

# Existence and global exponential stability of periodic solutions of cellular neural networks with distributed delays and impulses on time scales

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**Abstract**—In this paper, by using Mawhin's continuation theorem of coincidence degree and a method based on delay differential inequality, some sufficient conditions are obtained for the existence and global exponential stability of periodic solutions of cellular neural networks with distributed delays and impulses on time scales. The results of this paper generalized previously known results.

**Keywords**—periodic solutions, global exponential stability, coincidence degree,  $M$ -matrix

## I. INTRODUCTION

**S**TABILITY and periodicity of cellular neural networks have been paid much attention in the past decades[1-10], due to its applicability in the image processing, pattern recognition and associative memories and so on.

It is well known that most widely studied and used neural networks can be classified as either continuous or discrete. However, there has been a somewhat new category of neural networks, which displays a combination of characteristics of both the continuous-time and discrete-time systems, these are called impulsive neural networks[11-14]. To our knowledge, not many authors discuss stability and periodicity of cellular neural networks with delays and impulses. Recently, Yongkun Li and Zhiwei Xing have studied the existence and global exponential stability of the periodic solution of the following cellular neural networks with time delays and impulses [15]:

$$\begin{cases} \frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^n [b_{ij}(t)f_j(x_j(t)) \\ + c_{ij}(t)f_j(x_j(t - \tau_j(t)))] + I_i(t), \\ t \geq 0, t \neq t_k, i = 1, 2, \dots, n, \\ \Delta x_i(t_k) = J_i(x_i(t_k)) = -\gamma_{ik}x_i(t_k), i = 1, 2, \dots, n, \\ k = 1, 2, \dots, \end{cases}$$

However, in most situations, delays are variable, and in fact unbounded. So, in this paper, we will study the existence and global exponential stability of the periodic solution of cellular neural networks of the following with mixed delays and impulses:

$$\begin{cases} \frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^n [a_{ij}(t)f_j(x_j(t)) \\ + b_{ij}(t)f_j(x_j(t - \tau_{ij}(t)))] \\ + c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)f_j(x_j(s))ds + I_i(t), \\ t \geq 0, t \neq t_k, i = 1, 2, \dots, n, \\ \Delta x_i(t_k) = J_i(x_i(t_k)) = -\gamma_{ik}x_i(t_k), \\ i = 1, 2, \dots, n, k = 1, 2, \dots, \end{cases} \quad (1)$$

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where  $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ ,  $i = 1, 2, \dots, n$  are the impulses at moments  $t_k$  and  $0 < t_1 < t_2 < \dots$  is a strictly increasing sequence such that  $\lim_{t \rightarrow \infty} t_k = +\infty$ ;  $x_i(t)$  ( $i = 1, 2, \dots, n$ ) is the state of neuron and  $n$  is the number of neurons;  $A(t) = (a_{ij}(t))_{n \times n}$ ,  $B(t) = (b_{ij}(t))_{n \times n}$  and  $C(t) = (c_{ij}(t))_{n \times n}$  are connection matrix functions,

$I(t) = (I_1(t), I_2(t), \dots, I_n(t))^T : R^+ \mapsto R^n$  is continuous periodic function with period  $\omega > 0$ ,  $f(x) = (f_1(x_1), f_2(x_2), \dots, f_n(x_n))^T$  is the activation function of the neurons,  $F(t) = \text{diag}(a_1(t), a_2(t), \dots, a_n(t))$  with  $a_i(t) > 0$  ( $i = 1, 2, \dots, n$ ). The delays  $0 \leq \tau_{ij}(t) \leq \tau(i, j = 1, 2, \dots, n)$  are bounded functions. Kernel function  $k_{ij} : [0, \infty) \rightarrow [0, \infty)$  ( $i, j = 1, 2, \dots, n$ ) are all piecewise continuous functions on  $[0, \infty)$ , and satisfy  $\int_0^\infty k_{ij}(s)ds = 1$ ,  $i, j = 1, 2, \dots, n$ .

As usual in the theory of impulsive differential equations, at the points of discontinuity  $t_k$  of the solution  $t \mapsto x_i(t)$  we assume that  $x_i(t_k) \equiv x_i(t_k^-)$ . It is clear that, in general, the derivatives  $x_i'(t_k)$  do not exist. On the other hand, according to the first equality of (1) there exists the limits  $x_i'(t_k^\pm)$ . So, we assume that  $x_i'(t_k) \equiv x_i'(t_k^-)$ ,  $i = 1, 2, \dots, n$ ;  $k = 1, 2, \dots$ .

The initial conditions of system (1) are of the form  $x_i(s) = \phi_i(s) \neq 0$ ,  $s \leq 0$ ,  $i = 1, 2, \dots, n$ . where  $\phi_i$  is bounded and continuous function on  $(-\infty, 0]$ .

