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Exact Solution of Some Helical Flows of Newtonian Fluids

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Abstract—This paper deals with the helical flow of a Newtonian fluid in an infinite circular cylinder, due to both longitudinal and rotational shear stress. The velocity field and the resulting shear stress are determined by means of the Laplace and finite Hankel transforms and satisfy all imposed initial and boundary conditions. For large times, these solutions reduce to the well-known steady-state solutions.

Keywords—Newtonian fluids, Velocity field, Exact solutions, Shear stress, Cylindrical domains.

I. INTRODUCTION

THE study on the flow of an incompressible Newtonian fluid in an infinite circular cylinder is not only of fundamental theoretical interest but it also occurs in many applied problems. The flow in an infinite circular cylinder, starting from rest, has applications in the food industry. The starting solutions for the motion of the second grade fluids due to longitudinal and torsional oscillations of a circular cylinder have been established in [1]. Other recent results regarding helical flows of non-Newtonian fluids have been obtained Fetecau et al [2]. Vieru et al [3], by means of the Laplace transform and Cauchy's residue theorem, have determined the starting solutions for the oscillating motion of a Maxwell fluid.

The corresponding solutions for a Newtonian fluid, performing the same motion, are obtained from the general solutions as a particular case. Other interesting solutions for different oscillating motions of non-Newtonian fluids have also been obtained by Hayat et al [4,5].

The aim of this paper is to study the flow of a Newtonian fluid in an infinite circular cylinder of radius R. The motion is produced by the cylinder that at the initial moment is subjected to both longitudinal and torsional time dependent shear stresses. The exact solutions of the problems with initial and boundary conditions are determined by means of the finite Hankel and Laplace transforms. The solutions obtained in this paper can be used to make a comparison between the flows of Newtonian and non-Newtonian fluids.

I. GOVERNING EQUATIONS

Let us consider an incompressible Newtonian fluid at rest is situated in an infinite circular cylinder of radius R. We consider the helical flow of the fluid and assume that the velocity field, in a system of cylindrical coordinates (r , θ , z), is of the form [6]

$$v = v\left(r,t\right) = v\left(r,t\right)e_z + \omega(r,t)e_\theta, \tag{1}$$

where e_z and e_θ are the unit vectors in the z and θ directions respectively.

The constitutive equations and the governing equations are [2,3]

$$\frac{\partial v(r,t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)v(r,t),\ r \in (0,R),\ t > 0, \tag{2}$$

$$\tau_1(r,t) = \mu\frac{\partial v(r,t)}{\partial r}, \tag{3}$$

$$\frac{\partial \omega(r,t)}{\partial t} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\omega(r,t),$$

$$, r \in (0,R),\ t > 0, \tag{4}$$

$$\tau_2(r,t) = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\omega(r,t), \tag{5}$$

where $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity of the fluid, μ is the dynamic viscosity of the fluid, ρ is the constant density of the fluid, $\tau_1\left(r,t\right) = S_{rz}(r,t)$ and $\tau_2\left(r,t\right) = S_{r\theta}(r,t)$ are the shear stresses which are different from zero.

The appropriate initial and boundary conditions are

$$v(r,0) = 0,\ \ \omega(r,0) = 0,\ \ r \in [0,R), \tag{6}$$

$$\tau_1\left(R,t\right) = \mu\left(\frac{\partial v(r,t)}{\partial r}\right)_{r=R} = \mu\frac{\partial v\left(R,t\right)}{\partial r} = f_1 t^a, t > 0, \tag{7}$$

$$\tau_2\left(R,t\right) = \mu\left[\frac{\partial \omega(r,t)}{\partial r} - \frac{\omega(r,t)}{r}\right]_{r=R} =$$

$$= \mu\left(\frac{\partial \omega\left(R,t\right)}{\partial r} - \frac{\omega(R,t)}{R}\right) = f_2 t^a,\ t > 0, \tag{8}$$

where f is a constant and $a \geq 0$.

