{i=1}^n \lambda_i L_j^f(c_{ji} e^{\varepsilon \sigma_{ji}} + p_{ji} \int_0^\tau e^{\varepsilon s} ds + \mu_j (\varepsilon - b_j)) < 0 \end{cases} (10)$$

We construct Lyapunov functional as follows:

$$V(t) = V_1(t) + V_2(t)$$

$$V_1(t) = \sum_{i=1}^n \lambda_i \left[ \sum_{j=1}^p L_j^f |c_{ji}| \int_{t-\sigma_{ji}}^t |y_j(r) - y_j^*| e^{\varepsilon(r+\sigma_{ji})} dr \right]$$

$$+ \sum_{i=1}^n \lambda_i [|x_i(t) - x_i^*| e^{\varepsilon t}]$$

$$+ \sum_{j=1}^p L_j^f |p_{ji}| \int_0^\tau \int_{t-s}^t |y_j(r) - y_j^*| e^{\varepsilon(r+s)} dr ds].$$

$$V_2(t) = \sum_{j=1}^p \mu_j [|y_j(t) - y_j^*| e^{\varepsilon t}]$$

$$+ \sum_{i=1}^n L_i^g |q_{ij}| \int_0^\tau \int_{t-s}^t |x_i(r) - x_i^*| e^{\varepsilon(r+s)} dr ds]$$

$$+ \sum_{j=1}^p \mu_j \left( \sum_{i=1}^n L_i^g |d_{ij}| \int_{t-\delta_{ij}}^t |x_i(r) - x_i^*| e^{\varepsilon(r+\delta_{ij})} dr \right).$$

$$D^+ V_1(t) = \sum_{i=1}^n \lambda_i e^{\varepsilon t} \left[ \left( \sum_{j=1}^p L_j^f |c_{ji}| e^{\varepsilon \sigma_{ji}} |y_j(t) - y_j^*| \right) \right.$$

$$\left. - \left( \sum_{j=1}^p L_j^f |c_{ji}| |y_j(t - \sigma_{ji}) - y_j^*| \right) \right]$$

$$+ \sum_{i=1}^n \lambda_i e^{\varepsilon t} \{ D^+ |x_i(t) - x_i^*| + \varepsilon |x_i(t) - x_i^*| \}$$

$$+ \sum_{j=1}^p L_j^f |p_{ji}| |y_j(t) - y_j^*| \int_0^\tau e^{\varepsilon s} ds$$

$$- \sum_{j=1}^p L_j^f |p_{ji}| \int_0^\tau |y_j(t-s) - y_j^*| ds \}.$$

$$\begin{aligned}
 D^+V_2(t) &= \sum_{j=1}^p \mu_j e^{\varepsilon t} [D^+ |y_j(t) - y_j^*| + \varepsilon |y_j(t) - y_j^*| \\
 &+ \sum_{i=1}^n L_i^g |q_{ij}| |x_i(t) - x_i^*| \int_0^\tau e^{\varepsilon s} ds \\
 &- \sum_{i=1}^n L_i^g |q_{ij}| \int_0^\tau |x_i(t-s) - x_i^*| ds] \\
 &+ \sum_{j=1}^p \mu_j e^{\varepsilon t} [(\sum_{j=1}^p L_i^g |d_{ij}| e^{\varepsilon \delta_{ij}} |x_i(t) - x_i^*|) \\
 &- (\sum_{j=1}^p L_i^g |d_{ij}| |x_i(t - \delta_{ij}) - x_i^*|)]. \\
 D^+ |x_i(t) - x_i^*| &\leq -a_i |x_i(t) - x_i^*| \\
 &+ \sum_{j=1}^p |c_{ji}| |f_j(y_j(t - \sigma_{ji}) - f_j(y_j^*))| \\
 &+ \sum_{j=1}^p |p_{ji}| \int_0^\tau |f_j(y_j(t-s) - f_j(y_j^*))| ds \\
 &\leq -a_i |x_i(t) - x_i^*| + \sum_{j=1}^p L_j^f |c_{ji}| |y_j(t - \sigma_{ji}) - y_j^*| \\
 &+ \sum_{j=1}^p L_j^f |p_{ji}| \int_0^\tau |y_j(t-s) - y_j^*| ds. \\
 D^+ |y_j(t) - y_j^*| &\leq -b_j |y_j(t) - y_j^*| \\
 &+ \sum_{i=1}^n |d_{ij}| |g_i(x_i(t - \delta_{ij}) - g_i(x_i^*))| \\
 &+ \sum_{i=1}^n L_i^g \int_0^\tau |q_{ij}| |x_i(t-s) - x_i^*| ds \\
 &\leq -b_j |y_j(t) - y_j^*| + \sum_{i=1}^n L_i^g |d_{ij}| |x_i(t - \delta_{ij}) - x_i^*| \\
 &+ \sum_{i=1}^n L_i^g \int_0^\tau |q_{ij}| |x_i(t-s) - x_i^*| ds.
 \end{aligned} \tag{11}$$

