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Elastic-Plastic Transition in a Thin Rotating Disc with Inclusion

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***Abstract*—**Stresses for the elastic-plastic transition and fully plastic state have been derived for a thin rotating disc with inclusion and results have been discussed numerically and depicted graphically. It has been observed that the rotating disc with inclusion and made of compressible material requires lesser angular speed to yield at the internal surface whereas it requires higher percentage increase in angular speed to become fully plastic as compare to disc made of incompressible material.

***Keywords*—**Angular speed, Elastic-Plastic, Inclusion, Rotating disc, Stress, Transition.

I. Introduction

ROTATING discs form an essential part of the design of rotating machinery, namely rotors, turbines, compressors, flywheel and computer's disc drive etc. The analysis of thin rotating discs made of isotropic material has been discussed extensively by Timoshenko and Goodier [1] in the elastic range and by Chakrabarty [2] and Heyman [3] for the plastic range. Their solution for the problem of fully plastic state does not involve the plane stress condition, that is to say, we can obtain the same stresses and angular velocity required by the disc to become fully plastic without using the plane stress condition (i.e. $T_{zz}=0$). Gupta and Shukla [4] obtained a different solution for the fully plastic state by using Seth's transition theory and plane stress condition. This theory [5] does not require any assumptions like an yield condition, incompressibility condition and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. It utilizes the concept of generalized strain measure and asymptotic solution at critical points or turning points of the differential equations defining the deformed field and has been successfully applied to a large number of problems [4, 7-15]. Seth [6] has defined the generalized principal strain measure as,

$$e_{ii}=\int_0^{\overset{A}{e_{ii}}}\left[1-2\overset{A}{e}_{ii}\right]^{\frac{n}{2}-1} d\overset{A}{e}_{ij}=\frac{1}{n}\left[1-\left(1-2\overset{A}{e}_{ii}\right)^{\frac{n}{2}}\right],\ (i=1,2,3) \tag{1}$$

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where n is the measure and $\overset{A}{e}_{ii}$ is the Almansi finite strain components.

In this paper, the plastic stresses have been derived through the asymptotic solution of principal stress respectively and we analyses the elastic-plastic transition in a thin rotating disc with shaft by using Seth's transition theory. Results have been discussed numerically and depicted graphically.

II. Governing Equations

We consider a thin disc of constant density with central bore of radius a and external radius b. The annular disc is mounted on a shaft. The disc is rotating with angular speed ω about an axis perpendicular to its plane and passed through the center. The thickness of disc is assumed to be constant and is taken to be sufficiently small so that it is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero. The displacement components in cylindrical polar co- ordinate are given by [6]

$$u=r(1-\beta);\ v=0;\ w=dz, \tag{2}$$

where β is function of $r=\sqrt{x^2+y^2}$ only and d is a constant. The finite strain components are given by Seth [6] as,

$$\overset{A}{e}_{rr}\equiv\frac{\partial u}{\partial r}-\frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2=\frac{1}{2}\left[1-\left(r\beta'+\beta\right)^2\right],\ \overset{A}{e}_{\theta\theta}\equiv\frac{u}{r}-\frac{u^2}{2r^2}=\frac{1}{2}\left[1-\beta^2\right],$$
$$\overset{A}{e}_{zz}\equiv\frac{\partial w}{\partial z}-\frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2=\frac{1}{2}\left[1-(1-d)^2\right],\ \overset{A}{e}_{r\theta}=\overset{A}{e}_{\theta z}=\overset{A}{e}_{zr}=0, \tag{3}$$

where $\beta'=d\beta/dr$.

Substituting equation (3) in equation (1), the generalized components of strain are

$$e_{rr}=\frac{1}{n}\left[1-\left(r\beta'+\beta\right)^n\right],\ e_{\theta\theta}=\frac{1}{n}\left[1-\beta^n\right],\ e_{zz}=\frac{1}{n}\left[1-(1-d)^n\right],$$
$$e_{r\theta}=e_{\theta z}=e_{zr}=0, \tag{4}$$

where $\beta'=d\beta/dr$.

The stress –strain relations for isotropic material are given by [16],

$$T_{ij}=\lambda\delta_{ij}I_1+2\mu e_{ij},\quad (i,j=1,2,3) \tag{5}$$

where T_{ij} and e_{ij} are the stress and strain components, λ and μ are lame's constants and $I_1 = e_{kk}$ is the first strain invariant, δ_{ij} is the Kronecker's delta.

