Effects of Slip Condition and Peripheral Layer on Couple Stress Fluid Flow through a Channel with Mild Stenosis

Gurju Awgichew and G.Radhakrishnamacharya

Abstract—Steady incompressible couple stress fluid flow through two dimensional symmetric channel with stenosis is investigated. The flow consisting of a core region to be a couple stress fluid and a peripheral layer of plasma (Newtonian fluid). Assuming the stenosis to be mild, the equations governing the flow of the proposed model are solved using the slip boundary condition and closed form expressions for the flow characteristics (the dimensionless resistance to flow and wall shear stress at the maximum height of stenosis) are derived. The effects of various parameters on these flow variables have been studied. It is observed that the resistance to flow as well as the wall shear stress increase with the height of stenosis, viscosity ratio and Darcy number. However, the trend is reversed as the slip and the couple stress parameter increase.

Keywords—Stenosis, Couple stress fluid, Slip condition, Peripheral layer.

I. Introduction

THE term stenosis denotes the narrowing of an artery due to the development of arteriosclerotic plaques or other types of abnormal tissue development. This can cause circulatory disorders by reducing or occluding the blood supply which may result in serious consequences (cerebral strokes, myocardial infarction). Hence, the mathematical modelling of this type of flows may help in proper understanding and prevention of arterial diseases. The actual reason for the formation of stenosis is not known but many researchers have studied its effect on the flow characteristics (Young[1]; Zendehbudi and Moayeri [2]; Radhakrishnamacharya and Srinivasa Rao[3]) by assuming blood as a Newtonian fluid. But blood shows a non-Newtonian behaviour at low shear rates in tubes of smaller diameters (Whitmore[4]; Forrestor and Young[5]; Shukla et al.[6]; Misra and Ghosh[7]; Jain et al.[8] and Gupta et al.[9]). The non-Newtonian behaviour of blood is mainly due to the suspension of red blood cells in plasma. When neutrally buoyant corpuscles are contained in a fluid and there exists a velocity gradient due to shearing stress, corpuscles have rotatory motion. Furthermore, it is observed that corpuscles have spin angular momentum, in addition to orbital angular momentum. As a result, the symmetry of stress tensor is lost in the fluid motion that is subjected to spin angular momentum. The fluid that has neutrally buoyant corpuscles, when observed macroscopically, exhibits non-Newtonian behaviour, and its constitutive equation is expressed by Stokes[10]. This

represents the simplest generalization of the classical viscous fluid theory that sustains couple stresses and the body couples. The important feature of these fluids is that the stress tensor is not symmetric and their accurate flow behaviour cannot be predicted by the classical Newtonian theory. The main effect of couple stresses will be to introduce a size dependent effect that is not present in the classical viscous theories. The importance of consideration of couple stress effects in studies of physiological and some other fluids was indicated by Cowin[11]. Studies on the couple stress fluid behaviour are very useful, because such studies bear the potential to better explain the behaviour of rheologically complex fluids, such as liquid crystals, polymeric suspensions that have long-chain molecules, lubrication as well as human/sub-human blood (Stokes[10]). Sankad and Radhakrishnamacharya[12], Srinavasacharya and Srikanth[13] and Naeem et al.[14] studied the flow of couple stress fluid under different conditions.

It is experimentally observed that when blood flows through narrow tubes, there exists a cell-free plasma layer near the wall (Bugliarello and Hyden[15]) and (Bugliarello and Sevilla[16]). In view of their experiments, it is preferable to represent the flow of blood through narrow tubes by a two layered model instead of one layered model. Shukla el al.[17] have considered a two layered model in which the peripheral plasma layer and the core are both Newtonian fluids. Shukla et al.[18] have also carried out a two layered model in which fluids in both regions are non-Newtonian in character and examined the influence of the peripheral layer viscosity on the resistance to flow. Chaturani and Kaloni[19], Chaturani and Ponalagusamy[20] and Ponalagusamy and Tamil Selvi[21] have also contributed towards the flow of blood represented by a two layered model.

In the present study, the effect of slip on flow through two dimensional symmetric channel with mild stenosis has been investigated. The flow region is assumed to consist of a core with couple stress fluid and a peripheral layer with Newtonian fluid. Assuming the stenosis to be mild and using the slip boundary condition, the equations governing the flow have been solved and analytical expressions for the resistance to flow and the wall shear stress at the maximum height of stenosis have been derived. The effects of couple stress parameter, viscosity ratio, Darcy number, slip parameter and height of stenosis on the flow characteristics have been investigated and shown graphically.