Throughout this paper, we impose the following assumptions:

(H<sub>1</sub>) The delays  $0 \leq \tau_{ij}(t) \leq \tau(i, j = 1, 2, \dots, n)$  are bounded continuous  $\omega$ -periodic functions.

(H<sub>2</sub>)  $a_i(t)$ ,  $i = 1, 2, \dots, n$  are positive and bounded continuous  $\omega$ -periodic functions, and  $0 \leq \underline{a}_i \leq a_i(t) \leq \bar{a}_i$ .

(H<sub>3</sub>) kernel function  $k_{ij}$ ,  $i, j = 1, 2, \dots, n$  are all piecewise continuous functions, and satisfy  $\int_0^\infty k_{ij}(s)ds = 1$ .

(H<sub>4</sub>) There exist positive constants  $M_j > 0$  such that  $|f_j(x)| \leq M_j$  for  $j = 1, 2, \dots, n$ ,  $x \in R$ .

(H<sub>5</sub>)  $a_{ij}(t)$ ,  $b_{ij}(t)$ ,  $c_{ij}(t)$ ,  $i, j = 1, 2, \dots, n$  are bounded continuous  $\omega$ -periodic functions.

(H<sub>6</sub>) There exists a positive integer  $q$  such that  $t_{k+q} = t_k + \omega$ ,  $\gamma_{i(k+q)} = \gamma_{ik}$ , for  $k = 1, 2, \dots$ ,  $i = 1, 2, \dots, n$ .

(H<sub>7</sub>)  $\prod_{0 \leq t_k < t} (1 - \gamma_{ik})$ ,  $i = 1, 2, \dots, n$  are  $\omega$ -periodic functions.

(H<sub>8</sub>)  $f_i \in C(R, R)$ ,  $j = 1, 2, \dots, n$  are Lipschitzian with Lipschitz constants  $L_j > 0$ ,

$|f_j(x) - f_j(y)| \leq L_j |x - y|$  for all  $x, y \in R$ .

For convenience, we introduce the following notations:

$$\bar{a}_{ij} = \sup\{|a_{ij}(t)|, t \in [0, \omega]\}, \bar{b}_{ij} = \sup\{|b_{ij}(t)|, t \in [0, \omega]\}, \bar{c}_{ij} = \sup\{|c_{ij}(t)|, t \in [0, \omega]\}, \bar{I}_i = \sup\{|I_i(t)|, t \in [0, \omega]\} \\ N_i = (\int_0^\omega \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-2} dt)^{\frac{1}{2}}, i, j = 1, 2, \dots, n.$$

The organization of this paper is as follows. In Section II, we introduce some notations and definitions, and state some preliminary results needed in later sections. In Section III, we then study the existence of periodic solutions of system (1) by using the continuation theorem of coincidence degree proposed by Gaines and Mawhin [16]. In Section IV, we shall derive sufficient conditions to ensure that the periodic solution of (1) is globally exponentially stable.

## II. PRELIMINARIES

In this section, we shall introduce some notations and definitions, and state some preliminary results.

Consider the impulsive system[15]:

$$\begin{cases} x'(t) = f(t, x(t), x(t - \tau_1(t)), \dots, x(t - \tau_n(t))), \\ t \neq t_k, k = 1, 2, \dots, \\ \Delta x(t) |_{t=t_k} = J_k(x(t_k^-)) = -\gamma_{ik} x_i(t_k), \end{cases} \quad (2)$$

where  $x \in R^n, f: R \times R^n \rightarrow R^n$  is continuous and  $f(t + \omega, x(t - \tau_1(t)), \dots, x(t - \tau_n(t))) = f(t, x(t - \tau_1(t)), \dots, x(t - \tau_n(t)))$ ,  $J_k: R^n \rightarrow R^n, k = 1, 2, \dots, n$  are continuous;  $\tau_i \in C(R, [0, \tau]), i = 1, 2, \dots, n$  are  $\omega$ -periodic functions and  $t - \tau_i(t) \rightarrow \infty$ , as  $t \rightarrow \infty, i = 1, 2, \dots, n$ , and there exists a positive integer  $q$  such that  $t_{k+q} = t_k + \omega, J_{k+q}(x) = J_k(x)$  with  $t_k \in R, t_{k+1} > t_k, \lim_{k \rightarrow \infty} t_k = \infty, \Delta x(t) |_{t=t_k} = x(t_k^+) - x(t_k^-)$ . For  $t_k \neq 0 (k = 1, 2, \dots), [0, \omega] \cap \{t_k\} = \{t_1, t_2, \dots, t_q\}$ . As we know,  $\{t_k\}$  are called points of jump.