II. CALCULATION OF THE VELOCITY FIELD

In order to determine the exact solutions of the problems (2)--(8), the Laplace and finite Hankel transform method is used. Applying the Laplace transform [7,8] to (2)--(8) and using the initial conditions (6) we obtain the following problems:

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$$q\bar{v}(r,q)=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q), \tag{9}$$

$$\frac{\partial\bar{v}(R,q)}{\partial r}=\frac{f_1}{\mu}\frac{\Gamma(a+1)}{q^{a+1}}, \tag{10}$$

$$q\bar{\omega}(r,q)=\nu\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\bar{\omega}(r,q), \tag{11}$$

$$\frac{\partial\bar{\omega}(R,q)}{\partial r}-\frac{\bar{\omega}(R,q)}{R}=\frac{f_2}{\mu}\frac{\Gamma(a+1)}{q^{a+1}}, \tag{12}$$

where $\bar{v}(r,q)=\int_0^\infty v(r,t)e^{-qt}dt,$ and $\bar{\omega}(r,q)=\int_0^\infty \omega(r,t)e^{-qt}dt,$ are the Laplace transforms of the functions $v(r,t)$, respectively $\omega(r,t)$.

We define the following finite Hankel transforms of functions $\bar{v}(r,q)$ respectively $\bar{\omega}(r,q)$ [7]

$$\bar{v}_H(r_{0n},q)=\int_0^R r\bar{v}(r,q)J_0(rr_{0n})dr, \tag{13}$$

where $r_{0n},\ n=1,2,3,\cdots$ are the positive roots of the transcendental equation $J_1(Rr)=0,$ respectively

$$\bar{\omega}_H(r_{1n},q)=\int_0^R r\bar{\omega}(r,q)J_1(rr_{1n})dr, \tag{14}$$

where $r_{1n},\ n=1,2,3,\cdots$ are the positive roots of the transcendental equation $J_2(Rr)=0$.

In the above relations $J_\nu(\cdot)$ is the first-kind, ν order Bessel function.

Using the following formulae [7--10]

$$\frac{d}{dr}J_0[u(r)]=-J_1[u(r)]u'(r),$$

$$\frac{d}{dr}J_1[u(r)]=\left\{J_0[u(r)]-\frac{1}{u(r)}J_1[u(r)]\right\}u'(r), \tag{15}$$

we obtain that

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\bar{v}(r,q)J_0(rr_{0n})dr=$$

$$=RJ_0(Rr_{0n})\frac{\partial\bar{v}(R,q)}{\partial r}-r^2_{0n}\bar{v}_H(r_{0n},q), \tag{16}$$

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\bar{\omega}(r,q)J_1(rr_{1n})dr=RJ_1(Rr_{1n})\times$$

$$\times\left(\frac{\partial\bar{\omega}(R,q)}{\partial r}-\frac{\bar{\omega}(R,q)}{R}\right)-r^2_{1n}\bar{\omega}_H(r_{1n},q). \tag{17}$$

Applying the Hankel transform to (9), (11) and using the boundary conditions (10), (12) and (16), (17) we find that

$$\bar{v}_H(r_{0n},q)=Rf_1J_0(Rr_{0n})\Gamma(a+1)\times$$

$$\times\frac{1}{\rho q^{a+1}\left(q+\nu r^2_{0n}\right)}, \tag{18}$$

and

$$\bar{\omega}_H(r_{1n},q)=Rf_2J_1(Rr_{1n})\Gamma(a+1)\frac{1}{\rho q^{a+1}\left(q+\nu r^2_{1n}\right)}. \tag{19}$$

Now, for a more suitable presentation of the final results, we rewrite (18) and (19) in the equivalent forms

$$\bar{v}_H(r_{0n},q)=\bar{v}_{1H}(r_{0n},q)+\bar{v}_{2H}(r_{0n},q), \tag{20}$$

where

$$\bar{v}_{1H}(r_{0n},q)=\frac{Rf_1J_0(Rr_{0n})}{\mu r^2_{0n}}\frac{\Gamma(a+1)}{q^{a+1}},$$

$$\bar{v}_{2H}(r_{0n},q)=-\frac{Rf_1J_0(Rr_{0n})}{\mu r^2_{0n}}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2_{0n}}, \tag{21}$$

respectively,

$$\bar{\omega}_H(r_{1n},q)=\bar{\omega}_{1H}(r_{1n},q)+\bar{\omega}_{2H}(r_{1n},q), \tag{22}$$

$$\bar{\omega}_{1H}(r_{1n},q)=\frac{Rf_2J_1(Rr_{1n})}{\mu r^2_{1n}}\frac{\Gamma(a+1)}{q^{a+1}},$$

$$\bar{\omega}_{2H}(r_{1n},q)=-\frac{Rf_2J_1(Rr_{1n})}{\mu r^2_{1n}}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2_{1n}}. \tag{23}$$