From inequalities (10) and (11), we have

$$\begin{aligned}
 D^+V_1(t) &\leq \sum_{i=1}^n \lambda_i e^{\varepsilon t} [(\varepsilon - a_i) \|x_i(t) - x_i^*\| \\
 &+ \sum_{j=1}^m L_j^f |c_{ji}| e^{\varepsilon \sigma_{ji}} |y_j(t) - y_j^*| \\
 &+ \sum_{j=1}^m L_j^f |p_{ji}| |y_j(t) - y_j^*| \int_0^\tau e^{\varepsilon s} ds]. \\
 D^+V_2(t) &\leq \sum_{j=1}^m \mu_j e^{\varepsilon t} [(\varepsilon - b_j) |y_j(t) - y_j^*| \\
 &+ \sum_{i=1}^n L_i^g |d_{ij}| e^{\varepsilon \delta_{ij}} |x_i(t) - x_i^*| \\
 &+ \sum_{i=1}^n L_i^g |q_{ij}| |x_i(t) - x_i^*| \int_0^\tau e^{\varepsilon s} ds]. \\
 D^+V(t) &\leq e^{\varepsilon t} [\sum_{i=1}^n \lambda_i (\varepsilon - a_i) \\
 &+ \sum_{j=1}^p \mu_j \sum_{i=1}^n L_i^g (d_{ij} e^{\varepsilon \delta_{ij}} + |q_{ij}| \int_0^\tau e^{\varepsilon s} ds)] |x_i(t) - x_i^*| \\
 &+ e^{\varepsilon t} [\sum_{j=1}^p \mu_j (\varepsilon - b_j) + \sum_{i=1}^n \lambda_i \sum_{j=1}^p L_j^f (|c_{ji}| e^{\varepsilon \sigma_{ji}} \\
 &+ |p_{ji}| \int_0^\tau e^{\varepsilon s} ds)] |y_j(t) - y_j^*| \\
 &= e^{\varepsilon t} \sum_{i=1}^n [\lambda_i (\varepsilon - a_i) + \sum_{j=1}^p \mu_j L_i^g (|d_{ij}| e^{\varepsilon \delta_{ij}} \\
 &+ |q_{ij}| \int_0^\tau e^{\varepsilon s} ds)] |x_i(t) - x_i^*| \\
 &+ e^{\varepsilon t} \sum_{j=1}^p [\mu_j (\varepsilon - b_j) + \sum_{i=1}^n \lambda_i L_j^f (|c_{ji}| e^{\varepsilon \sigma_{ji}} \\
 &+ |p_{ji}| \int_0^\tau e^{\varepsilon s} ds)] |y_j(t) - y_j^*| \leq 0.
 \end{aligned}$$