**Definition 2.1.**<sup>[15]</sup> A function  $x \in ([0, \infty), R)$  is said to be a solution of system (2) in  $[0, \infty)$  satisfying the initial value condition  $x(s) = \phi(s) \neq 0, s \in (-\infty, 0]$ , where  $\phi \in C((-\infty, 0], R^n)$ , if the following conditions are satisfied

(i)  $x(t)$  is absolutely continuous in each interval  $(t_k, t_{k+1}) \subset [0, \infty)$ ;

(ii) for any  $t_k \in [0, \infty), k = 1, 2, \dots, x(t_k^+)$  and  $x(t_k^-)$  exist and  $x(t_k^-) = x(t_k)$ ;

(iii)  $x(t)$  satisfies (2) for almost everywhere in  $[0, \infty)$  and at impulsive points  $\{t_k\}$  situated in  $[0, \infty)$  may have discontinuity of the first kind.

**Definition 2.2.**<sup>[14]</sup> The periodic solution of system (2) is said to be globally exponentially stable (GES), if there exist constants  $\alpha > 0$  and  $\beta > 0$  such that

$$|x_i(t) - x_i^*| \leq \beta \|\phi - x^*\| e^{-\alpha t}$$

for all  $t \geq 0$ , where

$$\|\phi - x^*\| = \sup_{s \in (-\infty, 0]} (\sum_{i=1}^n |\phi_i(s) - x_i^*|).$$

Consider the nonimpulsive delay differential system

$$\begin{aligned} \frac{dy_i(t)}{dt} = & -a_i(t)y_i(t) + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n [a_{ij}(t) \\ & \times f_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})y_j(t)) \\ & + b_{ij}(t)f_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) \\ & + c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds] \\ & + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} I_i(t), \quad t \geq 0, i = 1, 2, \dots, n. \end{aligned} \quad (3)$$

with initial conditions  $y_i(s) = \phi_i(s) \neq 0, s \in (-\infty, 0], i = 1, 2, \dots, n$ .

**Lemma 2.1.** Assume  $(H_7)$  holds, then

(i) if  $y = (y_1, \dots, y_n)$  is a solution of (3), then

$x = (\prod_{0 \leq t_k < t} (1 - \gamma_{1k})y_1, \dots, \prod_{0 \leq t_k < t} (1 - \gamma_{nk})y_n)$  is a solution of (1);

(ii) if  $x = (x_1, \dots, x_n)$  is a solution of (1), then

$y = (\prod_{0 \leq t_k < t} (1 - \gamma_{1k})^{-1}x_1, \dots, \prod_{0 \leq t_k < t} (1 - \gamma_{nk})^{-1}x_n)$  is a solution of (3).

**Proof.** The proof is similar to that of Theorem 2.1 in [14] and will be omitted here.

Let  $X, Y$  be real Banach spaces,  $L: Dom L \subset X \rightarrow dim Y$  be a linear mapping, and  $N: X \rightarrow Y$  be a continuous mapping. The mapping  $L$  will be called a Fredholm mapping of index zero if  $dim Ker L = codim Im L < +\infty$  and  $Im L$  is closed in  $Y$ . If  $L$  is a Fredholm mapping of index zero and there exist continuous projectors  $P: X \rightarrow X$  and  $Q: Y \rightarrow Y$  such that  $Im P = Ker L, Ker Q = Im(I - Q)$ , it follows that mapping  $L|_{Dom L \cap Ker P}: (I - P)X \rightarrow Im L$  is invertible. We denote the inverse of that mapping by  $K_P$ . If  $\Omega$  is an open bounded subset of  $X$ , the mapping  $N$  will be called  $L$ -compact on  $\bar{\Omega}$ , if  $QN(\bar{\Omega})$  is bounded and  $K_P(I - Q)N: \bar{\Omega} \rightarrow X$  is compact. Since  $Im Q$  is isomorphic to  $Ker L$ , there exists an isomorphism  $J: Im Q \rightarrow Ker L$ .

Now, we introduce Mawhin's continuation theorem as follows.

**Lemma 2.2.**<sup>[16]</sup> Let  $\Omega \subset X$  be an open bounded set and let  $N: X \rightarrow Y$  be a continuous operator which is  $L$ -compact on  $\bar{\Omega}$ . Assume

(a) for each  $\lambda \in (0, 1), x \in \partial\Omega \cap Dom L, Lx \neq \lambda Nx$ ,

(b) for each  $x \in \partial\Omega \cap Ker L, QNx \neq 0$ , and  $deg(JQN, \Omega \cap Ker L, 0) \neq 0$ .

Then  $Lx = Nx$  has at least one solution in  $\bar{\Omega} \cap Dom L$ .

**Definition 2.3.**<sup>[15]</sup> Let the  $n \times n$  matrix  $A = (a_{ij})_{n \times n}$  have nonpositive off-diagonal elements and all principal minors of  $A$  are positive, then  $A$  is said to be an  $M$ -matrix.

