Exact Solutions of Steady Plane Flows of an Incompressible Fluid of Variable Viscosity Using (ξ, ψ)- Or (η, ψ)- Coordinates

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Abstract**—**The exact solutions of the equations describing the steady plane motion of an incompressible fluid of variable viscosity for an arbitrary state equation are determined in the (ξ,ψ) – or (η,ψ) - coordinates where $\psi(x,y)$ is the stream function, ξ and η are the parts of the analytic function, $\varpi=\xi(x,y)+i\eta(x,y)$. Most of the solutions involve arbitrary function/ functions indicating that the flow equations possess an infinite set of solutions.

Keywords**—**Exact solutions, Fluid of variable viscosity, Navier-Stokes equations, Steady plane flows

I. Introduction

NAEEM and Nadeem [1] extended Martin's [2] approach to study the steady plane flows of an incompressible fluid of variable viscosity for an arbitrary state equation. Naeem and Nadeem determined some new exact solutions to the flow equations and also indicated applicability of some of the solutions to physically possible situations. In Martin's approach a natural curvilinear coordinate system (ϕ,ψ) in the physical plane (x,y) is introduced in which ψ = constant are the streamlines and ϕ = constant is an arbitrary family of curves. In Martin's approach, the transformed system of flow equations becomes undetermined and is due to arbitrariness of the coordinate lines ϕ = constant. The system can be made determinate in a number of ways. Naeem and Nadeem [1] made the system determinate by making system orthogonal, in which case coefficient F of the first fundamental element ds^2 is zero. Naeem and Ali [3], following Martin's approach made the system governing the motion of fluid in [1] determined by taking $\phi=x$.

Recently Labropulu and Chandna [4] extended Martin's approach to study the steady plane infinitely conducting MHD aligned flows and made their system of flow equations determinate by taking $\phi(x,y)=\xi(x,y)$ or $\phi(x,y)=\eta(x,y)$ where $\xi(x,y)$ and $\eta(x,y)$ are the real and imaginary parts of an analytic function ϖ. Labropulu and Chandna obtained exact solutions for the flows when the stream line pattern is of the form $\frac{\eta-f(\xi)}{g(\xi)}=$ Constant or $\frac{\xi-k(\eta)}{m(\eta)}=$ Constant.

In the present work, we extend Labropulu and Chandna approach to study the steady plane flows of an incompressible fluid of variable viscosity for arbitrary state equation and present some exact solutions. The most of the solutions contain arbitrary function(s) allowing us to construct an infinite set of solutions to flow equations. The plan of this is as follows:

In the next section description of basic flow equations are discussed. Section-III presents the flow equations in the physical plane and Martin's system (ϕ,ψ). The coefficients E, F, G of first fundamental ds^2 are also given in (ξ,ψ) - and (η,ψ) - coordinate system. In Section-IV, exact solutions to flow equations are determined.

II. Basic Flow Equations

The basic non-dimensional equations governing the steady plane motion of an incompressible fluid of variable viscosity in the presence of an unknown external force with no heat addition are:

$$u_x+v_y=0 \tag{1}$$

$$uu_x+vu_y=-p_x+\frac{1}{R_e}[(2\mu u_x)_x+(\mu(u_y+v_x))_y]+\lambda^* f_1 \tag{2}$$

$$uv_x+vv_y=-p_y+\frac{1}{R_e}[(2\mu v_y)_y+(\mu(u_y+v_x))_x]+\lambda^* f_2 \tag{3}$$

$$uT_x+vT_y=\frac{1}{R_eP_r}(T_{xx}+T_{yy})+\frac{E_c}{R_e}[2\mu(u_x{}^2+v_y^2)+\mu(u_y+v_x)^2] \tag{4}$$

$$\mu=\mu(T) \tag{5}$$

where u,v are the velocity components, p the pressure, μ the fluid viscosity, T the fluid temperature, R_e the Reynolds number, P_r the Prandtl number and E_c the Eckert number, ρ the density of the fluid and $f_1, f_2,$ are the components of the external force. In (2) and (3) λ^* is a non-dimensional

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number, and in case of motion under the gravitational force, λ^*, is called the Froude number (F_r).

We define the following functions:

$$\omega = v_x - u_y \tag{6}$$

$$L = p + \frac{(u^2+v^2)}{2} - \frac{2\mu u_x}{R_e} \tag{7}$$

In term of these functions, the system (1-5) is replaced by the following system:

$$v_x + u_y = 0 \tag{8}$$

$$-v\omega = -L_x + \frac{[\mu(u_y+v_x)]_y}{R_e} + F_1 \tag{9}$$

$$u\omega = -L_y - \frac{4(\mu u_x)_y}{R_e} + \frac{[\mu(u_y+v_x)]_x}{R_e} + F_2 \tag{10}$$

$$\omega = v_x - u_y \tag{11}$$

$$uT_x + vT_y = \frac{1}{R_e P_r}(T_{xx} + T_{yy}) + \frac{E_c}{R_e}[2\mu(u_x{}^2 + v_y^2) + \mu(u_y + v_x)^2] \tag{12}$$

$$\mu = \mu(T) \tag{13}$$

of six equations in eight unknowns $u,, v, L, \mu, T, \omega, F_1, F_2$ as functions of x, y. The advantage of this system over the original system is that the order of (2) and (3) has decreased from two to one. In (9) and (10) for convenience we have put

$$F_1 = \lambda^* f_1, F_2 = \lambda^* f_2$$

Equation (8) implies the existence of a stream function $\psi(x,y)$ such that

$$u = \psi_y, v = -\psi_x \tag{14}$$

Let $\psi(x,y)$ = constant defines the family of streamlines. Let us assume $\phi(x,y)$ = constant to be some arbitrary family of curves such that it generates with $\psi(x,y)$ = constant a curvilinear net (ϕ,ψ) in the physical plane.