Case

$$\begin{aligned}
 |x_i(t_k^+) - x_i^*| &= |\alpha_{ik}| |x_i(t_k) - x_i^*| \leq |x_i(t_k) - x_i^*|, \\
 |y_j(t_k^+) - y_j^*| &= |\beta_{jk}| |y_j(t_k) - y_j^*| \leq |y_j(t_k) - y_j^*|.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 V(t_k^+) &= \sum_{i=1}^n \lambda_i (\sum_{j=1}^p L_j^f |c_{ji}| \int_{t_k - \sigma_{ji}}^{t_k} |y_j(r) - y_j^*|) \\
 &\times e^{\varepsilon(r + \sigma_{ji})} dr + \sum_{i=1}^n \lambda_i [|x_i(t_k^+) - x_i^*| e^{\varepsilon t_k^+} \\
 &+ \sum_{j=1}^p L_j^f |p_{ji}| \int_0^\tau \int_{t_k - s}^{t_k} |y_j(r) - y_j^*| e^{\varepsilon(r+s)} dr ds] \\
 &+ \sum_{j=1}^p \mu_j [|y_j(t_k^+) - y_j^*| e^{\varepsilon t_k^+} \\
 &+ \sum_{i=1}^n L_i^g |q_{ij}| \int_0^\tau \int_{t_k - s}^{t_k} |x_i(r) - x_i^*| e^{\varepsilon(r+s)} dr ds \\
 &+ \sum_{j=1}^p \mu_j \sum_{i=1}^n L_i^g |d_{ij}| \int_{t_k - \delta_{ij}}^{t_k} |x_i(r) - x_i^*| e^{\varepsilon(r + \delta_{ij})} dr] \leq V(t_k).
 \end{aligned}$$

Combining with the above discussion, for any  $t \geq 0$

$$V(t) \leq V(t_k^+) \leq V(t_k) \leq V(t_{k-1}^+) \leq \dots \leq V(0).$$

On the other hand, from the expression of  $V(t)$ , we have

$$\begin{aligned}
 V(t) &\geq e^{\varepsilon t} \sum_{i=1}^n \lambda_i |x_i(t) - x_i^*| + e^{\varepsilon t} \sum_{j=1}^p \mu_j |y_j(t) - y_j^*| \\
 &\geq e^{\varepsilon t} \gamma \left[ \sum_{i=1}^n |x_i(t) - x_i^*| + \sum_{j=1}^p |y_j(t) - y_j^*| \right] \\
 &= e^{\varepsilon t} \gamma \| (x(t), y(t)) - (x^*, y^*) \| \\
 &= e^{\varepsilon t} \gamma \| z(t) - z^* \|,
 \end{aligned} \tag{12}$$

where  $\gamma = \min \{ \min_{1 \leq i \leq n} \{ \lambda_i \}, \min_{1 \leq j \leq p} \{ \mu_j \} \}$

$$\begin{aligned}
 V(0) &= \sum_{i=1}^n \lambda_i (\sum_{j=1}^p L_j^f |c_{ji}| \int_{-\sigma_{ji}}^0 |y_j(r) - y_j^*| e^{\varepsilon(r + \sigma_{ji})} dr) \\
 &+ \sum_{i=1}^n \lambda_i |x_i(0) - x_i^*| \\
 &+ \sum_{j=1}^p L_j^f |p_{ji}| \int_0^\tau \int_{-s}^0 |y_j(r) - y_j^*| e^{\varepsilon(r+s)} dr ds \\
 &+ \sum_{j=1}^p \mu_j [|y_j(0) - y_j^*| \\
 &+ \sum_{i=1}^n L_i^g |q_{ij}| \int_0^\tau \int_{-s}^0 |x_i(r) - x_i^*| e^{\varepsilon(r+s)} dr ds] \\
 &+ \sum_{j=1}^p \mu_j (\sum_{i=1}^n L_i^g |d_{ij}| \int_{-\delta_{ij}}^0 |x_i(r) - x_i^*| e^{\varepsilon(r + \delta_{ij})} dr) \\
 &= \sum_{i=1}^n \lambda_i (\sum_{j=1}^p L_j^f |c_{ji}| \int_{-\sigma_{ji}}^0 |\psi_j(r) - y_j^*| e^{\varepsilon(r + \sigma_{ji})} dr) \\
 &+ \sum_{i=1}^n \lambda_i [|\varphi_i(0) - x_i^*| \\
 &+ \sum_{j=1}^p L_j^f |p_{ji}| \int_0^\tau \int_{-s}^0 |\psi_j(r) - y_j^*| e^{\varepsilon(r+s)} dr ds] \\
 &+ \sum_{j=1}^p \mu_j [| \psi_j(0) - y_j^* | \\
 &+ \sum_{i=1}^n L_i^g |q_{ij}| \int_0^\tau \int_{-s}^0 |\varphi_i(r) - x_i^*| e^{\varepsilon(r+s)} dr ds] \\
 &+ \sum_{j=1}^p \mu_j (\sum_{i=1}^n L_i^g |d_{ij}| \int_{-\delta_{ij}}^0 |\varphi_i(r) - x_i^*| e^{\varepsilon(r + \delta_{ij})} dr) \\
 &\leq \sum_{i=1}^n [\lambda_i + \sum_{j=1}^p \mu_j L_i^g (q_{ij} \int_0^\tau \int_{-s}^0 e^{\varepsilon(r+s)} dr ds
 \end{aligned}$$