**Lemma 2.3.**<sup>[17]</sup> Let  $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$  be a solution of the differential inequality:

$$x'(t) \leq Ax(t) + Bx(t), t \geq t_0,$$

where

$$\bar{x}(t) = \left( \sup_{t-\tau \leq s \leq t} \{x_1(s)\}, \sup_{t-\tau \leq s \leq t} \{x_2(s)\}, \dots, \sup_{t-\tau \leq s \leq t} \{x_n(s)\} \right)^T, A = (a_{ij})_{n \times n}, B = (b_{ij})_{n \times n}.$$

If

$$(A_1) a_{ij} \geq 0 (i \neq j), b_{ij} \geq 0, i, j = 1, 2, \dots, n; \sum_{j=1}^n \bar{x}_j(t_0) > 0;$$

$$(A_2) \text{ The matrix } -(A+B) \text{ is an } M\text{-matrix.}$$

Then there always exist constants  $\lambda > 0, r_i > 0$  ( $i = 1, 2, \dots, n$ ) such that

$$x_i(t) \leq r_i \sum_{j=1}^n \bar{x}_j(t_0) e^{\lambda(t-t_0)}.$$

### III. EXISTENCE OF PERIODIC SOLUTIONS

In this section, based on the Mawhin's continuation theorem, we shall study the existence of at least one periodic solution of (1). For convenience, we introduce the following notations:

$$\begin{aligned} G_i^y &= G_i(t, y_1(t), \dots, y_n(t)) \\ &= -a_i(t)y_i(t) + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n [a_{ij}(t) \\ &\times f_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})y_j(t)) \\ &+ b_{ij}(t)f_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) \\ &+ c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds] \\ &+ \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} I_i(t), \quad t \geq 0, \end{aligned}$$

where  $y = (y_1, y_2, \dots, y_n)^T$  is  $\omega$ -periodic function,  $i = 1, 2, \dots, n$ . Our main result of this section is as follows.

**Theorem 3.1.** Suppose (H<sub>1</sub>)-(H<sub>7</sub>) hold, then the system (1) has at least one  $\omega$ -periodic solution.

**Proof.** According to Lemma 2.1, we need only to prove that the nonimpulsive delay differential system (3) has an  $\omega$ -periodic solution. In order to use the continuation theorem of coincidence degree theory to establish the existence of a solution of (3), we take

$$Y = Z = \{y(t) \in C(R, R^n) : y(t + \omega) = y(t), t \in R, y = (y_1, y_2, \dots, y_n)^T\}$$

with the norm

$$\|y\| = \sum_{k=1}^n |y_k|_0, \quad |y_k|_0 = \sup_{t \in [0, \omega]} |y_k(t)|, \quad k = 1, 2, \dots, n$$

then  $Y$  and  $Z$  are Banach spaces.

Set

$$Ly = y' \text{ and } Py = \frac{1}{\omega} \int_0^\omega y(t)dt, \quad y \in Y; \quad Qz = \frac{1}{\omega} \int_0^\omega z(t)dt, \quad z \in Z$$

and

$$Ny = (G_1^y(t), G_2^y(t), \dots, G_n^y(t))^T, \quad y \in Y.$$

Obviously,  $\text{Ker} L = \{y \in Y, y = h, h \in R^n\}$ ,  $\text{Im} L = \{y \in Y, \int_0^\omega y(s)ds = 0\}$  and

$$\dim \text{Ker} L = n = \text{codim Im} L.$$

So,  $\text{Im} L$  is closed in  $Z$  and  $L$  is a Fredholm mapping of

index zero. It is easy to show that  $P$  and  $Q$  are continuous projectors satisfying

$$\text{Im} P = \text{Ker} L, \quad \text{Im} L = \text{Ker} Q = \text{Im}(I - Q).$$

Furthermore, through an easy computation, we can find that the inverse  $K_P : \text{Im} L \rightarrow \text{Ker} P \cap \text{Dom} L$  of  $L_P$  has the form

$$K_P(z) = \int_0^t z(s)ds - \frac{1}{\omega} \int_0^\omega \int_0^t z(s)dsdt.$$

Thus

$$QNy = \left( \frac{1}{\omega} \int_0^\omega G_1^y(t)dt, \dots, \frac{1}{\omega} \int_0^\omega G_n^y(t)dt \right)^T, \quad y \in Y$$

and

$$\begin{aligned} K_P(I - Q)Ny &= \begin{pmatrix} \int_0^t G_1^y(s)ds \\ \vdots \\ \int_0^t G_j^y(s)ds \\ \vdots \\ \int_0^t G_n^y(s)ds \end{pmatrix} \\ &- \begin{pmatrix} \frac{1}{\omega} \int_0^\omega \int_0^t G_1^y(s)dsdt \\ \vdots \\ \frac{1}{\omega} \int_0^\omega \int_0^t G_j^y(s)dsdt \\ \vdots \\ \frac{1}{\omega} \int_0^\omega \int_0^t G_n^y(s)dsdt \end{pmatrix} - \begin{pmatrix} (\frac{t}{\omega} - \frac{t}{2}) \int_0^\omega G_1^y(s)ds \\ \vdots \\ (\frac{t}{\omega} - \frac{t}{2}) \int_0^\omega G_j^y(s)ds \\ \vdots \\ (\frac{t}{\omega} - \frac{t}{2}) \int_0^\omega G_n^y(s)ds \end{pmatrix} \end{aligned}$$