Let

$$x = x(\phi,\psi), y = y(\phi\psi) \tag{15}$$

define the curvilinear net in (x,y) - plane and let the squared element of arc length along any curve be

$$ds^2 = E(\phi,\psi)d\phi^2 + 2F(\phi,\psi)d\phi d\psi + G(\phi,\psi)d\psi^2 \tag{16}$$

where

$$\left.\begin{aligned} E &= x_\phi^2 + y_\phi^2 \\ F &= x_\phi x_\psi + y_\phi y_\psi \\ G &= x_\psi^2 + y_\psi^2 \end{aligned}\right\} \tag{17}$$

Equation (15) can be solved to obtain

$$\phi = \phi(x,y), \psi = \psi(x,y) \tag{18}$$

such that

$$\left.\begin{aligned} x_\phi &= J\psi_y, x_\psi = -J\phi_y \\ y_\phi &= -J\psi_x, y_\psi = J\phi_x \end{aligned}\right\} \tag{19}$$

provided that $0 < |J| < \infty$, where J is the transformation Jacobian, and is defined as

$$J = x_\phi y_\psi - x_\psi y_\phi \tag{20}$$

If α is the angle of inclination of the tangent to the coordinate line ψ = constant directed in the sense of increasing ϕ, we have from differential geometry, the following results:

$$\left.\begin{aligned} &J = \pm W \\ &x_\phi = \sqrt{E}Cos\alpha, x_\psi = \frac{FCos\alpha - JSin\alpha}{\sqrt{E}} \\ &y_\phi = \sqrt{E}Sin\alpha, y_\psi = \frac{FSin\alpha + JCos\alpha}{\sqrt{E}} \\ &\alpha_\phi = \frac{J}{E}\Gamma_{11}^2, \alpha_\psi = \frac{J}{E}\Gamma_{12}^2 \end{aligned}\right\} \tag{21}$$

where

$$\left.\begin{aligned} \Gamma_{11}^2 &= \frac{[-FE_\phi + 2EF_\phi - EE_\psi]}{2W^2} \\ \Gamma_{12}^2 &= \frac{[EG_\phi - FE_\psi]}{2W^2} \end{aligned}\right\} \tag{22}$$

$$W = \sqrt{EG - F^2}$$

The three functions E, F, G of ϕ,ψ satisfy the Gauss equation:

$$K = \frac{\left(\frac{W\Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\Gamma_{12}^2}{E}\right)_\phi}{W} = 0 \tag{23}$$

where K is the Gaussian curvature.

III. TRANSFORMATION OF BASIC FLOW EQUATIONS IN THE STREAMLINED COORDINATE SYSTEM (ϕ,ψ)

If the arbitrary curve $\phi(x,y)$ = constant and the streamlines $\psi(x,y)$ = constant generate a curvilinear net in the physical frame, the system of equations (8-13) is transformed to the following system:

$$q = \frac{\sqrt{E}}{W} \tag{24}$$

$$\begin{aligned} J\omega = &-JL_\psi + \left(\frac{(F^2 - J^2)Sin2\alpha}{2E} + \frac{FJCos2\alpha}{E}\right)A_\phi - (FSin\alpha Cos\alpha + JCos^2\alpha)A_\psi \\ &+ \left(\frac{2FJSin2\alpha - F^2Cos2\alpha + J^2Cos2\alpha}{E}\right)M_\phi + (FCos\alpha - JSin2\alpha)M_\psi \\ &+ \frac{J}{\sqrt{E}}\{F(F_1Cos\alpha + F_2Sin\alpha) + J(F_2Cos\alpha + F_1Sin\alpha)\} \end{aligned} \tag{25}$$

$$\begin{aligned} 0 = &-JL_\phi + (FSin\alpha Cos\alpha - JSin^2\alpha)A_\phi - ESin\alpha Cos\alpha A_\psi \\ &+ (JSin2\alpha - FCos2\alpha)M_\phi + ECos2\alpha M_\psi + J\sqrt{E}(F_1Cos\alpha + F_2Sin\alpha) \end{aligned} \tag{26}$$

$$\frac{\left(\frac{GT_\phi}{J} - \frac{FT_\psi}{J}\right)_\phi + \left(\frac{ET_\psi}{J} - \frac{FT_\phi}{J}\right)_\psi}{JR_eP_r} = -\frac{E_cR_e(A^2 + 4M^2)}{4\mu} + \frac{qT_\phi}{\sqrt{E}} \tag{27}$$

$$\omega = \frac{\left(\frac{F}{W}\right)_\phi - \left(\frac{E}{W}\right)_\psi}{W} \tag{28}$$

$$K = \frac{\left(\frac{W\Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\Gamma_{12}^2}{E}\right)_\phi}{W} = 0 \tag{29}$$

$$\mu = \mu(T) \tag{30}$$

which in ϕ and ψ are considered as independent variables. This is a system of seven equations in ten unknown functions

$E, F, G, W, L, T, q, \mu, F_1, F_2$. In (25-27), the functions A and M are given by

$$A = \frac{4\mu}{JR_e}\left[\begin{array}{l}\left(\frac{FSin\alpha + JCos\alpha}{\sqrt{E}}\right)(q_\phi Cos\alpha - qSin\alpha\alpha_\phi) - \\ \sqrt{E}Sin\alpha(q_\psi Cos\alpha - qSin\alpha\alpha_\psi)\end{array}\right] \tag{31}$$

$$M = \frac{\mu}{JR_e}[q_\phi\left(\frac{-FCos2\alpha + JSin2\alpha}{\sqrt{E}}\right) + q_\psi\sqrt{E}Cos2\alpha + q\left(\frac{FSin2\alpha + JCos2\alpha}{\sqrt{E}}\right)\alpha_\phi - q\sqrt{E}Sin2\alpha\alpha_\psi] \tag{32}$$

Recently Labropulu and Chandna [4] presented a new approach for the determination of exact solutions of steady plane infinitely conducting MHD aligned flows. In their approach (ξ,ψ) –coordinate net or (η,ψ) – coordinate is used to obtain exact solution of these flows where coordinates $\psi(x, y)$ is the stream function and $\xi(x,y)$ and $\eta(x,y)$ are the real and imaginary parts of an analytic function $\varpi = \xi(x, y) + i\eta(x, y)$, Labropulu and Chandna following Martin's transform their flow equations in (ϕ,ψ)-system where ψ = constant is represents family of streamline and ϕ = constant is an arbitrary family of curves. The system of flow equations becomes undetermined due to arbitrariness of the coordinate lines ϕ= constant. Labropulu and Chandna made the system determinate by taking $\phi = \xi(x, y)$ or $\phi = \eta(x, y)$ where $\xi(x, y)$ and $\eta(x, y)$ are real and imaginary part of the analytical functions ϖ as outlined blow:

Let

$$\varpi = \xi + i\eta \tag{33}$$

be an analytic function of $z = x + iy$ where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$. Since ϖ is analytic function of x and y, then real and imaginary part must satisfy Cauchy-Riemann equations:

$$\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}, \frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x} \tag{34}$$