Using (13) and (14), after a straightforward calculation we obtain the following function-Hankel transform pairs:

$$f(r)=\frac{r^2}{2R},\ f_H(r_{0n})=\frac{RJ_0(Rr_{0n})}{r^2_{0n}}. \tag{24}$$

$$g(r)=\frac{r^3}{2R^2},\ g_H(r_{1n})=\frac{RJ_1(Rr_{1n})}{r^2_{1n}}. \tag{25}$$

The inverse Hankel transforms of functions $\bar{v}_{2H}(r_{0n},q)$ and $\bar{\omega}_{2H}(r_{1n},q)$ are [7]

$$\bar{v}_{2H}(r_{0n},q)=\frac{2}{R^2}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{J^2_0(Rr_{0n})}\bar{v}_{2H}(r_{0n},q)=$$

$$=\frac{2f_1}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r^2_{0n}J^2_0(Rr_{0n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2_{0n}}, \tag{26}$$

respectively,

$$\bar{\omega}_{2H}(r_{1n},q)=2\sum_{n=1}^{\infty}\frac{r^2_{1n}J_1(rr_{1n})}{\left[\left(1+r^2_{1n}\right)R^2-1\right]J^2_1(Rr_{1n})}\bar{\omega}_{2H}(r_{1n},q)$$

$$=\frac{2Rf_2}{\mu}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{\left[\left(1+r^2_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2_{1n}}, \tag{27}$$

Applying the inverse Hankel transform to (20)--(23), using (24)-(27) we obtain the following expressions for the Laplace transforms of functions $v(r,t)$ and $\omega(r,t)$:

$$\bar{v}(r,q)=\frac{r^2 f_1}{2\mu R}\frac{\Gamma(a+1)}{q^{a+1}}-$$

$$-\frac{2f_1}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r^2{}_{0n}J_0(Rr_{0n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2{}_{0n}}, \qquad (28)$$

and

$$\bar{\omega}(r,q)=\frac{r^3 f_2}{2\mu R^2}\frac{\Gamma(a+1)}{q^{a+1}}-\frac{2Rf_2}{\mu}$$

$$\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2{}_{1n}}. \qquad (29)$$

To obtain the velocity fields $v(r,t)=L^{-1}\{\bar{v}(r,q)\}$ and $\omega(r,t)=L^{-1}\{\bar{\omega}(r,q)\}$ we introduce the following notations:

$$A_1(q)=\frac{\Gamma(a+1)}{q^a},\ a>0,$$

$$B_1(r_{0n},q)=\frac{1}{q+\nu r^2{}_{0n}},\ B_2(r_{1n},q)=\frac{1}{q+\nu r^2{}_{1n}}. \qquad (30)$$

The inverse Laplace transforms of the above functions are [9]

$$a_1(t)=at^{a-1},\ a>0,\ b_1(r_{0n},t)=\exp\left(-\nu r^2{}_{0n}t\right),$$

$$b_2(r_{1n},t)=\exp\left(-\nu r^2{}_{1n}t\right). \qquad (31)$$

Applying the inverse Laplace transform to (28) and (29), using (31) and the convolution theorem we get:

If $a>0$ then

$$v(r,t)=\frac{r^2 f_1}{2\mu R}t^a-$$

$$-\frac{2f_1}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r^2{}_{0n}J_0(Rr_{0n})}a_1(t)*b_1(r_{0n},t), \qquad (32)$$

and

$$\omega(r,t)=\frac{r^3 f_2}{2\mu R^2}t^a-\frac{2Rf_2}{\mu}$$

$$\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}a_1(t)*b_2(r_{1n},t), \qquad (33)$$

Where

$$a(t)*b(t)=\int_0^t a(t)b(t-\tau)d\tau=\int_0^t a(t-\tau)b(t)d\tau$$

represents the convolution product of functions a and b.