Clearly,  $QN$  and  $K_P(I - Q)N$  are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that  $QN(\bar{\Omega})$ ,  $K_P(I - Q)N(\bar{\Omega})$  are relatively compact for any open bounded set  $\Omega \subset Y$ . Therefore,  $N$  is  $L$ -compact on  $\bar{\Omega}$  for any open bounded set  $\Omega \subset Y$ .

Now we reach the position to search for an appropriate open, bounded subset  $\Omega$ , for the application of the continuation theorem. Corresponding to the operator equation  $Ly = \lambda Ny$ ,  $\lambda \in (0, 1)$ , we have

$$\begin{aligned} y_i'(t) &= \lambda \{ -a_i(t)y_i(t) + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n [a_{ij}(t) \\ &\times f_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})y_j(t)) \\ &+ b_{ij}(t)f_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) \\ &+ c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds] \\ &+ \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} I_i(t) \}, \\ y \in Y \quad i &= 1, 2, \dots, n. \end{aligned} \quad (4)$$

Suppose that  $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in Y$  is a solution of system (4) for some  $\lambda \in (0, 1)$ . Integrating  $y_i(t)y_i'(t)$  over the interval  $[0, \omega]$ , we obtain

$$\begin{aligned} 0 &= \frac{1}{2} y_i^2(t) \Big|_0^\omega = \int_0^\omega y_i(t)y_i'(t)dt \\ &= \lambda \int_0^\omega \{ -a_i(t)y_i(t)y_i'(t) + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} y_i(t) \\ &\times \sum_{j=1}^n [a_{ij}(t)f_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})y_j(t)) \end{aligned}$$

$$\begin{aligned}
& + b_{ij}(t)f_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) \\
& + c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds] \\
& + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} y_i(t) I_i(t) \} dt
\end{aligned}$$

That is

$$\begin{aligned}
\int_0^\omega a_i(t)y_i^2(t)dt &= \int_0^\omega \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \\
&\times y_i(t) \sum_{j=1}^n [a_{ij}(t)f_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})y_j(t)) \\
&+ b_{ij}(t)f_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) \\
&+ c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds] \\
&+ \int_0^\omega (1 - \gamma_{ik})^{-1} y_i(t) I_i(t) dt, \quad i = 1, 2, \dots, n
\end{aligned}$$

Obviously

$$\begin{aligned}
\int_{-\infty}^t k_{ij}(t-s)ds &= - \int_{-\infty}^t k_{ij}(t-s)d(t-s) \\
&= - \int_{+\infty}^0 k_{ij}(u)du = \int_0^{+\infty} k_{ij}(u)du = 1.
\end{aligned}$$

From conditions (H<sub>2</sub>), (H<sub>4</sub>) and (H<sub>5</sub>), it follows that

$$\begin{aligned}
a_i \int_0^\omega |y_i^2(t)| dt &\leq \int_0^\omega \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} |y_i(t)| \\
&\times \sum_{j=1}^n [|a_{ij}(t)| |f_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})y_j(t))| \\
&+ |b_{ij}(t)| |f_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t)))| \\
&+ |c_{ij}(t)| \int_{-\infty}^t |k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds| \\
&+ \int_0^\omega (1 - \gamma_{ik})^{-1} |y_i(t)| |I_i(t)| dt] \\
&\leq \int_0^\omega \sum_{j=1}^n (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij}) M_j \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} |y_i(t)| dt \\
&+ \bar{I}_i \int_0^\omega \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} |y_i(t)| dt \\
&\leq (\sum_{j=1}^n (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij}) M_j + \bar{I}_i) (\int_0^\omega \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-2} dt)^{\frac{1}{2}} \\
&\times (\int_0^\omega |y_i(t)|^2 dt)^{\frac{1}{2}} \\
&= N_i (\sum_{j=1}^n (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij}) M_j + \bar{I}_i) (\int_0^\omega |y_i(t)|^2 dt)^{\frac{1}{2}}, \\
&i = 1, 2, \dots, n.
\end{aligned}$$