The equations

$$\xi = \xi(x, y), \eta = \eta(x, y) \tag{35}$$

can be solved to get

$$x = x(\xi,\eta), y = y(\xi,\eta) \tag{36}$$

such that

$$\frac{\partial x}{\partial\xi} = J^*\frac{\partial\eta}{\partial y}, \frac{\partial x}{\partial\eta} = -J^*\frac{\partial\xi}{\partial y}, \frac{\partial y}{\partial\xi} = -J^*\frac{\partial\eta}{\partial x}, \frac{\partial y}{\partial\eta} = J^*\frac{\partial\xi}{\partial x} \tag{37}$$

provided that $0 < \left|J^*\right| < \infty$ where J^* is given by

$$J^* = \frac{\partial(x, y)}{\partial(\xi,\eta)} = \frac{\partial x}{\partial\xi}\frac{\partial y}{\partial\eta} - \frac{\partial x}{\partial\eta}\frac{\partial y}{\partial\xi} \tag{38}$$

Employing (35), (36) and (39) in $ds^2 = dx^2 + dy^2$, we get

$$J^* = \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 = \left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2 \tag{39}$$

Equation (39), employing (35) and (37), yields

$$ds^2 = J^*(d\xi^2 + d\eta^2) \tag{40}$$

To analyze whether a family of curves $\frac{\eta - f(\xi)}{g(\xi)} =$ Constant, can or can't be streamlines in (ξ,ψ) – coordinate net, they assumed affirmative so that their exist some function $\gamma(\psi)$ such that

$$\frac{\eta - f(\xi)}{g(\xi)} = \gamma(\psi), \gamma'(\psi) \neq 0 \tag{41}$$

$$ds^2 = J^*\{1 + [f'(\xi) + g'(\xi)\gamma(\psi)]^2\}d\xi^2 + 2J^*\{f'(\xi) + g'(\xi)\gamma(\psi)\} g(\xi)\gamma'(\psi)d\xi d\psi + J^*g^2(\xi)\gamma'^2 d\psi^2 \tag{42}$$

$$E = J^*\left\{1 + [f'(\xi) + g'(\xi)\gamma(\psi)]^2\right\} \tag{43}$$

$$G = J^*g^2(\xi)\gamma'^2(\psi) \tag{44}$$

$$F = J^*[f'(\xi) + g'(\xi)\gamma(\psi)]g(\xi)\gamma'(\psi) \tag{45}$$

$$W = J^*g(\xi)\gamma'(\psi) \tag{46}$$

$$J = J^*g(\xi)\gamma'(\psi) \tag{47}$$

Similarly to analyze whether a family of curves $\frac{\xi - k(\eta)}{m(\eta)} =$ Constant, can or can't be streamlines in (η,ψ)-coordinate net, they assumed affirmative so that their exist some function $\gamma(\psi)$ such that

$$\frac{\xi - k(\eta)}{m(\eta)} = \gamma(\psi), \gamma'(\psi) \neq 0 \tag{48}$$

$$E = J^*\left\{1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2\right\} \tag{49}$$

$$G = J^*m^2(\eta)\gamma'^2(\psi) \tag{50}$$

$$F = J^*[k'(\eta) + m'(\eta)\gamma(\psi)]m(\eta)\gamma'(\psi) \tag{51}$$

$$W = J^*m(\eta)\gamma'(\psi) \tag{52}$$

$$J = -J^*m(\eta)\gamma'(\psi) \tag{53}$$

For both family of streamlines there exists some function $\gamma(\psi)$ [4] such that

$$\eta - f(\xi) = g(\xi)\gamma(\psi), \gamma'(\psi) \neq 0 \tag{54}$$

$$\xi - k(\eta) = m(\eta)\gamma(\psi), \gamma'(\psi) \neq 0 \tag{55}$$

Now in the next section, we determine the solutions of the (24-30) by assuming $\phi = \xi$ or $\phi = \eta$ and utilizing (42-55) for the family of streamlines in (ξ,ψ) and (η,ψ) coordinate net, respectively.

IV. Solutions

In this section, we assume analytic function $\varpi = \xi + i\eta$ and determine the solutions of the (24-30) for the flow characterized by the family of the stream lines

$\frac{\xi - k(\eta)}{m(\eta)} =$ Constant & $\frac{\eta - f(\eta)}{g(\xi)} =$ Constant

In the absence of external force.

(i) Assume

$$\varpi = \xi + i\eta = \ln z \tag{56}$$

where $z = x + iy$. The (56) yields.

$$\left.\begin{array}{l}\xi = \frac{1}{2}\ln(x^2 + y^2) \\ \eta = \tan^{-1}\left(\frac{y}{x}\right)\end{array}\right\} \tag{57}$$

OR

$$\left.\begin{array}{l}x = e^\xi Cos\eta \\ y = e^\xi Sin\eta\end{array}\right\} \tag{58}$$

A Example1 (Flows with ξ = constant as streamlines)

We let [4]

$$\xi = \gamma(\psi), \gamma'(\psi) \neq 0 \tag{59}$$

where $\gamma(\psi)$ is an unknown function and ξ is given by (57). Comparing (59) with (55), we get

$$k(\eta) = 0, m(\eta) = 1 \tag{60}$$

Utilizing (60) in (49-53), we get

$$\left.\begin{aligned} &E = J^* \\ &F = 0 \\ &G = J^* \gamma'^2(\psi) \\ &J = -J^* \gamma'(\psi) \\ &W = J^* \gamma'(\psi) \end{aligned}\right\} \tag{61}$$

where

$$J^* = e^{2\gamma(\psi)} \tag{62}$$

Equations (24-29), utilizing (61) and (62), become

$$q = \frac{1}{\gamma'(\psi) e^{\gamma(\psi)}} \tag{63}$$

$$\omega = -L_\psi - \frac{\gamma' Sin 2\eta}{2} A_\eta - Sin^2 \eta A_\psi + \gamma' Cos 2\eta M_\eta + Sin 2\eta M_\psi \tag{64}$$

$$0 = -L_\eta - A_\eta Cos^2 \eta - \frac{Sin\eta Cos\eta}{\gamma'} A_\psi - Sin 2\eta M_\eta + \frac{Cos 2\eta}{\gamma'} M_\psi \tag{65}$$

$$-\gamma'^3 T_{\eta\eta} + \gamma'^2 R_e P_r T_\eta - \gamma' T_{\psi\psi} + \gamma'' T_\psi = \frac{\gamma'^3 e^{2\gamma} E_c P_r R_e^2 \left(A^2 + 4M^2\right)}{4\mu} \tag{66}$$

$$\omega = \frac{\gamma''}{\gamma'^3 e^{2\gamma}} \tag{67}$$

where in (64-66), the functions A and M are given by

$$A = \frac{2\mu}{R_e e^{2\gamma(\psi)} \gamma'(\psi)} \left(2 + \frac{\gamma''(\psi)}{\gamma'^2(\psi)}\right) Sin 2\eta \tag{68}$$

$$M = -\frac{\mu}{R_e e^{2\gamma(\psi)} \gamma'(\psi)} \left(2 + \frac{\gamma''(\psi)}{\gamma'^2(\psi)}\right) Cos 2\eta \tag{69}$$