$$\begin{aligned}
 & + d_{ij} \int_{-\delta_{ij}}^0 e^{\varepsilon(r+\delta_{ij})} dr) \times \sup_{-\delta \leq s \leq 0} \sum_{i=1}^n |\varphi_i(s) - x_i^*| \\
 & + \sum_{j=1}^p [\mu_j + \sum_{j=1}^p \lambda_i L_j^f(c_{ji} \int_{-\sigma_{ji}}^0 e^{\varepsilon(r+\sigma_{ji})} dr \\
 & + p_{ji} \int_0^\tau \int_{-s}^0 e^{\varepsilon(r+s)} dr ds)] \times \sup_{-\sigma \leq s \leq 0} \sum_{j=1}^p |\psi_j(s) - y_j^*|. \quad (13)
 \end{aligned}$$

From (12) and (13), we obtain

$$e^{\varepsilon t} \gamma \|z(t) - z^*\| \leq V(t) \leq V(0) \leq M(\varepsilon) \gamma \|\phi - z^*\|,$$

so that

$$\|z(t) - z^*\| \leq M(\varepsilon) e^{-\varepsilon t} \|\phi - z^*\| \quad \text{for all } t \geq 0. \quad (14)$$

where  $M(\varepsilon) = \gamma^{-1} \max\{\sum_{i=1}^n \{\lambda_i + \sum_{j=1}^p \mu_j L_i^g[|q_{ij}| \int_0^\tau \int_{-s}^0 e^{\varepsilon(r+s)} dr ds + |d_{ij}| \int_{-\delta_{ij}}^0 e^{\varepsilon(r+\delta_{ij})} dr]\}, \sum_{j=1}^p \mu_j + \sum_{j=1}^p \lambda_i L_j^f[|c_{ji}| \int_{-\sigma_{ji}}^0 e^{\varepsilon(r+\sigma_{ji})} dr + |p_{ji}| \int_0^\tau \int_{-s}^0 e^{\varepsilon(r+s)} dr ds]\}$ . Thus, from (14) we directly obtain that system (1) is globally exponentially stable, the proof is completed.

*Corollary 1.* Under assumptions (H1) and (H2), system (1) is globally exponentially stable, if the following conditions are satisfied:

$$(1) |\alpha_{ik}| \leq 1, |\beta_{jk}| \leq 1, i = 1, 2, \dots, n, j = 1, 2, \dots, p, k = 1, 2, \dots.$$

$$(2) \begin{cases} \sum_{j=1}^p L_i^g(d_{ij} + q_{ij}\tau) - a_i < 0, j = 1, 2, \dots, p. \\ \sum_{i=1}^n L_j^f(c_{ji} + p_{ji}\tau) - b_j < 0, i = 1, 2, \dots, n. \end{cases}$$

*Remark 1.* Obviously, when there is no impulse in system (1), it reduces to the following model:

$$\begin{cases} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^p c_{ji} f_j(y_j(t - \sigma_{ji})) \\ + \sum_{j=1}^p \int_0^\tau p_{ji} f_j(y_j(t - s)) ds + r_i, t \geq 0, \\ y'_j(t) = -b_j y_j(t) + \sum_{i=1}^n d_{ij} g_i(x_i(t - \delta_{ij})) \\ + \sum_{i=1}^n \int_0^\tau q_{ij} g_j(x_i(t - s)) ds + s_j, t \geq 0, \end{cases} \quad (1^*)$$

by the process of proof of Theorems 1. Theorems 1 can guarantee that the system(1\*) has a unique equilibrium point. It is easy to show that the following corollaries hold.