Hence,

$$\begin{aligned}
(\int_0^\omega |y_i^2(t)| dt)^{\frac{1}{2}} &\leq \frac{N_i}{a_i} (\sum_{j=1}^n (\bar{a}_{ij} + \bar{b}_{ij} + \bar{c}_{ij}) M_j + \bar{I}_i) \\
&:= S_i, \quad i = 1, 2, \dots, n
\end{aligned} \quad (5)$$

Let  $t_i \in [0, \omega] \neq t_k, k = 1, 2, \dots, m$ . such that  $|y_i(t_i)| = \inf_{t \in [0, \omega]} |y_i(t)|, i = 1, 2, \dots, n$ . Then, by (5), we have

$$|y_i(t_i)| \sqrt{\omega} = |y_i(t_i)| (\int_0^\omega dt)^{\frac{1}{2}} \leq (\int_0^\omega |y_i^2(t)| dt)^{\frac{1}{2}} \leq S_i$$

thus,

$$|y_i(t_i)| \leq \frac{S_i}{\sqrt{\omega}} \quad (6)$$

From (6), and since  $y_i(t) = y_i(t_i) + \int_{t_i}^t y_i'(s)ds$ , it follows that

$$|y_i(t)| \leq \frac{S_i}{\sqrt{\omega}} + \int_0^\omega |y_i'(t)| dt \quad (7)$$

On the other hand, from (4) and conditions (H<sub>2</sub>), (H<sub>4</sub>), (H<sub>5</sub>), (H<sub>7</sub>), we have

$$\begin{aligned}
\int_0^\omega |y_i'(t)| dt &< \bar{a}_i \int_0^\omega |y_i(t)| dt + (\sum_{j=1}^n |a_{ij}(t)| + |b_{ij}(t)| \\
&+ |c_{ij}(t)| M_j + \bar{I}_i) \int_0^\omega \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} dt \\
&\leq \bar{a}_i \sqrt{\omega} (\int_0^\omega |y_i(t)|^2 dt)^{\frac{1}{2}} + (\sum_{j=1}^n (\bar{a}_{ij} + \bar{b}_{ij} \\
&+ \bar{c}_{ij}) M_j + \bar{I}_i) \sqrt{\omega} \int_0^\omega \prod_{0 \leq t_k < t} ((1 - \gamma_{ik})^{-2} dt)^{\frac{1}{2}} \\
&= \bar{a}_i \sqrt{\omega} (\int_0^\omega |y_i(t)|^2 dt)^{\frac{1}{2}} + N_i \sqrt{\omega} (\sum_{j=1}^n (\bar{a}_{ij} + \bar{b}_{ij} \\
&+ \bar{c}_{ij}) M_j + \bar{I}_i)
\end{aligned}$$

Together with (5), we get

$$\begin{aligned}
\int_0^\omega |y_i'(t)| dt &< \bar{a}_i \sqrt{\omega} S_i + N_i \sqrt{\omega} (\sum_{j=1}^n (\bar{a}_{ij} + \bar{b}_{ij} \\
&+ \bar{c}_{ij}) M_j + \bar{I}_i) := D_i.
\end{aligned} \quad (8)$$

in view of (7) and (8), we obtain

$$|y_i(t)| < \frac{S_i}{\sqrt{\omega}} + D_i := R_i, \quad i = 1, 2, \dots, n. \quad (9)$$

Denote  $A = \sum_{i=1}^m R_i + K$ , where  $K$  is a sufficiently large positive constant, clearly,  $A$  is independent of  $\lambda$ . Now, take  $\Omega = \{y \in Y : \|y(t)\| < A\}$ . It is clear that  $\Omega$  satisfies the requirement (a) in Lemma 2.2.

When  $y \in \partial\Omega \cap \text{Ker} L, y = (y_1, y_2, \dots, y_n)^T$  is a constant vector in  $R^n$  with  $\|y\| = A$ . Then  $QNy =$

$(\frac{1}{\omega} \int_0^\omega G_1^y dt, \dots, \frac{1}{\omega} \int_0^\omega G_n^y dt)$ ,  $y \in Y$  where

$$\begin{aligned} G_i^y = & -a_i(t)y_i(t) + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n [a_{ij}(t) \\ & \times f_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})y_j(t)) \\ & + b_{ij}(t)f_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})y_j(t - \tau_{ij}(t))) \\ & + c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)f_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})y_j(s))ds] \\ & + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} I_i(t), i = 1, 2, \dots, n \end{aligned}$$

Take  $J : ImQ \rightarrow KerL, r \rightarrow r$ . Then, if necessary, we can let  $K$  be greater such that  $y^T JQN y < 0$ . So, for any  $y \in \partial\Omega \cap KerL, QNy \neq 0$ . Furthermore, let  $\phi(\gamma; y) = -\gamma y + (1 - \gamma)JQN y$ , then for any  $y \in \partial\Omega \cap KerL, y^T \phi(\gamma; y) < 0$ , we get

$$\deg\{JQN, \Omega \cap KerL, 0\} = \deg\{-y, \Omega \cap KerL, 0\} \neq 0.$$

So, condition (b) of Lemma 2.2 is also satisfied. We now know that  $\Omega$  satisfies all the requirements in Lemma 2.2. Therefore, (3) has at least one  $\omega$ -periodic solution. As a sequence system (1) has at least one  $\omega$ -periodic solution. The proof is complete.