In order to determine the solutions of (64-66), we make use of the compatibility condition $L_{\eta\psi} = L_{\psi\eta}$ and this yield

$$\gamma'^2 Z_{\eta\eta} - \gamma' \left(2 - \frac{\gamma''(\psi)}{\gamma'^2(\psi)}\right) Z_\psi - Z_{\psi\psi} = 0 \tag{70}$$

where

$$Z = \frac{\mu}{R_e e^{2\gamma} \gamma'} \left(2 + \frac{\gamma''}{\gamma'^2}\right) \tag{71}$$

Equation (70) is the equation which the viscosity μ and $\gamma(\psi)$ must satisfy.

Equation (70) possesses many solutions and we consider only those solutions for which the exact solution of (66) can be determined. These solutions are for the following cases:

Case I $\quad \gamma'' = 0$

Case II $\quad \gamma'' \neq 0$

We study these cases separately

Case I

When $\gamma'' = 0$ we get

$$\gamma = a\psi + b \tag{72}$$

The (70) provides

$$a^2 \chi_{\eta\eta} - 2a\chi_\psi - \chi_{\psi\psi} = 0 \tag{73}$$

where

$$\chi(\eta, \psi) = \frac{2e^{-2\gamma} \mu(\eta, \psi)}{aR_e} \tag{74}$$

Two solutions of (73) are determined which employing (74) give us

$$\mu = \begin{cases} \dfrac{at_1 R_e}{2e^{-2\gamma}} & (75) \\[2ex] \dfrac{aR_e e^{2\gamma}}{2}\left(\dfrac{t_2 \eta^2}{2a^2} + t_3\eta + \dfrac{t_2\psi}{2a} + t_6 e^{-2a\psi} + t_7\right) & (76) \end{cases}$$

where $t_1(\neq 0), t_2(\neq 0), t_3(\neq 0), t_4, t_5, t_6, t_7$ are arbitrary constant, and $t_7 = t_4 + \frac{t_5}{2a} - \frac{t_2}{4a^2}$

The temperature distribution T for μ given by (75) satisfies the equation

$$a^2 T_{\eta\eta} + T_{\psi\psi} - aR\ P_r T_\eta = -2aR_e P_r E_c t_1 \tag{77}$$

whose solution is

$$T = -aR_e P_r E_c t_1 \psi^2 - \frac{t_8 \psi^2}{2} + t_9 \psi - \frac{t_8 \eta}{aR_e P_r} + t_{12} e^{\frac{R_e P_r \eta}{a}} + t_{13} \tag{78}$$

where $t_{13} = t_{10} - \frac{t_8}{R_e^2 P_r^2} - \frac{t_{11} a}{R_e P_r}$

For μ given by (76), the temperature distribution T satisfies the equation

$$a^2 T_{\eta\eta} + T_{\psi\psi} - aR_e P_r T_\eta = -2aR_e P_r E_c \chi(\eta, \psi) \tag{79}$$

The solution of (79) is

$$T = e^{\frac{R_e P_r \eta}{a}} \left(\int e^{-\frac{R_e P_r \eta}{a}} \left(-\frac{2R_e P_r E_c}{a} \int y_2(\eta) d\eta + t_{16} \right) d\eta + t_{17} \right) - 2aR_e P_r E_c \iint y_1(\psi) d\psi d\psi + t_{14}\psi + t_{15} \tag{80}$$

where

$$y_1(\psi) = \frac{t_2 \psi}{2a} - \frac{t_2}{4a^2} + \frac{t_5}{2a} + t_6 e^{-2a\psi} \tag{81}$$

$$y_2(\eta) = \frac{t_2 \eta^2}{2a^2} + t_3\eta + t_4 \tag{82}$$

Case II

When $\gamma'' \neq 0$, the (70) possesses trivial and non-trivial solutions.

Now $Z = 0$ is the trivial solution of (70) and provides

$$2 + \frac{\gamma''}{\gamma'^2} = 0 \tag{83}$$

Equation (83) yields

$$\gamma(\psi) = \frac{1}{2} \ln(2\psi + c_1) + c_2 \tag{84}$$

where c_1, c_2 are arbitrary constants. Equations (68) and (69), employing (83), yield

$$\left.\begin{aligned} &A = 0 \\ &M = 0 \end{aligned}\right\} \tag{85}$$

Equations (64) and (65), utilizing (85) yields

$$L = \text{constant} = c_3 \text{ (say).} \tag{86}$$

We note that in this case the viscosity function μ is arbitrary.

Now the (66), on using (84) and (85), becomes

$$T_{\eta\eta} - (2\psi + c_1) R_e P_r T_\eta + (2\psi + c_1)^2 T_{\psi\psi} + 2(2\psi + c_1) T_\psi = 0 \tag{87}$$

The solution of (87) is

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$$T = c_4\eta + \frac{c_4 R_e P_r}{4}\left(2\psi + c_1\right) + c_5 \ln\left(2\psi + c_1\right) + c_6 \tag{88}$$

where $c_4(\neq 0), c_5(\neq 0), c_6$ are arbitrary constants.For a non-trivial solution of (70) we let

Z = constant = c_7 (say).

Then (71) yields

$$\mu = \frac{c_7 R_e \gamma' e^{2\gamma}}{\left(2 + \frac{\gamma''}{\gamma'^2}\right)}, 2 + \frac{\gamma''}{\gamma'^2} \neq 0 \tag{89}$$

Equations (64) and (65) on using $Z = c_7$, yield

$$L_\psi = -\omega \tag{90}$$

$$L_\eta = -2c_7\left(1 + Cos2\eta\right) \tag{91}$$

Equations (90) and (91) give

$$L = -\int \frac{\gamma''}{\gamma'^3 e^{2\gamma}} d\psi - 2c_7\left(\eta + \frac{Sin2\eta}{2}\right) + c_8 \tag{92}$$

where $c_7(\neq 0)$ and c_8 are arbitrary constants.