Equation (5) for this problem becomes

$$T_{rr} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right]+2\mu e_{rr},\ T_{\theta\theta} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right]+2e_{\theta\theta},$$
$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0, \tag{6}$$

where $\beta' = d\beta/dr$.

Substituting equation (3) in equation (5), the strain components in terms of stresses are obtained as [16]

$$e_{rr} = \frac{1}{2}\left[1-\left(r\beta'+\beta\right)^2\right] = \frac{1}{E}\left[T_{rr}-\left(\frac{1-C}{2-C}\right)T_{\theta\theta}\right],$$
$$e_{\theta\theta} = \frac{1}{2}\left[1-\beta^2\right] = \frac{1}{E}\left[T_{\theta\theta}-\left(\frac{1-C}{2-C}\right)T_{rr}\right],$$
$$e_{zz} = \frac{1}{2}\left[1-(1-d)^2\right] = -\frac{(1-C)}{E(2-C)}\left[T_{rr}-T_{\theta\theta}\right],$$
$$e_{r\theta} = e_{\theta z} = e_{zr} = 0, \tag{7}$$

where E is the Young's modulus and C is compressibility factor of the material in term of Lame's constant, there are given by $C = 2\mu/\lambda+2\mu$ and $E = \mu(3\lambda+2\mu)/(\lambda+\mu)$.

Substituting equation (4) in equation (6), we get the stresses as

$$T_{rr} = (2\mu/n)\left[3-2C-\beta^n\left\{1-C+(2-C)\left((r\beta'/\beta)+1\right)^n\right\}\right],$$
$$T_{\theta\theta} = (2\mu/n)\left[3-2C-\beta^n\left\{2-C+(1-C)\left[(r\beta'/\beta)+1\right]^n\right\}\right],$$
$$T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0, \tag{8}$$

Equations of equilibrium are all satisfied except

$$\frac{d}{dr}(rT_{rr}) - T_{\theta\theta} + \rho\omega^2 r^2 = 0, \tag{9}$$

where ρ is the density of the material of the disc.

Using equation (8) in equation (9), we get a non- linear differential equation in β as

$$(2-C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta} = \frac{n\rho\omega^2 r^2}{2\mu}+\beta^n\left[1-(P+1)^n-nP\left\{1-C+(2-C)(P+1)^n\right\}\right]. \tag{10}$$

where $r\beta' = \beta P$ (P is function of β and β is function of r).Transition or turning points of β in equation (10) are $P\to 1$ and $P\to\pm\infty$.The boundary conditions are:

$$u = 0 \text{ at } r = a \text{ and } T_{rr} = 0 \text{ at } r = b. \tag{11}$$

III. Solution Through the PrincipaL Stress

For finding the plastic stress, the transition function is taken through the principal stress (see Seth [7, 8], Hulsurkar [9] and Gupta et al. [10 - 15]) at the transition point $P\to\pm\infty$. We take the transition function R as

$$R = \frac{n}{2\mu}T_{\theta\theta} = \left[(3-2C)-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}\right] \tag{12}$$

Taking the logarithmic differentiation of equation (12) with respect to r and using equation (10), we get

$$\frac{d(\log R)}{dr} = -\frac{\beta^n\left(\frac{1-C}{2-C}\right)\left[\begin{matrix}1-(P+1)^n\\ -n(1-C)P+\frac{n\rho\omega^2r^2}{2\mu\beta^n}\end{matrix}\right]+(2-C)nP\beta^n}{r\left[3-2C-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}\right]} \tag{13}$$

Taking the asymptotic value of equation (13) at $P\to\pm\infty$ and integrating, we get

$$R = A_1 r^{-1/(2-C)} \tag{14}$$

where A_1 is a constant of integration, which can be determine by boundary condition. From equation (12) and (14), we have

$$T_{\theta\theta} = (2\mu/n)A_1 r^{-1/(2-C)} \tag{15}$$

Substituting equation (15) in equation (9) and integrating, we get

$$T_{rr} = \left\{\frac{2\mu(2-C)}{n(1-C)}\right\}A_1 r^{-1/(2-C)} - \frac{\rho\omega^2r^2}{3}+\frac{B_1}{r} \tag{16}$$

where B_1 is a constant of integration, which can be determine by boundary condition.