#### IV. GLOBAL EXPONENTIAL STABILITY OF THE PERIODIC SOLUTION

Suppose that  $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  is a periodic solution of system (1). In this section, we will use a technique of differential inequality to study the global exponential stability of this periodic solution.

**Theorem 4.1.** Assume (H<sub>1</sub>)-(H<sub>8</sub>) hold. Moreover, suppose that matrix

$F = \alpha\beta(A + B + C)L$  is an  $M$ -matrix, where  $F = \text{diag}(\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n)$ ,  $A = (\bar{a}_{ij})_{n \times n}$ ,  $B = (\bar{b}_{ij})_{n \times n}$ ,  $C = (\bar{c}_{ij})_{n \times n}$ ,  $L = \text{diag}(L_1, L_2, \dots, L_n)$ ,  $\alpha = \max_{1 \leq i \leq n} \{\sup_{t \in [0, \omega]} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})\}$ ,  $\beta = \max_{1 \leq i \leq n} \{\sup_{t \in [0, \omega]} \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1}\}$ . Then the  $\omega$ -periodic solution of system (1) is globally exponentially stable.

**Proof.** According to Theorem 3.1, we know that (1) has an  $\omega$ -periodic solution  $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$ . Suppose that  $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$  is an arbitrary solution of (1).

Let  $y(t) = x(t) - x^*$ , then (1) can be written as

$$\begin{cases} \frac{dy_i(t)}{dt} = -a_i(t)y_i(t) + \sum_{j=1}^n [a_{ij}(t)g_j(y_j(t)) \\ \quad + b_{ij}(t)g_j(y_j(t - \tau_{ij}(t))) \\ \quad + c_{ij}(t) \int_{-\infty}^0 k_{ij}(t-s)g_j(y_j(s))ds], \quad t \neq t_k \\ \Delta y_i(t_k) = -\gamma_{ik}y_i(t_k), \quad t \geq 0, \quad i = 1, 2, \dots, n, \\ \quad k = 1, 2, \dots, \end{cases} \quad (10)$$

where

$$g_j(y_j(t)) = f_j(x_j(t)) - f_j(x_j^*), \quad j = 1, 2, \dots, n$$

Due to the assumption of (H<sub>8</sub>), we know that  $0 \leq |g_i(y_i)| \leq$

$L_i |y_i|$ ,  $i = 1, 2, \dots, n$ . The initial condition of (10) is  $\Psi(s) = \phi(s) - x^*$ ,  $s \in (-\infty, 0]$ .

Also according to Lemma 2.1, we consider the following nonimpulsive delay differential system:

$$\begin{aligned} \frac{du_i(t)}{dt} = & -a_i(t)u_i(t) + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n [a_{ij}(t) \\ & \times g_j(\prod_{0 \leq t_k < t} (1 - \gamma_{jk})u_j(t)) \\ & + b_{ij}(t)g_j(\prod_{0 \leq t_k < t - \tau_{ij}(t)} (1 - \gamma_{jk})u_j(t - \tau_{ij}(t))) \\ & + c_{ij}(t) \int_{-\infty}^t k_{ij}(t-s)g_j(\prod_{0 \leq t_k < s} (1 - \gamma_{jk})u_j(s))ds], \\ & i = 1, 2, \dots, n. \end{aligned} \quad (11)$$

with initial conditions  $u(s) = \Psi(s) = \phi(s) - x^*$ ,  $s \in (-\infty, 0]$ .

Let  $z_i(t) = |u_i(t)|$ , then the upper right derivative  $D^+ z_i(t)$  along the solutions of system (11) is as follows:

$$\begin{aligned} D^+ z_i(t) = & D^+ |u_i(t)| = u_i(t)' \text{sgn}(u_i(t)) \\ \leq & -\underline{a}_i |u_i(t)| + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n [a_{ij}(t) \\ & \times |L_j |u_j(t)| \prod_{0 \leq t_k < t} (1 - \gamma_{jk}) \\ & + |b_{ij}(t) |L_j |\bar{u}_j(t)| \prod_{0 \leq t_k < t} (1 - \gamma_{jk}) \\ & + |c_{ij}(t) |L_j |u_j(t)| \prod_{0 \leq t_k < t} (1 - \gamma_{jk})] \\ \leq & -\underline{a}_i |u_i(t)| + \prod_{0 \leq t_k < t} (1 - \gamma_{ik})^{-1} \sum_{j=1}^n [\bar{a}_{ij} L_j \\ & \times |u_j(t)| \prod_{0 \leq t_k < t} (1 - \gamma_{jk}) \\ & + \bar{b}_{ij} L_j |\bar{u}_j(t)| \prod_{0 \leq t_k < t} (1 - \gamma_{jk}) \\ & + \bar{c}_{ij} L_j |u_j(t)| \prod_{0 \leq t_k < t} (1 - \gamma_{jk})], \quad i = 1, 2, \dots, n. \end{aligned}$$

