Estimation of R= P [$Y < X$] for Two-parameter Burr Type XII Distribution

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Abstract— In this article, we consider the estimation of P[Y < X], when strength, X and stress, Y are two independent variables of Burr Type XII distribution. The MLE of the R based on one simple iterative procedure is obtained. Assuming that the common parameter is known, the maximum likelihood estimator, uniformly minimum variance unbiased estimator and Bayes estimator of P[Y < X] are discussed. The exact confidence interval of the R is also obtained. Monte Carlo simulations are performed to compare the different proposed methods.

Keywords—Stress-Strength model; Maximum likelihood estimator; Bayes estimator; Burr type XII distribution.

I. INTRODUCTION

BURR introduced twelve different forms of cumulative distribution functions for modeling lifetime data or survival data [1]. Out of those twelve distributions, Burr Type XII and Burr Type X have received the maximum attention. Several authors considered different aspects of these two distributions, see for example, [2]- [7].

The Burr Type XII has the following distribution function for X > 0:

$$F(x; p,b) = 1 - \left(1 + x^b\right)^{-p}; \quad \text{for} \quad p > 0, b > 0 \qquad (1)$$

Therefore, the Burr Type XII has the density function for x > 0 as:

$$f(x; p,b) = pb\, x^{b-1}\left(1 + x^b\right)^{-(p+1)}; \quad \text{for} \quad p > 0, b > 0$$

In stress-strength model, the stress (Y) and the strength (X) are treated as random variables and the reliability of a component during a given period is taken to be the probability that its strength exceeds the stress during the entire interval. Due to the practical point of view of reliability stress-strength model, the estimation problem of R=P(Y<X) has attracted the attention of many authors. Ahmad et al. [8] and Surles & Padgett [9] considered the estimation of P[Y < X], where X and Y are Burr Type X random variables, Abd-Elfattah & Mandouh [10] considered the estimation of P[Y < X], when X and Y are independent Lomax random variables with known scale parameter and Recently Kundu and Gupta [11] have considered estimation of P(Y<X), when X and Y are independent generalized exponential Distribution.

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In the present article, the inference of R=P(Y<X), is studied when X and Y are two independent but not identically random variables belonging to as burr type XII distribution with two parameters. In Section (II), the point estimation of reliability R is obtained using maximum likelihood method. Also we discuss different estimation procedures of R if b is known in Section (III). Monte Carlo simulation results are presented in Section (IV) and finally we draw conclusions in Section (V).

II. MAXIMUM LIKELIHOOD ESTIMATOR OF R

Let X and Y are two independent Burr Type XII random variables with parameters p, b and q, b respectively. Therefore

$$R = P(Y < X) = \int_0^\infty \int_0^x f(x)f(y)\,dy\,dx$$

$$= \int_0^\infty \int_0^x pbx^{b-1}(1 + x^b)^{-(p+1)} qby^{b-1}(1 + y^b)^{-(q+1)} \,dy\,dx$$

$$= \frac{q}{p+q} \qquad (2)$$

Now to compute the MLE of R, first we obtain the *MLE's* of p and q. Let $X_1,...,X_n$ be a random sample from $\mathrm{BurrXII}(p,b)$ and $Y_1,...,Y_m$ be a random sample from $\mathrm{BurrXII}(q,b)$.

Therefore the log-likelihood function L of p, q and b for the observed sample is

$$L(p,q,b) = n\ln p + m\ln q + (n+m)\ln b$$

$$+ (b-1)\left(\sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j\right) - (p+1)\sum_{i=1}^{n} \ln\left(1 + x_i^b\right)$$

$$- (q+1)\sum_{j=1}^{m} \ln\left(1 + y_j^b\right) \qquad (3)$$

Differentiating partially with respect to p , q and b, setting the results equal to zero we get three nonlinear equations.