The temperature distribution in this case, satisfies

$$\gamma'^3 T_{\eta\eta} - \gamma'^2 R_e P_r T_\eta + \gamma' T_{\psi\psi} - \gamma'' T_\psi = -\gamma'^2 c_7 E_c P_r R_e \left(2 + \frac{\gamma''}{\gamma'^2}\right) \tag{93}$$

whose solution is

$$T = c_9\eta + \int e^{\int \frac{\gamma''}{\gamma'} d\psi}\left(\int e^{\int \frac{\gamma''}{\gamma'} d\psi}\left(c_9\gamma' R_e P_r - c_7\gamma'^2 E_c P_r R_e\left(2 + \frac{\gamma''}{\gamma'^2}\right)\right) d\psi\right) d\psi \tag{94}$$

$$+ c_{10}\int e^{\int \frac{\gamma''}{\gamma'} d\psi} d\psi + c_{11}$$

where $c_9(\neq 0), c_{10}, c_{11}$ are arbitrary constants. We mention that the function $\gamma(\psi)$ is arbitrary in this case, and therefore we can construct an infinite set of solutions to the flow equations.

B Example-2 (Flows with $\eta =$ constant as streamlines)

Assume [4]

$$\eta = \gamma(\psi) \tag{95}$$

where $\gamma(\psi)$ is unknown function and η is given by (57).Equations (96) and (54), give

$$f(\xi) = 0, g(\xi) = 1 \tag{96}$$

Equations (43-47), employing (96) yield

$$\left.\begin{aligned} E &= e^{2\xi} \\ F &= 0 \\ G &= e^{2\xi}\gamma'^2(\psi) \\ J &= e^{2\xi}\gamma'(\psi) \\ W &= e^{2\xi}\gamma'(\psi) \end{aligned}\right\} \tag{97}$$

Equations (24-29), employing (97), give

$$q = \frac{1}{e^{\xi}\gamma'(\psi)} \tag{98}$$

$$\omega = -L_\psi - Cos^2\gamma(\psi)A_\psi - Sin2\gamma(\psi)M_\psi - \frac{\gamma'(\psi)Sin2\gamma(\psi)}{2}A_\xi \tag{99}$$

$$+ \gamma'(\psi)Cos2\gamma(\psi)M_\xi$$

$$0 = -L_\xi - \frac{Cos\gamma(\psi)Sin\gamma(\psi)}{\gamma'(\psi)}A_\psi + \frac{Cos2\gamma(\psi)}{\gamma'(\psi)}M_\psi - Sin^2\gamma(\psi)A_\xi \tag{100}$$

$$+ Sin2\gamma(\psi)M_\xi$$

$$\gamma'(\psi)T_{\xi\xi} + \left(\frac{T_\psi}{\gamma'(\psi)}\right)_\psi = -\frac{e^{2\xi}\gamma'(\psi)E_c P_r R_e^2\left(A^2 + 4M^2\right)}{4\mu} + P_r R_e T_\xi \tag{101}$$

$$\omega = \frac{\gamma''(\psi)}{e^{2\xi}\gamma'^3(\psi)} \tag{102}$$

where the functions A and M are given by

$$A = \frac{4\mu}{e^{2\xi}\gamma'(\psi)R_e}\left(-Cos2\gamma(\psi) + \frac{\gamma''(\psi)}{2\gamma'^2(\psi)}Sin2\gamma(\psi)\right) \tag{103}$$

$$M = -\frac{2\mu}{e^{2\xi}\gamma'(\psi)R_e}\left(Sin2\gamma(\psi) + \frac{\gamma''(\psi)}{2\gamma'^2(\psi)}Cos2\gamma(\psi)\right) \tag{104}$$

Equations (99) and (100), employing (102-104), can be rewritten as

$$\frac{\gamma''}{e^{2\xi}\gamma'^3} = -L_\psi - 4\gamma' Sin2\gamma X - \gamma' Y_\xi + 4Cos^2\gamma X_\psi \tag{105}$$

$$- 4\gamma' Cos^2\gamma Y - Sin2\gamma Y_\psi$$

$$0 = -\gamma' L_\xi - 4\gamma' Sin^2\gamma X_\xi - 4\gamma' X - \gamma' Sin2\gamma Y_\xi - Y_\psi \tag{106}$$

where

$$X = \frac{\mu}{R_e e^{2\xi}\gamma'} \tag{107}$$

$$Y = \frac{\gamma''}{R_e e^{2\xi}\gamma'^3}\mu \tag{108}$$

Proceeding in the same manner as in example–1, a solution of (105) and (106) is

$$\mu = aR_e e^{\xi}\left(\int e^{\xi} Z_1(\xi)d\xi + Z_2(\psi)\right) \tag{109}$$

$$L = e^{-\xi}\left(g_1 + 4Cos^2\gamma Z_2(\psi)\right) + 2Cos2\gamma Z_3(\xi) - g_1 e^{-\xi} \tag{110}$$

$$-2Z_3(\xi) - 4\int Z_3(\xi)d\xi + g_2$$

provided $\gamma = a\psi + b$. In (109) and (110), $Z_1(\xi), Z_2(\psi)$ and $Z_3(\xi)$ are arbitrary functions and g_1, g_2 are arbitrary constants.