G. Radhakrishnamacharya, National Institute of Technology, Warangal-506004, A.P, India(corresponding author e-mail: grk.nitw@yahoo.com)
Gurju Awgichew, Antonymous Institute,(e-mail: gurjua@yahoo.com)

II. MATHEMATICAL FORMULATION

We consider steady and incompressible fluid flow in a channel, with Newtonian fluid in the peripheral layer and couple stress fluid in the core region. Cartesian coordinate system is chosen so that the x-axis coincides with the center line of the channel and the y-axis normal to it. The stenosis is supposed to be mild and develops in a symmetric manner. The geometry of the wall is taken as (Shukla et al.[6])

$$\frac{h}{h_0} = \begin{cases} 1 - \frac{\delta_s}{2h_0}(1 + \cos\frac{2\pi}{L_0}[x - d - \frac{L_0}{2}]), & d \le x \le d + L_0 \\ 1 & , \quad \text{otherwise} \end{cases}$$

(1)

where h_0 is the mean half width of the non-stenotic region of the channel, L is the length of the channel, L_0 is the length of stenosis and δ_s is the maximum height of stenosis.(Fig.1) The geometry of the interface between the peripheral and the core region is taken as(Shukla et al.[6])

$$\frac{h_1}{h_0} = \begin{cases} \beta - \frac{\delta_i}{2h_0}(1 + \cos\frac{2\pi}{L_0}[x - d - \frac{L_0}{2}]), & d \le x \le d + L_0 \\ \beta & , \quad \text{otherwise} \end{cases}$$

(2)

where β is the ratio of the central mean half width to the channel mean half width in the unobstructed region and δ_i is the maximum bulging of the interface at $x = d + L_0/2$ due to the presence of stenosis.(Fig.1) The appropriate equations
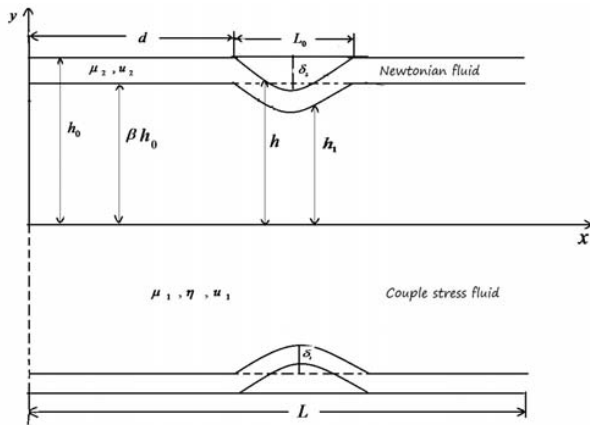

Fig. 1. Geometry of arterial stenosis with peripheral layer

describing the flow in the central region and peripheral layer are given as(Srinavasacharya and Srikanth[13])

$$\rho_1\left[\frac{\partial \overline{q}}{\partial t} + (\overline{q} \cdot \nabla)\overline{q}\right] = -\nabla p + \mu_1\nabla^2\overline{q} - \eta\nabla^4\overline{q}; 0 \le y \le h_1(x)$$

(3)

$$\rho_2\left[\frac{\partial \overline{q_1}}{\partial t} + (\overline{q_1} \cdot \nabla)\overline{q_1}\right] = -\nabla p + \mu_2\nabla^2\overline{q_1} ; h_1(x) \le y \le h(x)$$

(4)

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, ρ_1 and ρ_2 are the densities and

$\overline{q} = (u_1, v_1, 0)$ and $\overline{q_1} = (u_2, v_2, 0)$ are the velocity vectors of the fluids in the central region and peripheral layer respectively, p is the fluid pressure, t is time, μ_2 is the viscosity of plasma in the peripheral layer, μ_1 is the viscosity coefficient of the classical fluid in the core region and η is the couple stress fluid viscosity.