*Corollary 2.* Under assumptions (H1) and (H2), system (1\*) is globally exponentially stable if there exist positive constants such that

$$\begin{cases} \sum_{j=1}^p \mu_j L_i^g(d_{ij} + q_{ij}\tau) - \lambda_i a_i < 0, j = 1, 2, \dots, p. \\ \sum_{i=1}^n \lambda_i L_j^f(c_{ji} + p_{ji}\tau) - \mu_j b_j < 0, i = 1, 2, \dots, n. \end{cases}$$

#### IV. AN ILLUSTRATIVE EXAMPLE

Consider the following impulsive BAM neural network with mixed delays:

$$\begin{cases} x'_i(t) = -a_i x_i(t) + \sum_{j=1}^3 c_{ji} f_j(y_j(t - \sigma_{ji})) \\ + \sum_{j=1}^3 \int_0^\tau p_{ji} f_j(y_j(t - s)) ds + r_i, t \geq 0, t \neq t_k, \\ \Delta x_i(t) = (\alpha_{ik} - 1)x_i(t), i = 1, \dots, n, k = 1, \dots, t = t_k \\ y'_j(t) = -b_j y_j(t) + \sum_{i=1}^3 d_{ij} g_i(x_i(t - \delta_{ij})) \\ + \sum_{i=1}^3 \int_0^\tau q_{ij} g_j(x_i(t - s)) ds + s_j, t \geq 0, t \neq t_k, \\ \Delta y_j(t) = (\beta_{jk} - 1)y_j(t), j = 1, \dots, p, k = 1, \dots, t = t_k. \end{cases} \quad (15)$$

Where  $(a_1, a_2, a_3) = (1, 1, 1)^T, (b_1, b_2, b_3) = (1, 1, 1)^T, \sigma_{ji}, \delta_{ij} \in (0, \infty), \mu_j = \lambda_i, L_j^f = L_i^g = \tau = 1, i, j = 1, 2, 3. \alpha_{ik} = \frac{1}{2} \sin(1 + k), \beta_{jk} = \frac{2}{3} \cos 2k, f_j(x) = g_i(x) = \frac{1}{2}(|x + 1| - |x - 1|), c_{ji}, d_{ij}, p_{ji}, q_{ij}$  are positive and

$$M_{ji} = \frac{1}{a_i} (|c_{ji}| + |\tau p_{ji}|) = c_{ji} + p_{ji}$$

$$N_{ij} = \frac{1}{b_j} (|d_{ij}| + |\tau q_{ij}|) = d_{ij} + q_{ij}$$

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/18 & 0 \\ 5/2 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix},$$

$$\begin{pmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & N_{22} & N_{23} \\ N_{31} & N_{32} & N_{33} \end{pmatrix} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 1/32 \\ 0 & 2 & 1/3 \end{pmatrix}$$

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 4/9 \\ 1/2 \end{pmatrix}, \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -4/3 \\ 61/96 \end{pmatrix}$$

Simple computation shows that

$$K = (k_{ij})_{6 \times 6} = \begin{pmatrix} 0 & 0 & 0 & 1/2 & 1/18 & 0 \\ 0 & 0 & 0 & 5/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/32 & 0 & 0 & 0 \\ 0 & 2 & 1/3 & 0 & 0 & 0 \end{pmatrix}$$

Hence, by using matlab, it shows

$$\rho(K) = \max \text{eigenvalues}(K) = 0.602262 < 1.$$

Therefore, it follows that Theorem 1 and Theorem 2 that system (15) has a unique equilibrium

$$z^* = (x_1^*, x_2^*, x_3^*, y_1^*, y_2^*, y_3^*)^T = (1, 1, 1, 1, 1, 1)^T$$

from the given date, we obtain

$$\begin{cases} \sum_{j=1}^p \mu_j L_i^g(d_{ij} + q_{ij}\tau) - \lambda_i a_i = 0, j = 1, 2, \dots, p. \\ \sum_{i=1}^n \lambda_i L_j^f(c_{ji} + p_{ji}\tau) - \mu_j b_j < 0, i = 1, 2, \dots, n. \end{cases}$$

So that system (15) is globally exponentially stable.