$$\frac{\partial L}{\partial p} = \frac{n}{p} - \sum_{i=1}^{n} \ln\left(1 + x_i^b\right) = 0, \qquad (4)$$

$$\frac{\partial L}{\partial q} = \frac{m}{q} - \sum_{j=1}^{m} \ln\left(1 + y_j^b\right) = 0, \qquad (5)$$

$$\frac{\partial L}{\partial b} = \frac{n+m}{b} + \sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j$$

$$- (p+1)\sum_{i=1}^{n} \frac{x_i^b \ln x_i}{1+x_i^b} - (q+1)\sum_{j=1}^{m} \frac{y_j^b \ln y_j}{1+y_j^b} = 0 \qquad (6)$$

From (4), (5) and (6), we obtain

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} \ln(1+x_i^b)} \qquad (7)$$

$$\hat{q} = \frac{m}{\sum_{j=1}^{m} \ln(1+y_j^b)} \qquad (8)$$

and $\hat{b}$, can be obtained as the solution of the non-linear equation

$$h(b) = \frac{n+m}{b} + \sum_{i=1}^{n} \ln x_i + \sum_{j=1}^{m} \ln y_j$$

$$- \frac{n}{\sum_{i=1}^{n} \ln(1+x_i^b)} \sum_{i=1}^{n} \frac{x_i^b \ln x_i}{1+x_i^b} - \frac{m}{\sum_{j=1}^{m} \ln(1+y_j^b)} \sum_{j=1}^{m} \frac{y_j^b \ln y_j}{1+y_j^b}$$

$$- \sum_{i=1}^{n} \frac{x_i^b \ln x_i}{1+x_i^b} - \sum_{j=1}^{m} \frac{y_j^b \ln y_j}{1+y_j^b} = 0 \qquad (9)$$

Consequently, $\hat{b}$, can be obtained by solving the non-linear equation

$$u(b) = b \qquad (10)$$

Where

$$u(b) = (n+m)\Big(-\sum_{i=1}^{n} \ln x_i - \sum_{j=1}^{m} \ln y_j$$

$$+ \frac{n}{\sum_{i=1}^{n} \ln(1+x_i^b)} \sum_{i=1}^{n} \frac{x_i^b \ln x_i}{1+x_i^b} + \frac{m}{\sum_{j=1}^{m} \ln(1+y_j^b)} \sum_{j=1}^{m} \frac{y_j^b \ln y_j}{1+y_j^b}$$

$$+ \sum_{i=1}^{n} \frac{x_i^b \ln x_i}{1+x_i^b} + \sum_{j=1}^{m} \frac{y_j^b \ln y_j}{1+y_j^b} \Big)^{-1}$$

Since $\hat{b}$, is a fixed point solution of the non-linear equation (10), therefore, it can be obtained by using a simple iterative scheme as follows:

$$u(b_{(j)}) = b_{(j+1)} \qquad (11)$$

Where $b_{(j)}$ is the j- th iterate of $\hat{b}$. Once we obtain $\hat{b}$, $\hat{p}$ and $\hat{q}$ can be obtained from (7) and (8) respectively. Therefore, the MLE of R becomes

$$R = \frac{\hat{q}}{\hat{p}+\hat{q}} \qquad (12)$$

III. ESTIMATION OF R IF b IS KNOWN

In this section, we consider the estimation of R when b is known. Without loss of generality, we can assume that b = 1. Therefore, in this section it is assumed that $X_1,...,X_n$ is a random sample from $\text{BurrXII}(p,1)$ and $Y_1,...,Y_m$ is a random sample from $\text{BurrXII}(q,1)$ and based on the samples we want to estimate R. First, we consider the MLE of R and its distributional properties.