For the present problem, by neglecting body forces and body couples(Alemayehu and Radhakrishnamacharya[22]) and taking the restrictions for mild stenosis (Young[1]) , Eqs.(3) and (4) get reduced to

$$\frac{dp}{dx} = \mu_1\frac{\partial^2 u_1}{\partial y^2} - \eta\frac{\partial^4 u_1}{\partial y^4} \quad ; 0 \le y \le h_1(x)$$

(5)

$$\frac{dp}{dx} = \mu_2\frac{\partial^2 u_2}{\partial y^2} \quad ; \quad h_1(x) \le y \le h(x)$$

(6)

$$\frac{\partial u_1}{\partial x} = 0 \quad at \quad y = 0$$

(7)

$$u_2 = \frac{-h_0\sqrt{D_a}}{\alpha_1}\frac{\partial u_2}{\partial y} \quad at \quad y = \pm h(x)$$

(8)

$$\frac{\partial^2 u_1}{\partial y^2} = 0 \quad at \quad y = \pm h_1(x)$$

(9)

$$u_1 = u_2 \quad and \quad \tau_1 = \tau_2 \quad at \quad y = \pm h_1(x)$$

(10)

Here (8) is the Saffman's slip boundary condition(Bhatt and Sacheti[23]) and (9) indicates the vanishing of couple stress. Further, D_a is the permeability parameter (or Darcy number) and α_1 is the slip parameter, τ_1 and τ_2 are the shearing stress of the central and peripheral layers, respectively.

Solving (5) and (6), subject to the boundary conditions (7)-(10), the expressions for velocities u_1 and u_2 can obtained as

$$u_1(y) = -\frac{h_0^2}{2\mu_2}\frac{dp}{dx}\left[(\frac{h}{h_0})^2 - \overline{\mu_2}(\frac{y}{h_0})^2 - (1 - \overline{\mu_2})(\frac{h_1}{h_0})^2\right.$$
$$- \frac{2\,\overline{\mu_2}}{m^2} + \frac{2\sqrt{Da}}{\alpha_1}(\frac{h_1}{h_0}) - \frac{2Da}{\alpha_1^2}$$
$$\left. + \frac{2\,\overline{\mu_2}}{m^2}\frac{cosh(m(\frac{y}{h_0}))}{cosh(m(\frac{h_1}{h_0}))}\right]$$
$$for \quad 0 \le y \le h_1(x) \quad (11)$$

$$u_2(y) = -\frac{h_0^2}{2\mu_2}\frac{dp}{dx}\left[(\frac{h}{h_0})^2 - (\frac{y}{h_0})^2 + \frac{2\sqrt{Da}}{\alpha_1}(\frac{y}{h_0})\right.$$
$$\left. - \frac{2Da}{\alpha_1^2}\right] \quad for \quad h_1(x) \le y \le h(x) \quad (12)$$

where $\overline{\mu_2} = \mu_2/\mu_1$ and $m = h_0(\mu_1/\eta)^{1/2}$ is the couple stress parameter.

III. ANALYSIS

The flow flux Q , which is defined as

$$Q = 2\int_0^{h_1} u_1 dy + 2\int_{h_1}^h u_2 dy$$

(13)

can be obtained in the following form using (11) and (12),

$$Q = -\frac{2}{3\mu_2}\frac{h_0^3}{dx}\frac{dp}{dx}\Big[(\frac{h}{h_0})^3 - (1-\overline{\mu_2})(\frac{h_1}{h_0})^3$$
$$-\frac{3}{m^2}\frac{\overline{\mu_2}}{m^2}(\frac{h_1}{h_0}) + \frac{3\sqrt{Da}}{2\alpha_1}\Big((\frac{h_1}{h_0})^2 + (\frac{h}{h_0})^2\Big)$$
$$-\frac{3Da}{\alpha_1^2}(\frac{h}{h_0}) + \frac{3}{m^3}\frac{\overline{\mu_2}}{m^3}tanh(m(\frac{h_1}{h_0}))\Big] \quad (14)$$

Introducing the the following dimensionless quantities

$$x' = \frac{x}{L}, \quad L_0' = \frac{L_0}{L}, \quad d' = \frac{d}{L} \quad (15)$$

in (1) and (2), and from (14) the pressure gradient can be obtained as(after dropping the primes)

$$\frac{dp}{dx} = -\frac{3\mu_2 Q}{2h_0^3}\frac{1}{G} \quad (16)$$

where

$$G = (\frac{h}{h_0})^3 - (1-\overline{\mu_2})(\frac{h_1}{h_0})^3 - \frac{3}{m^2}\frac{\overline{\mu_2}}{m^2}(\frac{h_1}{h_0})$$
$$+\frac{3\sqrt{Da}}{2\alpha_1}\Big((\frac{h_1}{h_0})^2 + (\frac{h}{h_0})^2\Big) - \frac{3Da}{\alpha_1^2}(\frac{h}{h_0})$$
$$+\frac{3}{m^3}\frac{\overline{\mu_2}}{m^3}tanh(m(\frac{h_1}{h_0}))$$