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## REFERENCES

- [1] Kosto B. *Neural networks and fuzzy systems—a dynamical system approach to machine intelligence*. Englewood Cliffs (NJ):Prentice-Hall; 1992. p. 38-108.
- [2] Kosto B. *Bi-directional associative memories*. IEEE Trans Syst Man Cybernet 1988;18:49-60.
- [3] Kosto B. *Adaptive bi-directional associative memories*. Appl Opt 1987;26:4947-4960.
- [4] Kohonen T. *Self-organization and associative memory*. New York: Springer; 1988.
- [5] Cao J. *Periodic oscillatory solution of bidirectional associative memory networks with delays*. Phys Rev E 2000;61:1825-1828.
- [6] Cao J, Wang L. *Exponential stability and periodic oscillatory solution in BAM networks with delays*. IEEE Trans Neural Networks 2002;13(2):457-463.
- [7] Cao J. *Global asymptotic stability of delayed bi-directional associative memory neural networks*. Appl Math Computer 2003;142:333-339.
- [8] Cao J, J.L. Liang, J. Lam, *Exponential stability of high-order bidirectional associative memory neural networks with time delays*, Physical D 199 (2004) 425-436.
- [9] J. Cao, D. Ho, X. Huang, *LMI-based criteria for global robust stability of bidirectional associative memory networks with time delay*, Nonlinear Anal. 66 (2007) 1558-1572.
- [10] Liao XF, Yu JB. *Qualitative analysis of bidirectional associative memory with time delays*. Int J Circuit Theory Appl 1998;26:219-229.
- [11] Gopalsamy K, He XZ. *Delay-independent stability in bi-directional associative memory networks*. IEEE Trans Neural Networks 1994;5:998-1002.
- [12] Liang XB. *Global exponential stability and application of the Hopfield neural networks*. Sci China 1995;25(5):523-532.
- [13] Wei JJ, Ruan SG. *Stability and bifurcation in a neural network model with two delays*. Physical D 1999;130:255-272.
- [14] Jin C. *Stability analysis of discrete-time Hopfield BAM neural networks*. Acta Auto Silica 1999;25(5):606-612.
- [15] Xia YH, Cao J, Lin M. *New results on the existence and uniqueness of almost periodic solutions for BAM neural networks with continuously distributed delays*. Chaos, Solitons & Fractals 2007;314:928-936.
- [16] Cao J. *Global stability analysis in delayed cellular neural networks*. Phys Rev E 1999;59:5940-5944.
- [17] Cao J, Chen A, Huang X. *Almost periodic attractor of delayed neural networks with variable coefficients*. Phys Lett A 2005;340:104-120.
- [18] Kelly DG. *Stability in contractive nonlinear neural networks*. IEEE Trans Biomed Eng 1990;37:231-242.
- [19] Huang X, Cao J. *Almost periodic solutions of inhibitory cellular neural networks with time-vary delays*. Phys Lett A 2003;314:222-231.
- [20] Chen A, Cao J. *Existence and attractivity of almost periodic solutions for cellular networks with distributed delays and variable coefficients*. Appl Math Comp 2003;134:125-140.
- [21] Xia YH, Cao J. *Almost periodicity in an ecological model with M-predators and N-preys by 'pure-delay type' system*. Nonlinear Dyn 2005;39(3):275-304.
- [22] Xia YH, Cao J. *Global attractivity of a periodic ecological model with m-predators and n-preys by 'pure-delay type' system*. Comput Appl Math [to appear].
- [23] S. Arik, V. Tavsanoglu, *Global asymptotic stability analysis of bidirectional associative memory neural networks with constant time delays*, Neurocomputing 68 (2005) 161-171.
- [24] Haibo Gu, Haijun Jiang, Zhidong Teng, *Existence and globally exponential stability of periodic solution of BAM neural networks with impulses and recent-history distributed delays*, Neurocomputing 71 (2008) 813-822.
- [25] Yonghui Xia, Zhenkun Huang, Maoan Han. *Existence and globally exponential stability of equilibrium for BAM neural networks with impulses* Chaos, Solitons and Fractals 37 (2008) 588-597

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