The temperature distribution T *satisfies* the equation

$$a^2 T_{\xi\xi} + T_{\psi\psi} - aP_r R_e T_\xi = -\frac{4E_c P_r \mu}{e^{2\xi}} \tag{111}$$

where μ is given by (109). The solution of (111) is

$$T = -\frac{4E_c P_r R_e e^{\frac{P_r R_e \xi}{a}}}{a}\left(\int e^{-\frac{P_r R_e}{a}}\left(\int Z_3(\xi)d\xi\right)\right) - \frac{a_1 a}{P_r R_e} + a_2 e^{\frac{P_r R_e \xi}{a}}$$

$$- 4aE_c P_r R_e \int\left(\int Z_2(\psi)d\psi\right)d\psi - a_3\psi + a_4, fora = -R_e P_r$$

$$-\frac{4E_c P_r R_e e^{\frac{P_r R_e \xi}{a}}}{a}\left(\int e^{-\frac{P_r R_e \xi}{a}}\left(\int Z_3(\xi)d\xi\right)d\xi\right) - \frac{a_1 a}{P_r R_e} + a_2 e^{\frac{P_r R_e \xi}{a}}$$

$$+ a_5 Cos\sqrt{a^2 + aP_r R_e}\psi + a_6 Sin\sqrt{a^2 + aP_r R_e}\psi$$

$$+ \frac{1}{\sqrt{a^2 + aP_r R_e}}\left(\begin{aligned} &4a\left(Cos\left(\sqrt{a^2 + aP_r R_e}\psi\right)\int Z_2(\psi)Sin\left(\sqrt{a^2 + aP_r R_e}\psi\right)d\psi\right) \\ &-\left(\int Z_2(\psi)Cos\left(\sqrt{a^2 + aP_r R_e}\psi\right)d\psi\right)Sin\left(\sqrt{a^2 + aP_r R_e}\psi\right) \\ &E_c P_r R_e \end{aligned}\right) \tag{112}$$

for $a \neq -R_e P_r$

where $a_1, a_2, a_3, a_4, a_5, a_6$ are all arbitrary constants. We note that the expressions for μ, L and T involve arbitrary functions, and this allows us to construct a large number of solution to the flow equations.

(2) Assume

$$\varpi = \xi + i\eta = a^* z + b \tag{113}$$

where $a^* = a_1 + ia_2, b^* = b_1 + ib_2$

C Example 3 (Flows with $\eta-\lambda\xi=$ constant as streamlines)

Proceeding in the same manner as in examples 1 and 2, we find

$$\left.\begin{aligned} E &= \frac{1+\lambda^2}{a_1^2+a_2^2} \\ G &= \frac{\gamma'^2(\psi)}{a_1^2+a_2^2} \\ F &= \frac{\lambda\gamma'(\psi)}{a_1^2+a_2^2} \\ W &= \frac{\gamma'(\psi)}{a_1^2+a_2^2} \\ J^* &= \frac{1}{a_1^2+a_2^2} \\ J &= \frac{\gamma'(\psi)}{a_1^2+a_2^2} \end{aligned}\right\} \tag{114}$$

For this example, in (24-29), employing (114), become

$$q = \frac{\sqrt{(1+\lambda^2)(a_1^2+a_2^2)}}{\gamma'(\psi)} \tag{115}$$

$$\frac{(1+\lambda^2)\gamma''(\psi)}{\gamma'^2(\psi)} = -\gamma'(\psi)\beta_5 L_\psi + \beta_7\gamma'^2(\psi)A_\xi - \beta_6\gamma'(\psi)A_\psi + \beta_8\gamma'^2(\psi)M_\xi + \beta_9\gamma'(\psi)M_\psi \tag{116}$$

$$0 = -\gamma'(\psi)\beta_5 L_\xi + \beta_1\gamma'(\psi)A_\xi + \beta_3 A_\psi + \beta_2\gamma'(\psi)M_\xi + \beta_4 M_\psi \tag{117}$$

$$\left(\gamma'(\psi)T_\xi - \lambda T_\psi\right)_\xi + \left(\frac{1+\lambda^2}{\gamma'(\psi)}T_\psi - \lambda T_\xi\right)_\psi - P_r R_e T_\xi = -\frac{\beta_5\gamma'(\psi)E_c P_r R_e^2\left(A^2+4M^2\right)}{4\mu} \tag{118}$$

where

$$A = -\frac{4a_1 a_2\gamma''(\psi)}{R_e\gamma'^3(\psi)}\mu \tag{119}$$

$$M = -\frac{\left(a_1^2-a_2^2\right)\gamma''(\psi)}{R_e\gamma'^3(\psi)}\mu \tag{120}$$

and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9$ are given in ppendix-*B*.

Equations (116-118), employing (119) and (120), become

$$L_\psi = -\beta_{14}\frac{\gamma''}{\gamma'^3} + \beta_{12}\gamma' X_\xi + \beta_{13} X_\psi \tag{121}$$

$$L_\xi = -\beta_{10} X_\xi - \beta_{11}\frac{X_\psi}{\gamma'} \tag{122}$$

$$\gamma' T_{\xi\xi} - 2\lambda T_{\xi\psi} + \frac{1+\lambda^2}{\gamma'}T_{\psi\psi} - \frac{\gamma''(1+\lambda^2)}{\gamma'^2}T_\psi - P_r R_e T_\xi = -\frac{E_c P_r R_e^2\gamma' X^2}{\mu} \tag{123}$$

where

$$X(\xi,\psi) = \frac{\gamma''(\psi)}{R_e\gamma'^3(\psi)}\mu(\xi,\psi) \tag{124}$$

and $\beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}$ are given in appendix *B*.

On eliminating the generalized energy function *L* from (121) and (122), we obtain

$$\frac{\beta_{11}}{\gamma'}X_{\psi\psi} + \gamma'\beta_{12}X_{\xi\xi} - \frac{\beta_{11}\gamma''}{\gamma'^2}X_\psi + (\beta_{10}+\beta_{13})X_{\xi\psi} = 0 \tag{125}$$

Equation (125) is the compatibility equations for example (3). We found that this compatibility equation possesses solutions for the following possible cases for which equation (123) is exactly solvable.

Case I $\gamma''(\psi) = 0$

Case II $\gamma''(\psi) \neq 0$

We study these two cases separately as follows.

Case-I

When $\gamma''(\psi) = 0, \gamma(\psi) = a\psi + b$. Equation (125) is identically satisfied. The viscosity function μ is arbitrary and the generalized energy function L turns out to be constant.

The solution of (123), for any value of λ, is given by

$$T = m_1\xi\psi + \frac{(2\lambda a m_1 - m_2)\psi^2}{2(1+\lambda^2)} + m_3\psi + m_4 + e^{\frac{P_r R_e\xi}{a}}\int\left(\frac{m_2\xi}{a^2} + m_5\right)e^{-\frac{P_r R_e\xi}{a}}d\xi + m_6 e^{\frac{P_r R_e\xi}{a}} \tag{126}$$

where $m_1, m_2, m_3, m_4, m_5, m_6$ are all non-zero arbitrary constants.