Integrating (16) with respect to x, we get pressure difference along the total length of a channel as

$$\Delta p = \frac{3\mu_2 Q}{2h_0^3}\int_0^1 \frac{1}{G}dx \quad (17)$$

The resistance to flow, denoted by λ , is defined by

$$\lambda = \frac{\Delta p}{Q} = \frac{3\mu_2}{2h_0^3}\int_0^1 \frac{1}{G}dx \quad (18)$$

The shearing stress at the wall is given as

$$\tau_w = -\mu_2\frac{\partial u_2}{\partial y}\Big|_{y=h} \quad (19)$$

The wall shear stress at the maximum height of stenosis, i.e. at $x = d + L_0/2$ obtained from (19) as

$$\tau_s = \frac{3\mu_2 Q}{2h_0^2}\frac{A_1}{A_2} \quad (20)$$

where

$$\epsilon_1 = 1 - (\delta_s/h_0), \epsilon_2 = 1 - (\delta_i/h_0), A_1 = (\epsilon_1 + \frac{\sqrt{Da}}{\alpha_1}),$$
$$A_2 = (\epsilon_1)^3 - (1-\overline{\mu_2})(\epsilon_2)^3 - \frac{3\overline{\mu_2}}{m^2}(\epsilon_2) - \frac{3Da}{\alpha_1^2}(\epsilon_1)$$
$$+\frac{3\sqrt{Da}}{2\alpha_1}((\epsilon_1)^2 + (\epsilon_2)^2) + \frac{3\overline{\mu_2}}{m^3}tanh(m\ \epsilon_2).$$

Using $h_1 = \beta h$ and $\delta_i = \beta\delta_s$(Shukla et al.[6]) , expressions for the dimensionless resistance to flow $\overline{\lambda}$ and shearing stress $\overline{\tau_s}$ can be obtained as:

$$\overline{\lambda} = \overline{\mu_2}I\int_0^1 \frac{dx}{B_1} \quad (21)$$

$$\overline{\tau_s} = \frac{\overline{\mu_2}IA_1}{B_2} \quad (22)$$

where

$$I = 1 - \frac{3}{m^2} + \frac{3\sqrt{Da}}{\alpha_1} - \frac{3Da}{\alpha_1^2} + \frac{3}{m^3}tanh(m),$$
$$B_1 = \Big(1 - (1-\overline{\mu_2})\beta^3\Big)(\frac{h}{h_0})^3 + \frac{3\sqrt{Da}}{2\alpha_1}(\beta^2+1)(\frac{h}{h_0})^2$$
$$-3(\frac{\overline{\mu_2}\beta}{m^2} + \frac{Da}{\alpha_1^2})(\frac{h}{h_0}) + \frac{3\overline{\mu_2}}{m^3}tanh(m\beta(\frac{h}{h_0})),$$
$$B_2 = (1 + \frac{\sqrt{Da}}{\alpha_1})\Big[(1 - (1-\overline{\mu_2})\beta^3)\epsilon_1^3 - (\frac{3\overline{\mu_2}\beta}{m^2} + \frac{3Da}{\alpha_1^2})\epsilon_1$$
$$+\frac{3\sqrt{Da}}{2\alpha_1}(\beta^2+1)\epsilon_1^2 + \frac{3\overline{\mu_2}}{m^3}tanh(m\beta\epsilon_1)\Big],$$
$$\overline{\lambda} = \frac{\lambda}{\lambda_c}, \quad \overline{\tau_s} = \frac{\tau_s}{\tau_c}.$$

Terms λ_c and τ_c are the resistance to flow and the wall shear stress, respectively in the absence of peripheral layer with no stenosis.