A. MLE of R

Based on the above samples, it is clear that, the MLE of R namely $\hat{R}$ is given by

$$\hat{R} = \frac{\hat{q}}{\hat{q}+\hat{p}} \qquad (13)$$

Where

$$\hat{p} = \frac{n}{\sum_{i=1}^{n} ln\Big(1+x_i\Big)} \qquad (14)$$

$$\hat{q} = \frac{m}{\sum_{j=1}^{m} ln(1+y_j)} \qquad (15)$$

Therefore,

$$\hat{R} = \frac{m\sum_{i=1}^{n} ln(1+x_i)}{\left[m\sum_{i=1}^{n} ln(1+x_i) + n\sum_{j=1}^{m} ln(1+y_j) \right]} \qquad (16)$$

We considered

$$u = 2p\sum_{i=1}^{n} \ln(1+x_i) \sim \chi_{2n}^2$$

$$V = 2q\sum_{j=1}^{m} \ln(1+y_j) \sim \chi_{2m}^2$$

Therefore,

$$\hat{R} \underset{=}{d} \frac{U}{U+cV} \underset{=}{d} \frac{1}{1+\dfrac{p}{q}F}$$

or

$$\frac{R}{1-R} \times \frac{1-\hat{R}}{\hat{R}} \underset{=}{d} F \qquad (17)$$

Here $\underset{=}{d}$ indicates equivalent in distribution and $c = \dfrac{np}{mq}$.

The random variables U and V are independent and follow χ^2 distribution, with 2n and 2m degrees of freedom respectively. Moreover, F has an F distribution with 2m and 2n degrees of freedom. Therefore, the $100(1-\alpha)\%$ confidence interval of R can be obtained as:

$$\left[\frac{1}{1 + F_{2n,2m:1-\frac{\alpha}{2}}\left(\frac{1}{\hat{R}} - 1\right)} \quad , \quad \frac{1}{1 + F_{2n,2m:\frac{\alpha}{2}}\left(\frac{1}{\hat{R}} - 1\right)} \right] \qquad (18)$$

Where, $F_{2n,2m,\frac{\alpha}{2}}$ and $F_{2n,2m,1-\frac{\alpha}{2}}$ are the lower and upper $\dfrac{\alpha}{2}^{th}$ percentile points of a F distribution with 2n and 2m degrees of freedom.

B. UMVUE of R

In this subsection we obtain the UMVUE of R .When the common parameter is known, $\left(\sum\limits_{i=1}^{n} \ln(1 + x_i), \sum\limits_{j=1}^{m} \ln(1 + y_j) \right)$ is a jointly sufficient statistic for (p,q). Therefore using the results of Tong [12],[13] it follows that

$$\tilde{R} = 1 - \sum_{i=0}^{n-1}(-1)^i \frac{(m-1)!(n-1)!}{(m+i-1)!(n-i-1)!}\left(\frac{T_2}{T_1}\right)^i \text{ if } T_2 \le T_1 \quad (19)$$

or

$$\tilde{R} = 1 - \sum_{i=0}^{m-1}(-1)^i \frac{(m-1)!(n-1)!}{(m-i-1)!(n+i-1)!}\left(\frac{T_1}{T_2}\right)^i \text{ if } T_1 \le T_2 \quad (20)$$

Where $T_1 = \sum\limits_{i=1}^{m} \ln(1 + y_i)$ and $T_2 = \sum\limits_{i=1}^{n} \ln(1 + x_i)$

C. Bayes Estimation of R

In this subsection, we obtain the Bayes estimation of R under the assumptions that the parameters p and q are random variables for both the populations. It is assumed that p and q have independent gamma priors with the PDF's:

$$\pi(p) = \frac{\beta_1^{\alpha_1}}{\Gamma(\alpha_1)} p^{\alpha_1 - 1} e^{-\beta_1 p} \qquad p > 0 \qquad (21)$$

$$\pi(q) = \frac{\beta_2^{\alpha_2}}{\Gamma(\alpha_2)} q^{\alpha_2 - 1} e^{-\beta_2 q} \qquad q > 0 \qquad (22)$$

respectively. Here $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$. Therefore, p and q follow $Gamma(\alpha_1, \beta_1)$ and $Gamma(\alpha_2, \beta_2)$ respectively. The posterior PDF's of p and q are as follows:

$$\pi(p|x) \sim \Gamma\left(n + \alpha_1, \beta_1 + \sum_{i=1}^{n} \ln(1 + x_i) \right) \qquad (23)$$

$$\pi(q|x) \sim \Gamma\left(m + \alpha_2, \beta_2 + \sum_{j=1}^{m} \ln(1 + y_j) \right) \qquad (24)$$