If $a=0$ *then*

$$v(r,t)=\frac{r^2 f_1}{2\mu R}-$$

$$-\frac{2f_1}{\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r^2{}_{0n}J_0(Rr_{0n})}\exp\left(-\nu r^2{}_{0n}t\right), \qquad (34)$$

and

$$\omega(r,t)=\frac{r^3 f_2}{2\mu R^2}-$$

$$-\frac{2Rf_2}{\mu}\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\exp\left(-\nu r^2{}_{1n}t\right), \qquad (35)$$

IV. CALCULATION OF THE SHEAR STRESS

Applying the Laplace transform to (3) and (5) we find that

$$\bar{\tau}_1(r,q)=\mu\frac{\partial\bar{v}(r,q)}{\partial r}, \qquad (36)$$

$$\bar{\tau}_2(r,q)=\mu\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,q). \qquad (37)$$

Differentiating (28) and (29) with respect to r we find that [9,10]

$$\frac{\partial\bar{v}(r,q)}{\partial r}=\frac{rf_1}{\mu R}\frac{\Gamma(a+1)}{q^{a+1}}+$$

$$+\frac{2f_1}{\mu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r_{0n}J_0(Rr_{0n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2{}_{0n}}, \qquad (38)$$

$$\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{\omega}(r,t)=\frac{r^2 f_2}{2R^2}\frac{\Gamma(a+1)}{q^{a+1}}+\frac{2Rf_2}{\mu}$$

$$\sum_{n=1}^{\infty}\frac{r_{1n}J_2(rr_{1n})}{\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2{}_{1n}}, \qquad (39)$$

Introducing (38) and (39) into (36) and (37) we get

$$\bar{\tau}_1(r,q)=\frac{rf_1}{R}\frac{\Gamma(a+1)}{q^{a+1}}+$$

$$+\frac{2f_1}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r_{0n}J_0(Rr_{0n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2{}_{0n}}, \qquad (40)$$

$$\bar{\tau}_2(r,q)=\frac{r^2 f_2}{R^2}\frac{\Gamma(a+1)}{q^{a+1}}+2Rf_2$$

$$\sum_{n=1}^{\infty}\frac{r_{1n}J_2(rr_{1n})}{\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\frac{\Gamma(a+1)}{q^a}\frac{1}{q+\nu r^2{}_{1n}}. \qquad (41)$$

Using (31) and the convolution theorem, we find the shear stress $\tau_1(r,t)$, $\tau_2(r,t)$ under the forms

If $a>0$ then

$$\tau_1(r,t) = \frac{rf_1}{R}t^a + + \frac{2f_1}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r_{0n}J_0(Rr_{0n})}a_1(t)*b_1(r_{0n},t), \quad (42)$$

and

$$\tau_2(r,t) = \frac{r^2 f_2}{R^2}t^a + +2Rf_2\sum_{n=1}^{\infty}\frac{r_{1n}J_2(rr_{1n})}{\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}a_1(t)*b_2(r_{1n},t), (43)$$

If $a = 0$ *then*

$$\tau_1(r,t) = \frac{rf_1}{R} + \frac{2f_1}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r_{0n}J_0(Rr_{0n})}\exp\left(-\nu r^2{}_{0n}t\right), \quad (44)$$

and

$$\tau_2(r,t) = \frac{r^2 f_2}{R^2} + +2Rf_2\sum_{n=1}^{\infty}\frac{r_{1n}J_2(rr_{1n})}{\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\exp\left(-\nu r^2{}_{1n}t\right), (45)$$

Iit is important to point out that from (42)--(45), we have $\tau_1(R,t) = f_1 t^a$ respectively $\tau_2(R,t) = f_2 t^a$.

For $a = 1$, from (32), (33), (42) and (43) we get the following results:

$$v(r,t) = \frac{r^2 f_1}{2\mu R}t - -\frac{2f_1}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r^4{}_{0n}J_0(Rr_{0n})}\left[1-\exp\left(-\nu r^2{}_{0n}t\right)\right], \quad (46)$$

$$\omega(r,t) = \frac{r^3 f_2}{2\mu R^2}t + \frac{2Rf_2}{\nu\mu} \sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r^2{}_{1n}\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\left[1-\exp\left(-\nu r^2{}_{1n}t\right)\right], \quad (47)$$

$$\tau_1(r,t) = \frac{rf_1}{R}t + + \frac{2f_1}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r^3{}_{0n}J_0(Rr_{0n})}\left[1-\exp\left(-\nu r^2{}_{0n}t\right)\right], \quad (48)$$

$$\tau_2(r,t) = \frac{r^2 f_2}{R^2}t + \frac{2Rf_2}{\nu} \sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}\left[1-\exp\left(-\nu r^2{}_{1n}t\right)\right], (49)$$