IV. RESULTS AND DISCUSSION

The resistance to flow and the wall shear stress are the two important characteristics in the study of fluid flow through a stenosed artery. The expressions for resistance to the flow and wall shear stress, given by (21) and (22) respectively have been numerically evaluated using MATHEMATICA software for different values of relevant parameters and presented graphically.

Figs.2-7 show the effects of various parameters on the resistance to the flow. It can be observed that the resistance to the flow increases with the height of the stenosis (Figs.2-7). This result agrees with the previous results obtained by Shukla et al.[6]; Chaturani and Ponalagusamy[20], Maruthi Prasad and Radhakrishnamacharya[24]. Further, it can be noticed that the resistance to the flow increases with viscosity ratio ($\overline{\mu_2}$)(Fig.2), Darcy number (Fig.3), half width ratio(β) (Fig.4) and length of the stenosis (Fig.5) but decreases with couple stress parameter (Fig.6) and the slip parameter(Fig.7).

Figs.8-12 show the effects of various parameters on the wall shear stress at the maximum height of the stenosis. The wall shear stress increases with the height of the stenosis (Figs.8-12). This result agrees with previous results obtained by Shukla et al.[6], Maruthi Prasad and Radhakrishnamacharya[24], Chaturani and Ponalagusamy[20]. Moreover, the wall shear stress increases with viscosity ratio($\overline{\mu_2}$) (Fig.8), Darcy number (Fig.9) and half width ratio(β) (Fig.10) but decreases with the slip parameter(Fig.11) and the couple stress parameter(Fig.12). However, the decrease with couple stress parameter is not very significant.

V. CONCLUSION

A mathematical model for the steady flow of couple stress fluid in the core region of a two layered flow having mild stenosis in the lumen of the channel has been investigated. It has been shown that both the resistance to flow and the
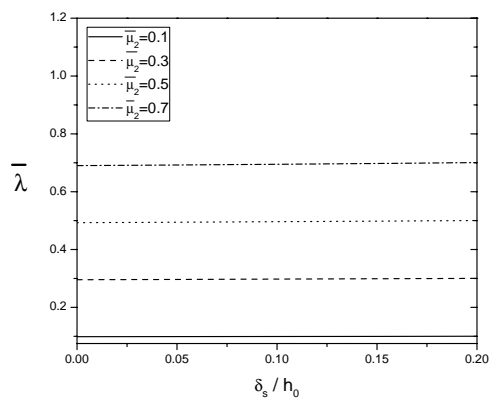
Fig. 2. Effect of $\overline{\mu_2}$ on $\overline{\lambda}$ ($d = 0.4, L_0 = 0.2, \alpha_1 = 0.02, Da = 0.005, m = 0.2, \beta = 0.95$)
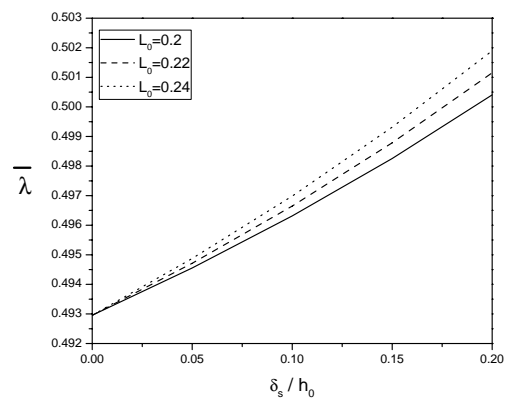

Fig. 5. Effect of L_0 on $\overline{\lambda}$ ($d = 0.4, \beta = 0.95, \alpha_1 = 0.02, \overline{\mu_2} = 0.5, m = 0.2, Da = 0.005$)
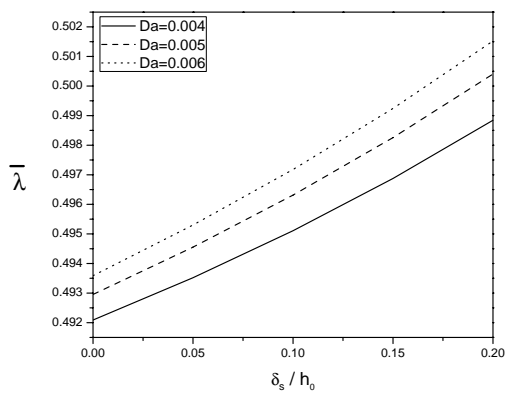