Assume that p and q are independent, using (23) and (24), the joint posterior density of p and q given the data as

$$\pi(p,q|x, y) = \frac{(T_1)^{n+\alpha_1} (T_2)^{m+\alpha_2}}{\Gamma(n+\alpha_1)\,\Gamma(m+\alpha_2)} \qquad (25)$$
$$\times p^{n+\alpha_1 - 1} q^{m+\alpha_2 - 1} \exp\{-pT_1 - qT_2\}$$

$$T_1 = \beta_1 + \sum_{i=1}^{n} \ln(1 + x_i) \text{ and } T_2 = \beta_2 + \sum_{j=1}^{m} \ln(1 + y_j)$$

Applying the transformations technique of random variables, let

$$r = \frac{q}{q+p} \quad \text{and} \quad u = q + p \qquad 0 < r < 1, u > 0$$

Then

$$\pi(u,r|x, y) = \frac{(T_1)^{n+\alpha_1} (T_2)^{m+\alpha_2}}{\Gamma(n+\alpha_1)\,\Gamma(m+\alpha_2)} u^{\alpha_1+\alpha_2+m+n-1}$$
$$\times (1-r)^{n+\alpha_1 - 1} r^{m+\alpha_2 - 1} \exp\{-u[(1-r)T_1 - rT_2]\}$$

integrate out u

$$\pi(r|x, y) = \frac{(T_1)^{n+\alpha_1} (T_2)^{m+\alpha_2}}{\Gamma(n+\alpha_1)\,\Gamma(m+\alpha_2)} r^{m+\alpha_2 - 1}$$
$$\times (1-r)^{n+\alpha_1 - 1} \frac{\Gamma(\alpha_1 + \alpha_2 + m + n)}{[(1-r)T_1 + rT_2]^{\alpha_1+\alpha_2+m+n}} \qquad (26)$$

Using equation (26), Bayes estimator of R, say $\hat{R}_{BS}$, under squared error loss function is

$$\hat{R}_{BS} = E(R|x, y) = \int_0^1 r\,\pi(r|x, y)\,dr$$

$$\hat{R}_{BS} = \frac{\Gamma(\alpha_1 + \alpha_2 + m + n)}{\Gamma(n+\alpha_1)\,\Gamma(m+\alpha_2)}$$
$$\times (T_1)^{n+\alpha_1}(T_2)^{m+\alpha_2} \int_0^1 \frac{r^{m+\alpha_2}(1-r)^{n+\alpha_1 - 1}}{[(1-r)T_1 + rT_2]^{\alpha_1+\alpha_2+m+n}}\,dr \qquad (27)$$

The computation of the $\hat{R}_{BS}$ is complicated as it can seen from equation (27).so, we will use the MATHCAD program to evaluate the value of $\hat{R}_{BS}$.

IV. SIMULATION STUDY

In this section we present results of some numerical experiments to compare the performance of the different estimators proposed in the section (III). We perform extensive Monte Carlo simulations to compare the performance of the different estimators, mainly with respect to their biases and mean squared errors. we consider the case when the common parameter b, is known. In this case we consider the following small sample size;

(n,m)= (10,10),(10,20),(10,30),(20,10),(20,20),(20,30), (30, 10), (30, 20), (30, 30) and we take p = 10 and q =5, 8 respectively. Without loss of generality, we take b = 1. All the results are based on 1000 replications. We obtain the estimates of R by using the MLE and UMVUE. We also compute the Bayes estimate of R as suggested in subsection C with the following configurations of (n,m)=(30,30) and $\alpha_1 = 0.1, \alpha_2 = 5,10$, $\beta_1 = 15,20,25$, $\beta_2 = 15,20,25$. We report the average estimates and average $MSE's$ of the $MLE's$ and $UMVUE's$ based on 1000 replications in Table 1 and Bayes estimator based on 1000 replications is reported in Table 2. From table 1 we can note that the mean square error decreasing by increasing sample size m with sample size n is constant and the mean square error increasing by decreasing sample size n with sample size m is constant and also it decreasing by increasing the both of them. The changes in mean square error of $\hat{R}$ due to change in p and q can be ignored and it is observed in Table 2 the mean square error of $\hat{R}_{BS}$ increasing by increasing α_2 that α_1 is constant.