IV. CONCLUSIONS

In this paper, the velocity fields and the associated shear stresses corresponding to the helical flow induced by an infinite circular cylinder in an incompressible Newtonian fluid, have been determined using Hankel and Laplace transforms. The solutions that have been obtained, written in terms of the Bessel functions, satisfy all imposed initial and boundary conditions and can be used to make a comparison between flows of Newtonian and non-Newtonian fluids. For $t \to \infty$ the solutions (30), (31) and (40)--(45) reduce to the steady-state solutions:

$$v_{0s}(r,t) = \frac{r^2 f_1}{2\mu R}, \ \omega_{0s}(r,t) = \frac{r^3 f_2}{2\mu R^2}, \quad (50)$$

$$\tau_{01s}(r,t) = \frac{rf_1}{R}, \ \tau_{02s}(r,t) = \frac{r^2 f_2}{R^2}, \quad (51)$$

$$v_{1s}(r,t) = \frac{r^2 f_1}{2\mu R}t - \frac{2f_1}{\nu\mu R}\sum_{n=1}^{\infty}\frac{J_0(rr_{0n})}{r^4{}_{0n}J_0(Rr_{0n})}, \quad (52)$$

$$\omega_{1s}(r,t) = \frac{r^3 f_2}{2\mu R^2}t - -\frac{2Rf_2}{\nu\mu}{}_2\sum_{n=1}^{\infty}\frac{J_1(rr_{1n})}{r^2{}_{1n}\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}, \quad (53)$$

$$\tau_{01s}(r,t) = \frac{rf_1}{R}t + \frac{2f_1}{\nu R}\sum_{n=1}^{\infty}\frac{J_1(rr_{0n})}{r^3{}_{0n}J_0(Rr_{0n})}, \quad (54)$$

$$\tau_{02s}(r,t) = \frac{r^2 f_2}{R^2}t + + \frac{2Rf_2}{\nu}\sum_{n=1}^{\infty}\frac{J_2(rr_{1n})}{r_{1n}\left[\left(1+r^2{}_{1n}\right)R^2-1\right]J_1(Rr_{1n})}. \quad (55)$$

REFERENCES

[1] C. Fetecau, Corina Fetecau, "Starting solutions for the motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder," Int. J. Eng. Sci. 44, (2006) 788—796.
[2] C. Fetecau, Corina Fetecau, D. Vieru, "On some helical flows of Oldroyd-B fluids," Acta Mechanica. 189, (2007) 53--63.
[3] D. Vieru, W. Akhtar, Corina Fetecau, C. Fetecau, "Starting solutions for the oscillating motion of a Maxwell fluid in cylindrical domains," Meccanica. 42, (2007) 573--583.
[4] T. Hayat, S. Asghar, A. M. Siddiqui, "Some unsteady unidirectional flows of a non-Newtonian fluid," Int. J. Eng. Sci. 38, (2000) 337--346.
[5] T. Hayat, A. M. Siddiqui, S. Asghar, "Some simple flows of an Oldroyd-B fluid," Int. J. Eng. Sci. 39, (2001) 135--147.
[6] W. P. Wood, "Transient viscoelastic flows in pipes of circular and annular cross- section," J. Non-Newtonian Fluid Mech. 100, (2001) 115--126.
[7] L. Debnath, D. Bhatta, "Integral Transforms and Their Applications "(second ed.), Chapman and Hall/CRC Press, Boca Raton London New York, 2007.
[8] V. Ditkin, A. Proudnicov, "Transformations Integrals et Calcul operational," Editions Mir-Moscou, 1987.
[9] N. W. McLachlan, "Bessel Functions for Engineers," Oxford Univeristy Press, London, 1955.
[10] M. Abramowitz, I. A. Stegun, "Handbook of Mathematical Functions, "NBS, Appl. Math. Series 55, Washington, D. C, 1964.

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