Fig. 3. Effect of Da on $\overline{\lambda}$ ($d = 0.4, L_0 = 0.2, \alpha_1 = 0.02, \overline{\mu_2} = 0.5, m = 0.2, \beta = 0.95$)
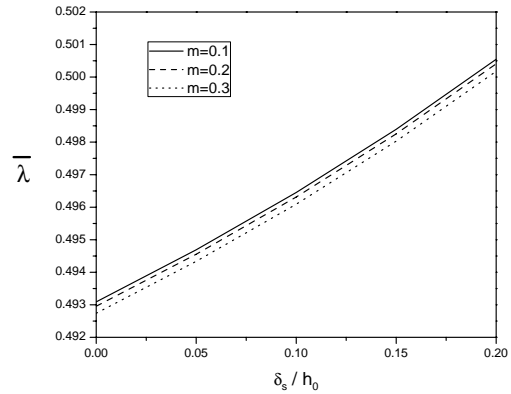

Fig. 6. Effect of m on $\overline{\lambda}$ ($d = 0.4, \beta = 0.95, \alpha_1 = 0.02, \overline{\mu_2} = 0.5, L_0 = 0.2, Da = 0.005$)
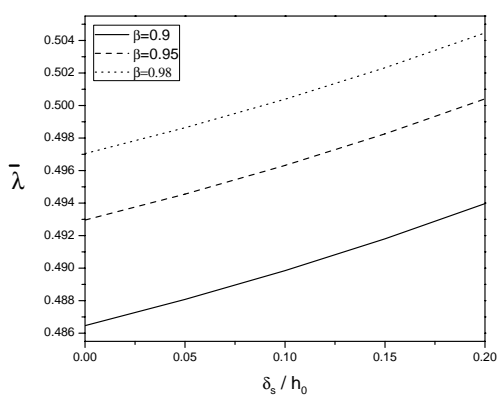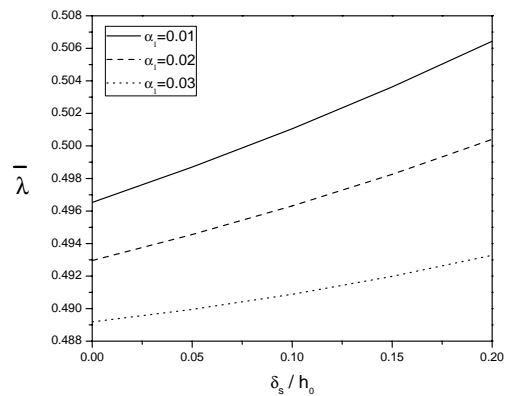