V. CONCLUSION

In this paper we compare different methods of estimating R = P(Y < X) when Y and X both follow Burr Type XII distribution with parameters (p,b) and (q,b), respectively. When the parameter b is unknown, it is observed that the $MLE's$ of the three unknown parameters can be obtained by solving one non-linear equation. We consider one simple iterative procedure to compute the $MLE's$ of the unknown parameters and in turn to compute the MLE of R. When the parameter b is known, we obtain maximum likelihood estimator and uniformly minimum variance unbiased estimator. We also obtain Bayes estimator under squared error loss function. It is observed that the MLE and UMVUE are quite similar in nature, although based on mean squared errors, the performance of the $MLE's$ are marginally better than the rest.

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TABLE I
BIASES AND MEAN SQUARED ERRORS OF THE MLES AND UMVUES

(n , m)	$q \rightarrow$ Methods $\downarrow$	5	8
(10,10)	MLE UMVUE	-0.0075(0.0260) -0.0083(0.0264)	0.0017(0.0396) 0.0059(0.0398)
(10,20)	MLE UMVUE	-0.0202(0.0189) -0.0152(0.0192)	-0.0340(0.0280) -0.0271(0.0282)
(10,30)	MLE UMVUE	-0.0358(0.0167) -0.0323(0.0168)	-0.0551(0.0253) -0.0495(0.0253)
(20,10)	MLE UMVUE	0.0225(0.0209) 0.0240(0.0213)	0.0376(0.0280) 0.0410(0.0281)
(20,20)	MLE UMVUE	-0.0045(0.0115) -0.0032(0.0118)	-0.0008(0.0159) -0.0045(0.0158)
(20,30)	MLE UMVUE	-0.0152(0.0095) 0.0118(0.0094)	-0.0054(0.0121) 0.0026(0.0128)
(30,10)	MLE UMVUE	0.0368(0.0207) 0.0460(0.0207)	0.0532(0.0295) 0.0554(0.0293)
(30,20)	MLE UMVUE	0.0098(0.0121) 0.0091(0.0123)	0.0107(0.0123) 0.0148(0.0125)
(30,30)	MLE UMVUE	0.0004(0.0094) 0.0001(0.0093)	-0.0030(0.0099) -0.0016(0.0101)

In each cell the first, second rows represent the average biases and mean squared errors of the $MLE`s$ , $UMVUE`s$.

TABLE II
BIASES AND MSEs OF THE BAYES ESTIMATORS OF R

(n,m)	α_1	α_2	β_1	β_2	$\hat{R}_{BS}$	MSE
(30,30)	0.1	5	15	15	0.4995	0.1593E-05
	0.1	5	15	20	0.5007	0.1769E-05
	0.1	5	15	25	0.5030	0.1757E-05
	0.1	5	20	15	0.5022	0.1716E-05
	0.1	5	20	20	0.5025	0.1803E-05
	0.1	5	20	25	0.5027	0.1716E-05
	0.1	5	25	15	0.5021	0.1756E-05
	0.1	5	25	20	0.5024	0.1768E-05
	0.1	5	25	25	0.5023	0.1758E-05
	0.1	10	15	15	0.5444	0.6641E-05
	0.1	10	15	20	0.5443	0.6625E-05
	0.1	10	15	25	0.5445	0.6761E-05
	0.1	10	20	15	0.5447	0.6745E-05
	0.1	10	20	20	0.5445	0.6722E-05
	0.1	10	20	25	0.5443	0.6719E-05
	0.1	10	25	15	0.5442	0.6708E-05
	0.1	10	25	20	0.5443	0.6705E-05
	0.1	10	25	25	0.5442	0.6675E-05