Case-II

When $\gamma''(\psi) \neq 0$, the function X in (124) can either be considered constant or non-constant. When, $X =$ constant$= m_7$ (say), the equation is identically satisfied and therefore

$$\mu = \frac{m_7 R_e\gamma'^3(\psi)}{\gamma''(\psi)} \tag{127}$$

Equation (123), employing (127), yields

$$\gamma' T_{\xi\xi} - 2\lambda T_{\xi\psi} + \frac{1+\lambda^2}{\gamma'}T_{\psi\psi} - \frac{\gamma''(1+\lambda^2)}{\gamma'^2}T_\psi - P_r R_e T_\xi = -\frac{E_c P_r R_e m_7\gamma'^4}{\gamma''}\left(\frac{\gamma''}{\gamma'^3}\right)^2 \tag{128}$$

whose solution is

$$T = \int e^{\int\frac{\gamma''}{\gamma'}d\psi}\left(\int e^{-\int\frac{\gamma''}{\gamma'}d\psi}\left(\frac{\gamma'}{1+\lambda^2}\right)\left(P_r R_e m_8 - \frac{E_c P_r R_e m_7\gamma'^4}{\gamma''}\left(\frac{\gamma''}{\gamma'^3}\right)^2\right)d\psi + m_9\right)d\psi + m_8\xi + m_{10} \tag{129}$$

where m_7, m_8, m_9, m_{10} are all non zero arbitrary constants. We note that in this case the function $\gamma(\psi)$ is arbitrary. The generalized energy function L can easily be determined from (121) and (122).

When *X* is not a constant, the solution of (125) is

$$X = m_{11}\xi + m_{12}\int e^{\frac{\beta_1}{\beta_{11}}\int\frac{\gamma''}{\gamma'}d\psi}d\psi + m_{13} \tag{130}$$

where m_{11}, m_{12}, m_{13} are non-zero arbitrary constants. The viscosity μ is given by

$$\mu = \frac{R_e\gamma'^3(\psi)}{\gamma''(\psi)}\left(m_{11}\xi + X_1(\psi)\right) \tag{131}$$

Equation (123), in this case becomes.

$$T_{\xi\xi} - \frac{2\lambda}{\gamma'}T_{\xi\psi} + \frac{1+\lambda^2}{\gamma'}T_{\psi\psi} - \frac{\gamma''(1+\lambda^2)}{\gamma'^3}T_\psi - \frac{P_r R_e}{\gamma'}T_\xi = -\frac{E_c P_e R_e\gamma''}{\gamma'^3}\left(m_{11}\xi + X_1(\psi)\right) \tag{132}$$

The two solutions of (132) are obtained and these are

(i)
$$T=\int\left(\gamma'\int\frac{\gamma'}{1+\lambda^2}\left(\begin{array}{l}-\dfrac{E_cP_rR_e\gamma''}{\gamma'^3}X_1(\psi)+\dfrac{P_rR_em_{14}}{\gamma'}\psi\\ +\dfrac{(P_rR_em_{16}+2\lambda m_{14})}{\gamma'}\end{array}\right)d\psi\right)d\psi \tag{133}$$
$$+\psi m_{17}+m_{18}+m_{15}\xi^2+m_{16}\xi+m_{14}\xi\psi$$
$$for\quad m_{14}=\frac{E_cR_eP_rm_{11}}{1+\lambda^2},m_{15}=0$$

(ii) and
$$T=e^{-\int\frac{2m_{15}P_rR_e}{(1+\lambda^2)m_{14}m_{19}+(2\psi m_{15}-E_cm_{11}m_{19})P_rR_e}d\psi}$$
$$(\int e^{\int\frac{2m_{15}P_rR_e}{(1+\lambda^2)m_{14}m_{19}+(2\psi m_{15}-E_cm_{11}m_{19})P_rR_e}d\psi}$$
$$\left(E_cR_eP_rX_1(\psi)+\left(\begin{array}{l}-2m_{15}+(2\lambda m_{14}+(\psi m_{14}+m_{16})P_rR_e))\\ \left(m_{19}+\dfrac{2\psi m_{15}P_rR_e}{(1+\lambda^2)m_{14}-E_cm_{11}P_rR_e}\right)\end{array}\right)\right)$$
$$\Big/\left(\left(1+\lambda^2\right)\left(m_{19}+\frac{2\psi m_{15}P_rR_e}{(1+\lambda^2)m_{14}-E_cm_{11}P_rR_e}\right)^2\right)d\psi)d\psi$$
$$m_{20}\int e^{-\int\frac{2m_{15}P_RR_e}{(1+\lambda^2)m_{14}m_{19}+(2\psi m_{15}-E_cm_{11}m_{19})P_rR_e}d\psi}d\psi$$
$$+m_{21}+\psi m_{17}+m_{15}\xi^2+m_{16}\xi+m_{14}\xi\psi$$
$$form_{14}=\frac{m_{11}E_cP_rR_e}{1+\lambda^2},m_{15}=0 \qquad 134)$$

For solution given by (133), the function γ is arbitrary, and for (134) it is given by
$$\gamma=\frac{(1+\lambda^2)m_{14}-E_cm_{11}P_rR_e}{2P_rR_em_{15}}\ln\left(\frac{2P_rR_em_{15}\psi}{(1+\lambda^2)m_{14}-E_cm_{11}P_rR_e}+m_{19}\right)+m_{20} \tag{135}$$

The constants $m_{16},m_{17},m_{18},m_{19},m_{29},m_{21}$ are all non zero arbitrary constants. In (131) and (132) $X_1(\psi)$ is given by
$$X_1(\psi)=m_{12}\int e^{\frac{\beta_1}{\beta_{11}}\int\frac{\gamma''}{\gamma'}d\psi}d\psi+m_{13} \tag{136}$$

The generalized energy function L can easily be determined in this case in the same manner as in examples 1 and 2.

V. Conclusion

Labropulu and Chandna approach is extended to study the steady plane flows of an incompressible fluid of variable viscosity for arbitrary state equation. The exact solutions to flow equations are determined. The solutions involve arbitrary function(s) indicating that flow equations possess an infinite set of solutions.