Hence

$$\begin{aligned} D^+ z_i(t) \leq & -\underline{a}_i |u_i(t)| + \beta \sum_{j=1}^n [\bar{a}_{ij} L_j |u_j(t)| \alpha \\ & + \bar{b}_{ij} L_j |\bar{u}_j(t)| \alpha + \bar{c}_{ij} L_j |u_j(t)| \alpha] \\ \leq & -\underline{a}_i |u_i(t)| + \alpha\beta \sum_{j=1}^n (\bar{a}_{ij} + \bar{c}_{ij}) L_j |u_j(t)| \\ & + \alpha\beta \sum_{j=1}^n \bar{b}_{ij} L_j |\bar{u}_j(t)| \\ \leq & -\underline{a}_i z_i(t) + \alpha\beta \sum_{j=1}^n (\bar{a}_{ij} + \bar{c}_{ij}) L_j z_j(t) \\ & + \alpha\beta \sum_{j=1}^n \bar{b}_{ij} L_j \bar{z}_j(t), \quad i = 1, 2, \dots, n. \end{aligned}$$

That is

$$D^+z_i(t) \leq (-F + \alpha\beta(A + C)L)z(t) + \alpha\beta BL\bar{z}(t), \quad t \geq 0, \quad i = 1, 2, \dots, n.$$

where  $F = \text{diag}(\underline{a}_1, \underline{a}_2, \dots, \underline{a}_n)$ ,  $A = (\bar{a}_{ij})_{n \times n}$ ,  $B = (\bar{b}_{ij})_{n \times n}$ ,  $C = (\bar{c}_{ij})_{n \times n}$ ,  $L = \text{diag}(L_1, L_2, \dots, L_n)$ .

By initial conditions  $x_i(s) = \phi_i(s) \neq 0$ ,  $s \in (-\infty, 0]$ ,  $i = 1, 2, \dots, n$ , we know that  $\bar{z}_i(0) > 0$ , according to Lemma 2.3, if the matrix  $-[-F + \alpha\beta(A + C)L + \alpha\beta BL] = F - \alpha\beta(A + B + C)L$  is an  $M$ -matrix, then there must exist constants  $\mu > 0$ ,  $r_i > 0$  ( $i = 1, 2, \dots, n$ ) such that

$$z_i(t) = |u_i(t)| \leq r_i \sum_{j=1}^n \bar{z}_j(0)e^{-\mu t} = r_i \sum_{j=1}^n |\bar{u}_j(0)| e^{-\mu t}, \quad i = 1, 2, \dots, n.$$

By initial conditions, we have  $\bar{u}(0) = \bar{\Psi}(0) = \bar{\phi}(0) - x^*$ , then the solution of (10) satisfies

$$\begin{aligned} |y_i(t)| &= \prod_{0 \leq t_k < t} (1 - \gamma_{ik}) |u_i(t)| \\ &\leq \prod_{0 \leq t_k < t} (1 - \gamma_{ik}) r_i \sum_{j=1}^n |\bar{u}_j(0)| e^{-\mu t} \\ &\leq \prod_{0 \leq t_k < t} (1 - \gamma_{ik}) r_i \sum_{j=1}^n |\bar{\phi}_j(0) - x_j^*| e^{-\mu t} \\ &\leq \alpha r_i \sum_{j=1}^n |\bar{\phi}_j(0) - x_j^*| e^{-\mu t} \\ &= \alpha r_i \sum_{i=1}^n |\bar{\phi}_i(0) - x_i^*| e^{-\mu t}, \quad i = 1, 2, \dots, n. \end{aligned}$$

That is

$$\begin{aligned} |x_i(t) - x_i^*| &\leq \alpha r_i \sum_{i=1}^n |\bar{\phi}_i(0) - x_i^*| e^{-\mu t} \\ &= \alpha r_i \left[ \sup_{s \in (-\infty, 0]} \left( \sum_{i=1}^n |\phi_i(s) - x_i^*| \right) \right] e^{-\mu t} \\ &= \alpha r_i \|\phi - x^*\| e^{-\mu t}, \quad i = 1, 2, \dots, n. \end{aligned}$$

From Definition 2.2, we can see the  $\omega$ -periodic solution  $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T$  of system (1) is globally exponentially stable.

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