Substituting equations (15) and (16) in second equation of (7), we get

$$\beta = \sqrt{1-\frac{2(1-C)}{E(2-C)}\left[\frac{\rho\omega^2r^2}{3}-\frac{B_1}{r}\right]} \tag{17}$$

Substituting equation (17) in equation (2), we get

$$u = r - r\sqrt{1-\frac{2(1-C)}{E(2-C)}\left[\frac{\rho\omega^2r^2}{3}-\frac{B_1}{r}\right]} \tag{18}$$

where $E = 2\mu(3-2C)/(2-C)$ is the Young's modulus in term of compressibility factor can be expressed as. Using boundary condition (11) in equations (16) and (18), we get

$$A_1 = \frac{\rho\omega^2 n(1-C)\left(b^3-a^3\right)}{6\mu(2-C)b^{1-C/2-C}},\ B_1 = \frac{\rho\omega^2a^3}{3} \tag{19}$$

Substituting the values of constant integration A_1 and B_1 from equation (19) in equations (15), (16), and (18) respectively, we get the transitional stresses and displacement as

$$T_{\theta\theta} = \frac{\rho\omega^2(1-C)\left(b^3-a^3\right)}{3r(2-C)}\left(\frac{r}{b}\right)^{1-C/2-C} \tag{20}$$

$$T_{rr} = \frac{\rho\omega^2}{3r}\left[\left(b^3-a^3\right)\left(\frac{r}{b}\right)^{1-C/2-C} - r^3 + a^3\right] \tag{21}$$

and $$u = r - r\sqrt{1-\frac{2}{E}\left(\frac{1-C}{2-C}\right)\frac{\rho\omega^2\left(r^3-a^3\right)}{3r}} \tag{22}$$

From equation (20) and (21), we get

$$T_{rr} - T_{\theta\theta} = \left(\frac{\rho\omega^2}{3}\right)\left[\frac{\left(b^3-a^3\right)}{(2-C)r}\left(\frac{r}{b}\right)^{1-C/2-C} - r^2 + \frac{a^3}{r}\right] \tag{23}$$

Initial yielding—

From equation (23), it is seen that $|T_{rr} - T_{\theta\theta}|$ is maximum at the internal surface (that is at $r = a$), therefore yielding will take place at the internal surface of the disc and equation (23) gives

$$\left|T_{rr} - T_{\theta\theta}\right|_{r=a} = \left|\frac{\rho\omega^2\left(b^3-a^3\right)}{3(2-C)a}\left(\frac{a}{b}\right)^{1-C/2-C}\right| \equiv Y(say).$$

and the angular speed necessary for initial yielding is given by

$$\Omega_i^2 = \frac{\rho\omega_i^2 b^2}{Y} = \left|\frac{3ab^2(2-C)}{\left(b^3-a^3\right)}\left(\frac{b}{a}\right)^{1-C/2-C}\right| \tag{24}$$

and $\omega_i = \Omega_i / b\sqrt{Y/\rho}$.

Fully-plastic state—

The disc becomes fully plastic $(C \to 0)$ at the external surface and equations (23) becomes,

$$\left|T_{rr} - T_{\theta\theta}\right|_{r=b} = \left|\frac{\rho\omega^2\left(b^3-a^3\right)}{6b}\right| \equiv Y^*.$$

Angular speed required for the disc to become fully plastic is given by

$$\Omega_f^2 = \frac{\rho\omega_f^2 b^2}{Y^*} = \left|\frac{6}{\left(1-a^3/b^3\right)}\right|, \tag{25}$$

where $\omega_f = \Omega_f / b\sqrt{Y^*/\rho}$.

We introduce the following non-dimensional components

$R = r/b$, $R_0 = a/b$, $\sigma_r = T_{rr}/Y$, $\sigma_\theta = T_{\theta\theta}/Y$, $Y/E = H$, $Y^*/E = H^*$ and $\bar{u} = u/b$.

Elastic-plastic transitional stresses and angular speed from equations (20), (21) and (24) in non-dimensional form become

$$\sigma_\theta = \frac{\Omega_i^2}{3R}\left[\frac{\left(1-R_0^3\right)(1-C)}{(2-C)}R^{1-C/2-C}\right] \tag{26}$$

$$\sigma_r = \frac{\Omega_i^2}{3R}\left[\left(1-R_0^3\right)R^{1-C/2-C} - R^3 + R_0^3\right] \tag{27}$$

and $$\Omega_i^2 = \frac{3(2-C)}{\left(1-R_0^3\right)}R_0^{1/(2-C)} \tag{28}$$

Using equations (17), (20) and (21) in first equation of equation (7), we get

$$H = \frac{\frac{2\left(1-R_0^3\right)(3-2C)R^{1-C/2-C}}{(1-C)(2-C)} + \frac{2\left(R_0^3-R^3\right)(2-C)}{(1-C)} - 6R^3}{\frac{\Omega_i^2}{3R}\left[\frac{4\left(R^3-R_0^3\right)\left(1-R_0^3\right)(3-2C)R^{1-C/2-C}}{(2-C)^2} - \left(\frac{1-C}{2-C}\right)\left(4R^3-R_0^3\right)^2 - 4\left(R^3-R_0^3\right)^2\right]}$$