Fig. 4. Effect of β on $\overline{\lambda}$ ($d = 0.4, L_0 = 0.2, \alpha_1 = 0.02, \overline{\mu_2} = 0.5, m = 0.2, Da = 0.005$)
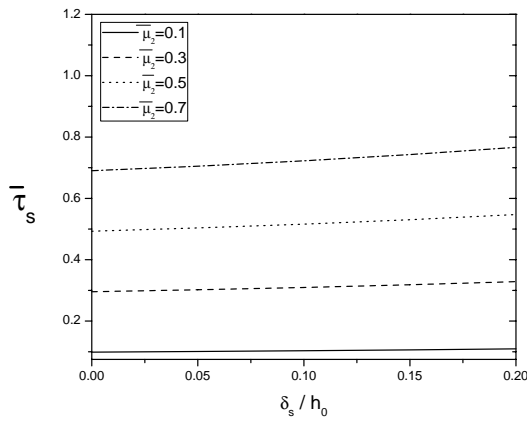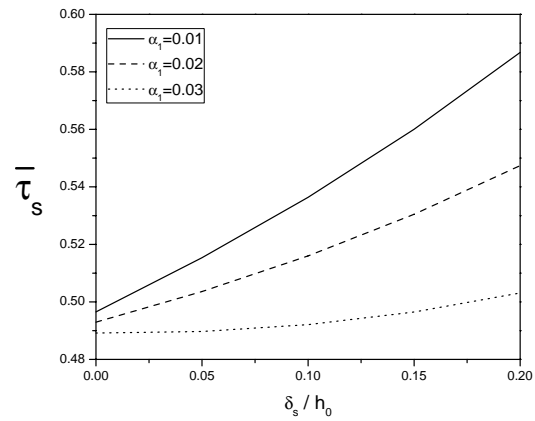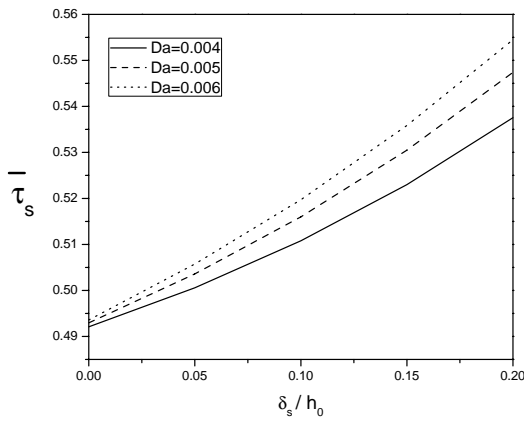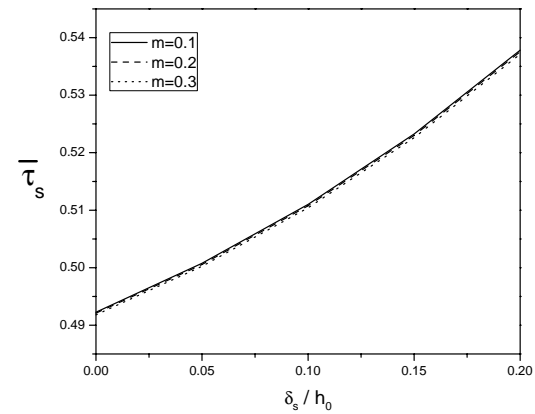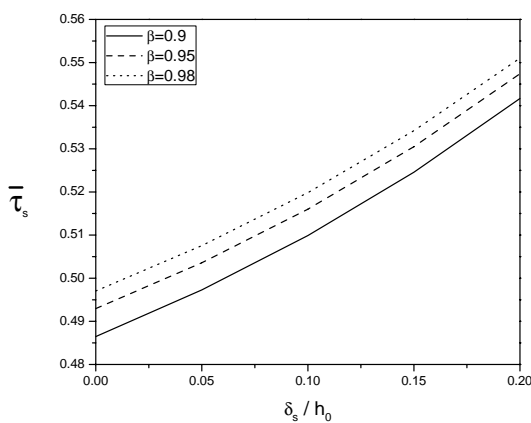

Fig. 7. Effect of α_1 on $\overline{\lambda}$ ($d = 0.4, \beta = 0.95, m = 0.2, \overline{\mu_2} = 0.5, L_0 = 0.2, Da = 0.005$)

Fig. 8.   Effect of $\overline{\mu_2}$ on $\overline{\tau_s}$ ($\beta = 0.95, m = 0.2, \alpha_1 = 0.02, Da = 0.005$)

Fig. 11.   Effect of α_1 on $\overline{\tau_s}$ ($Da = 0.005, m = 0.2, \beta = 0.95, \overline{\mu_2} = 0.5$)

Fig. 9.   Effect of Da on $\overline{\tau_s}$ ($\beta = 0.95, m = 0.2, \alpha_1 = 0.02, \overline{\mu_2} = 0.5$)

Fig. 12.   Effect of m on $\overline{\tau_s}$ ($Da = 0.005, \alpha_1 = 0.02, \beta = 0.95, \overline{\mu_2} = 0.5$)

wall shear stress at the maximum height of the stenosis decreases with the slip parameter. Further, it is observed that the resistance to flow and the wall shear stress at the maximum height of the stenosis increase with the viscosity ratio ($\overline{\mu_2}$).

Fig. 10.   Effect of β on $\overline{\tau_s}$ ($Da = 0.005, m = 0.2, \alpha_1 = 0.02, \overline{\mu_2} = 0.5$)

REFERENCES

[1] D. F. Young, Effects of a time-dependent stenosis on flow through a tube, J. Engrg. Ind.Trans.ASME.,vol.90, pp. 248-254, 1968.
[2] G.R.Zendehbudi and M.S.Moayeri, Comparison of physiological and simple plusatile flows through stenosed arteries, J. Biomech., vol.32, pp. 959-965,1999.
[3] G. Radhakrishnamacharya and P. Srinivasa Rao, Flow of a magnetic fluid through a non-uniform wavy tube, Proc.Nat.Acad.Sci.India, vol.76 , pp. 241-245, 2007.
[4] R.L.Whitmore, Rheology of the Circulation, Pergamon Press, New York ,1968.
[5] J.H. Forrester and D.F. Young, Flow through a converging-diverging tube and its implications in occlusive vascular disease, J. Biomech., vol.3, pp. 307-316, 1970.
[6] J. B. Shukla, R. S. Parihar and B. R. P. Rao, Effects of stenosis on non-Newtonian flow of the blood in an artery,Bull. Math. Biol., vol.42, pp. 283-294, 1980.