Appendix A

Martin's [2] introduced curvilinear coordinates ϕ,ψ in which curves $\psi=$ constant are the streamlines and the curves ϕ = constant are left arbitrary so that the physical coordinates can be replaced by ϕ,ψ

Martin assumed that
$$x=x(\phi,\psi),\ y=(\phi,\psi) \tag{A.1}$$
define a system of curvilinear coordinates in (x,y) – plane such that the Jacobian, $J=\dfrac{\partial(x,y)}{\partial(\phi,\psi)}$ of the transformation (A.1) is non-zero and finite. The first fundamental form of differential element is defined by
$$ds^2=E(\phi,\psi)d\phi^2+2F(\phi,\psi)d\phi d\psi+G(\phi,\psi)d\psi^2 \tag{A.2}$$
in which E, F, G are given by
$$\left.\begin{array}{l}E(\phi,\psi)=x_\phi^2+y_\phi^2\\ F(\phi,\psi)=x_\phi x_\psi+y_\phi y_\psi\\ G(\phi,\psi)=x_\psi^2+y_\psi^2\end{array}\right\} \tag{A.3}$$

Differentiating (A.1) with respect to x and y and solving the resulting equations for $\psi_x,\psi_y,\phi_x,\phi_y$ yields
$$\left.\begin{array}{l}x_\phi=J\psi_y,x_\psi=-J\phi_y\\ y_\phi=-J\psi_x,y_\psi=J\phi_x\end{array}\right\} \tag{A.4}$$
wherein
$$\begin{array}{l}J=\pm\sqrt{EG-F^2}\\ =\pm(x_\phi x_\psi-y_\phi y_\psi)=\pm W(say)\end{array} \tag{A.5}$$

Let α be the angle between the tangent vector at the point $p(x,y)$ [see Fig.1] to the coordinate line ψ = constant and the $x-axis$,

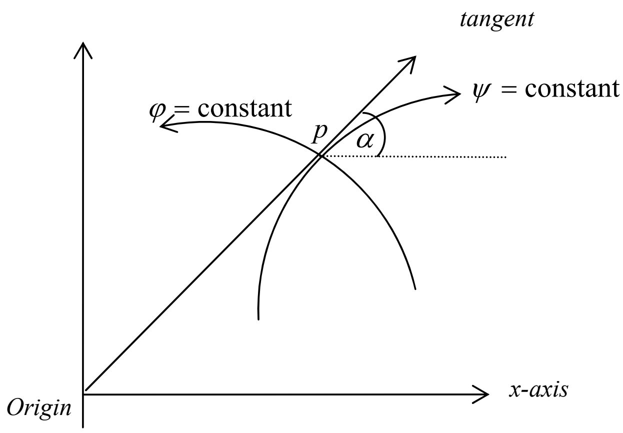

Fig. 1 (ϕ,ψ) coordinate system

then
$$Tan\alpha=\frac{y_\phi}{x_\phi} \tag{A.6}$$
Equation (A.4), on utilizing (A.3, A.6), gives
$$\left.\begin{array}{l}x_\phi=\sqrt{E}Cos\alpha,x_\psi=\dfrac{FCos\alpha-JSin\alpha}{\sqrt{E}}\\ y_\phi=\sqrt{E}Sin\alpha,y_\psi=\dfrac{FSin\alpha+JCos\alpha}{\sqrt{E}}\end{array}\right\} \tag{A.7}$$
The integrability conditions
$$x_{\phi\psi}=x_{\psi\phi},y_{\phi\psi}=y_{\psi\phi} \tag{A.8}$$
for variables x and y, yield

$$\left.\begin{aligned}\alpha_\phi &= \frac{J\Gamma_{11}^2}{E}\\ \alpha_\psi &= \frac{J\Gamma_{12}^2}{E}\end{aligned}\right\} \quad (A.9)$$

wherein

$$\left.\begin{aligned}\Gamma_{11}^2 &= \frac{[-FE_\phi + 2EF_\phi - EE_\psi]}{2W^2}\\ \Gamma_{12}^2 &= \frac{[EG_\phi - FE_\psi]}{2W^2}\end{aligned}\right\} \quad (A.10)$$

Equation (A.9), on employing integrability condition for $\alpha(\phi,\psi)$, $\alpha_{\phi\psi} = \alpha_{\psi\varphi}$ yields

$$K = \frac{\left(\frac{W\Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\Gamma_{12}^2}{E}\right)_\phi}{W} = 0 \quad (A.11)$$

wherein K is called the Gaussian Curvature and (A.11) is called the Gauss equations. This equation represents a necessary and sufficient condition that E, F, G, are coefficient of the first fundamental form in (A.2).

APPENDIX B

$$\beta_1 = -\frac{a_2(\lambda a_1 + a_2)}{(1+\lambda^2)(a_1^2 + a_2^2)^2} \quad (B\text{-}1)$$

$$\beta_2 = -\frac{(\lambda a_1^2 + 2a_1a_2 - \lambda a_2^2)}{(1+\lambda^2)(a_1^2 + a_2^2)^2} \quad (B\text{-}2)$$

$$\beta_3 = \frac{a_1a_2}{(a_1^2 + a_2^2)^2} \quad (B.3)$$

$$\beta_4 = \frac{(a_1^2 - a_2^2)}{(a_1^2 + a_2^2)^2} \quad (B.4)$$

$$\beta_5 = \frac{1}{(a_1^2 + a_2^2)} \quad (B.5)$$

$$\beta_6 = \frac{a_1(a_1 - \lambda a_2)}{(1+\lambda^2)(a_1^2 + a_2^2)^2} \quad (B.6)$$

$$\beta_7 = \frac{(\lambda a_1 + a_2)(a_1 - \lambda a_2)}{(1+\lambda^2)(a_1^2 + a_2^2)^2} \quad (B.7)$$

$$\beta_8 = -\frac{((-1+\lambda^2)a_1^2 + 4\lambda a_1a_2 - (-1+\lambda^2)a_2^2)}{(1+\lambda^2)(a_1^2 + a_2^2)^2} \quad (B.8)$$

$$\beta_9 = \frac{(\lambda a_1^2 + 2a_1a_2 - \lambda a_2^2)}{(1+\lambda^2)(a_1^2 + a_2^2)^2} \quad (B.9)$$

$$\beta_{10} = \frac{\beta_2(a_1^2 - a_2^2) + 4\beta_1a_1a_2}{\beta_5} \quad (B.10)$$

$$\beta_{11} = \frac{4\beta_3a_1a_2 + \beta_4(a_1^2 - a_2^2)}{\beta_5} \quad (B.11)$$

$$\beta_{12} = \frac{\beta_8(a_2^2 - a_1^2) - 4\beta_7a_1a_2}{\beta_5} \quad (B.12)$$

$$\beta_{13} = \frac{\beta_9(a_2^2 - a_1^2) + 4\beta_6a_1a_2}{\beta_5} \quad (B.13)$$

$$\beta_{14} = \frac{(1+\lambda^2)}{\beta_5} \quad (B.14)$$

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