The displacement component $\bar{u}$ from equation (22) is given by

$$\bar{u} = R - R\sqrt{1-2H\left(\frac{1-C}{2-C}\right)\left[\frac{\Omega_i^2}{3R}\left(R^3-R_0^3\right)\right]} \tag{29}$$

Stresses, displacement and angular speed for fully-plastic state $(C \to 0)$ are obtained from equations (26), (27), (29) and (25) as,

$$\sigma_\theta = \frac{\Omega_f^2}{6R}\left[\left(1-R_0^3\right)R^{1/2}\right] \tag{30}$$

$$\sigma_r = \frac{\Omega_f^2}{3R}\left[\left(1-R_0^3\right)R^{1/2} - R^3 + R_0^3\right] \tag{31}$$

$$\bar{u} = R - R\sqrt{1-H^*\left[\frac{\Omega_f^2}{3R}\left(R^3-R_0^3\right)\right]} \tag{32}$$

where $$H^* = \frac{\left[3\left(1-R_0^3\right)R^{1/2} + 4\left(R_0^3-R^3\right) - 6R^3\right]}{\frac{\Omega_f^2}{3R}\left[3\left(R^3-R_0^3\right)\left(1-R_0^3\right)R^{1/2} - \frac{1}{2}\left(4R^3-R_0^3\right)^2 - 4\left(R^3-R_0^3\right)^2\right]}.$$

and $$\Omega_f^2 = \frac{\rho\omega_f^2 b^2}{Y^*} = \left|\frac{6}{\left(1-R_0^3\right)}\right| \tag{33}$$

TABLE I
ANGULAR SPEED REQUIRED FOR INITIAL YIELDING AND FULLY PLASTIC STATE

	Compressibility	Poisson's Ratio $\left(\nu = \frac{1-C}{2-C}\right)$	Angular Speed required for initial yielding Ω_i^2	Angular Speed required for fully-plastic state Ω_f^2	Percentage increase in Angular speed $\left(\sqrt{\Omega_f^2/\Omega_i^2}-1\right)\times 100$
$0.5 \le R \le 1$	0	0.5	4.848732	6.85714	18.92069 %
	0.25	0.4285	4.037701	6.85714	30.31801 %
	0.5	0.333	3.239797	6.85714	45.4812 %
	0.75	0.2	2.461496	6.85714	66.9061 %

IV. Numerical Illustration and Discussion

In Fig. 1, curves have been drawn between angular speed Ω_i^2 and various radii ratios $R_0 = a/b$ for $C = 0\,(\nu = 0.5)$, $0.25\,(\nu = 0.4285)$, $0.5\,(\nu = 0.333)$, $0.75\,(\nu = 0.2)$. It has been observed that the rotating disc made of incompressible material with inclusion required higher angular speed to yield at the internal surface as compared to disc made of compressible material and a much higher angular speed is required with the increase in radii ratio. It can be seen from Table I, that for Isotropic compressible material, higher percentage increase in angular speed is required to become fully plastic as compared to rotating disc made of incompressible material. In Fig. 3, curve have been drawn between stresses, displacement and radii ratio $R = r/b$ for fully plastic state. It has been observed that the radial stress is maximum at the internal surface. Similar graph was also obtained by Güven [17], for rotating disc with rigid inclusion in to the account for $\nu = 0.333$ for linear strain hardening material behavior into account.