[7] J.C.Misra and S.K. Ghosh S K, A mathematical model for the study of blood flow through a channel with permeable walls, Acta Mechanica, vol.12, pp. 137-153, 1997.

[8] N.Jain , S.P.Singh and M. Gupta, Steady flow of blood through an atherosclerotic artery: A non-Newtonian model, International Journal of Applied Mathematics and Mechanics,vol.8, pp. 52-63, 2012.

[9] S. Gupta, M. Gupta and S.P. Singh SP, Effect of radial viscosity variation on non-Newtonian flow of blood in a stenosed artery.International Journal of Applied Mathematics and Mechanics, vol.8, pp. 51-61, 2012.

[10] V. K. Stokes, Couple Stresses in Fluids, Phys. Fluids., vol.9, pp. 1709-1715,1966.

[11] S.C.Cowin, The theory of polar fluids, Advances in Applied Mechanics, Academic Press, New York, pp. 279-347, 1974.

[12] G. C. Sankad and G. Radhakrishnamacharya, Effect of Magnetic field on the peristaltic transport of couple stress fluid in a channel with wall properties, Int. J. Biomath., vol.4, pp. 365-378,2011.

[13] D. Srinivasacharya and D. Srikanth, Effect of couple stresses on the pulsatile flow through a constricted annulus,Comptus Rendus Mecanique., vol.336, pp. 820-827,2008.

[14] R.K. Naeem, S.Younus and Dania, Inverse solutions for unsteady incompressible couple stress fluid flows, International Journal of Applied Mathematics and Mechanics, vol.6, pp.1-17, 2010.

[15] G. Bugliarello and J.W. Hyden, Detailed characteristics of the flow of blood in vitro, Trans.Soc.Rheol., vol.7, pp. 209-230, 1963.

[16] G.Bugliarello and J.Sevilla, Velocity distribution and other characteristics of steady and pulsatile blood flow in fine glass tubes, Biorheology, vol.7, pp. 85-107,1970.

[17] J.B.Shukla, R.S.Parihar and B.R.P.Rao BRP, Effect of peripheral layer viscosity on blood flow through the artery with mild stenosis, Bull.Math.Biol., vol.42, pp. 797-805, 1980.

[18] J.B. Shukla, R.S. Parihar and B.R.P.Rao, Biorheological aspects of blood flow through artery with mild stenosis: Effect of peripherial layer, Biorheology, vol.17, pp. 403-410, 1980.

[19] P.Chaturani and P.N. Kaloni, Two-layered poiseuille flow model for blood flow through arteries of small diameter and arterioles, Biorheology, vol.13, pp. 243-250, 1976.

[20] P. Chaturani and R.Ponalagusamy, A two-layered model for blood flow through stenosed arteries, Proc. Of 11th National Conf. on fluid mechanics and fluid power,B.H.E.L.(R and D),Hydrabad, India, pp. 6-22, 1982.

[21] R.Ponalagusamy and R.Tamil Selvi, A study on two layered model (Casson-Newtonian) for blood through an arterial stenosis: Axially variable slip velocity at the wall, J.Franklin Inst., vol.348, pp. 2308-2321, 2011.

[22] H. Alemayehu and G. Radhakrishnamacharya, Dispersion of solute in peristaltic motion of couple stress fluid in the presence of magnetic field, Int. J of Engineering and Appl. Sciences., vol.7, pp. 156-160, 2011.

[23] B. S. Bhatt and N. C. Sacheti, On the analogy in slip flows, Indian Journal of Pure and Applied Mathematics., vol.10, pp. 303-306, 1979.

[24] K. Maruthi Prasad and G.Radhakrishnamacharya, Flow of Herschel-Bulkley fluid through an inclined tube of non-uniform cross-section with multiple stenosis, Arch. Mech., vol.60, pp. 161-172, 2008.