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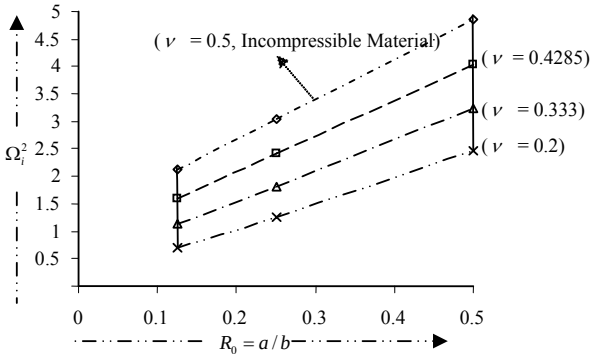

Fig. 1 Angular speed required for initial yielding at the internal surface of the rotating disc

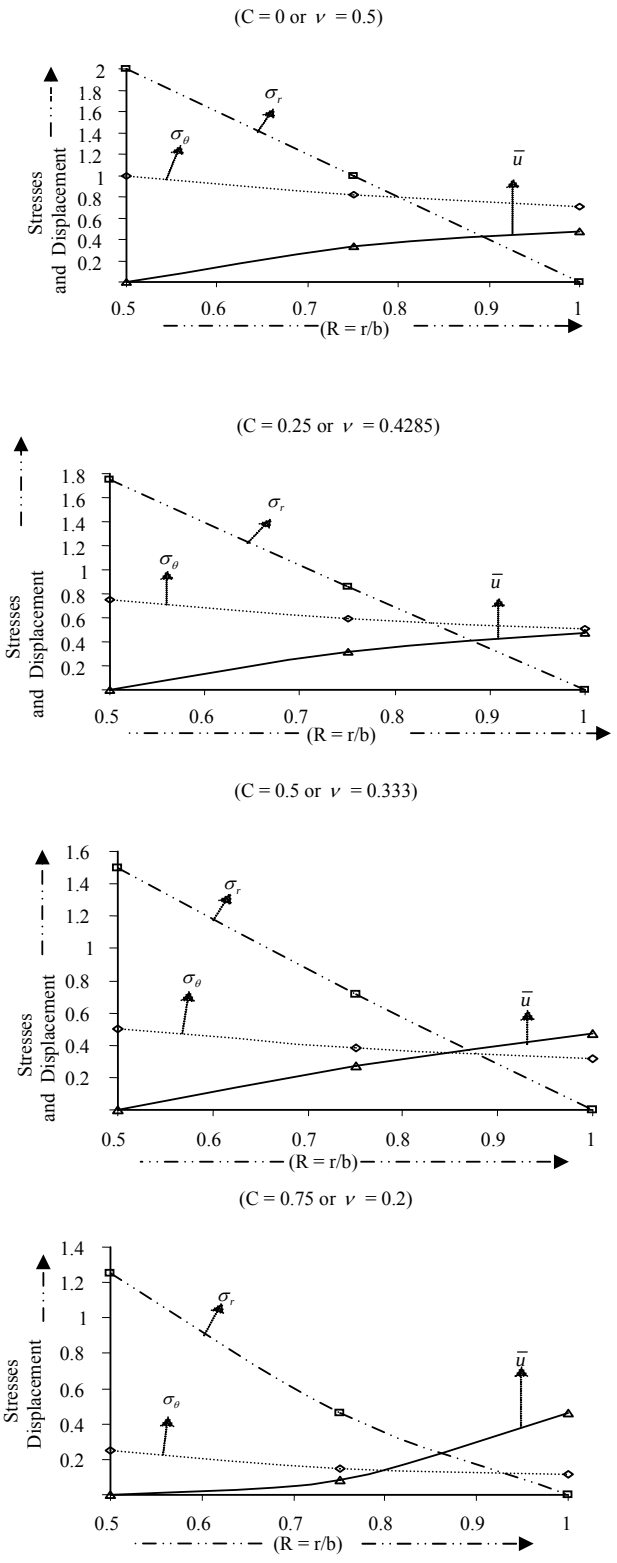

Fig. 2 Stresses and Displacement at the elastic-plastic Transition state

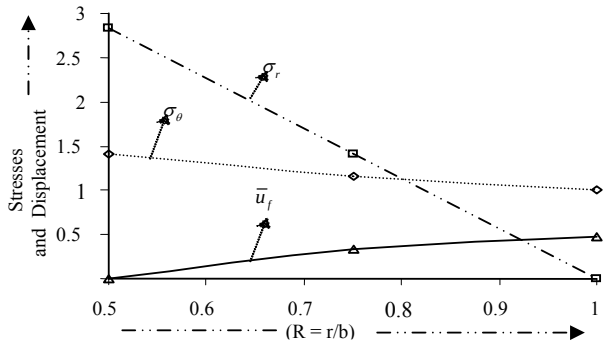

Fig. 3 Stresses and displacement for fully plastic state

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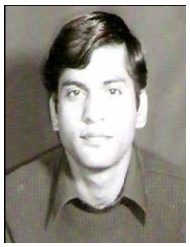

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