Dynamics of a Discrete Three Species Food Chain System

Kejun Zhuang, Zhaohui Wen

Abstract—The main purpose of this paper is to investigate a discrete time three–species food chain system with ratio dependence. By employing coincidence degree theory and analysis techniques, sufficient conditions for existence of periodic solutions are established.

Keywords—Food chain; ratio–dependent; coincidence degree; periodic solutions

I. Introduction

IN the past decade, the food chain systems in population dynamics have attracted new attraction because of their complex dynamical properties [1–7]. Many researchers focused on the the global stability, chaos, Hopf bifurcation, periodic solutions and permanence of those models governed by differential and difference equations.

Recently, Wang and Pang proposed the following three species food chain model with Holling II–type function response in [7]:

$$
\begin{cases}
\dot{x}(t) = r_1 x(t) - a_1 x^2(t) - a_2 x(t) y(t), \\
\dot{y}(t) = r_2 y(t) - \frac{d_1 y^2(t)}{x(t)} - \frac{by(t)z(t)}{\delta + y(t)}, \\
\dot{z}(t) = \frac{kby(t)z(t)}{\delta + y(t)} - d_2 z(t),
\end{cases}
\tag{1}
$$

where all the coefficients are positive constants. The second species predate on the first species and the top species predate on the middle species. The detailed ecological meanings of this system can be found in [7].

Taking account of environmental periodic variation and time delay effect, the modification of (1) is the non–autonomous differential equations

$$
\begin{cases}
\dot{x}(t) = r_1(t)x(t) - a_1(t)x^2(t) - a_2(t)x(t)y(t), \\
\dot{y}(t) = r_2(t)y(t) - \frac{d_1(t)y(t)y(t-\tau)}{x(t-\tau)} - \frac{b(t)y(t)z(t)}{\delta(t)+y(t)}, \\
\dot{z}(t) = \frac{k(t)b(t)y(t-\tau)z(t)}{\delta(t)+y(t-\tau)} - d_2(t)z(t).
\end{cases}
$$

However, it is known that the discrete time model are more appropriate than the continuous ones when the populations have non–overlapping generations. Discrete time models can also provide efficient computational models of continuous for numerical simulations. Following the method in [8], we consider the following discrete analogue with the help of

Kejun Zhuang is with the Institute of Applied Mathematics, School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu 233030, P.R.China, e–mail: zhkj123@163.com

Zhaohui Wen is with the Institute of Applied Mathematics, School of Statistics and Applied Mathematics, Anhui University of Finance and Economics, Bengbu 233030, P.R.China, e–mail: wzh590624@sina.com

differential equations with piecewise constant arguments

$$
\begin{cases}
x(k+1) = x(k)\exp\{r_1(k) - a_1(k)x(k) - a_2(k)y(k)\}, \\
y(k+1) = y(k)\exp\{r_2(k) - \sum_{l=0}^{m} \frac{d_{1l}(k)y(k-l)}{x(k-l)} \\
\qquad - \frac{b(k)z(k)}{\delta(k)+y(k)}\}, \\
z(k+1) = z(k)\exp\{\sum_{l=0}^{m} \frac{h_l(k)y(k-l)}{\delta_l(k)+y(k-l)} - d_2(k)\},
\end{cases}
\tag{2}
$$

where all the coefficients are positive $\omega-$periodic sequences and k is an integer. In the following, we shall explore the existence of periodic solutions for system (2).

II. Preliminaries

For simplicity, we use the following notations throughout this paper,

$$
I_\omega = \{0, 1, 2, \ldots, \omega - 1\}, \quad \bar{f} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} f(k).
$$

According to the Theorem 2.1 in [9], we can easily obtain the following special case.

Lemma 2.1 ([9]). Let $k_1, k_2 \in I_\omega$ and $k \in \mathbb{Z}$. If $g : \mathbb{Z} \to \mathbb{R}$ is $\omega-$periodic, then

$$
g(k) \le g(k_1) + \frac{1}{2}\sum_{s=0}^{\omega-1}|g(s+1) - g(s)|
$$

and

$$
g(k) \ge g(k_2) - \frac{1}{2}\sum_{s=0}^{\omega-1}|g(s+1) - g(s)|,
$$

the constant factor $\frac{1}{2}$ is the best possible.

Now, we introduce some concepts and a useful result from [10].

Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$, then it follows that $L|\operatorname{Dom} L \cap \ker P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Next, we state the Mawhin's continuation theorem, which is a main tool in the proof of our theorem.

Lemma 2.2 (Continuation Theorem). Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose

(a) for each $\lambda \in (0,1)$, every solution u of $Lu = \lambda Nu$ is such that $u \notin \partial\Omega$;

(b) $QNu \neq 0$ for each $u \in \partial\Omega \cap \ker L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then the operator equation $Lu = Nu$ has at least one solution lying in $\mathrm{Dom}\, L \cap \bar{\Omega}$.

III. MAIN RESULTS

We now prove our results on the existence of positive periodic solutions of system (2).

Theorem 3.1. Assume that

$$\bar{r}_2 \omega e^{L_1} > e^{M_2} \sum_{k=0}^{\omega-1} \sum_{l=0}^{m} d_{1l}(k),$$

where $M_2 = \ln \frac{\bar{r}_1}{\bar{a}_2} + \bar{r}_2 \omega$, $L_1 = \ln \frac{\sum_{k=0}^{\omega-1} \sum_{l=0}^{m} d_{1l}(k)e^{L_2}}{\bar{r}_2 \omega} - \bar{r}_1 \omega$ and $L_2 = \ln \frac{\bar{d}_2 \omega}{\sum_{k=0}^{\omega-1} \sum_{l=0}^{m} \frac{h_l(k)}{\delta_l(k)}} - \bar{r}_2 \omega$. Then system (2) has at least one positive $\omega-$periodic solution.

Proof Let $x(k) = e^{u_1(k)}$, $y(k) = e^{u_2(k)}$ and $z(k) = e^{u_3(k)}$, then system (2) is equivalent to the following form,

$$
\begin{cases}
u_1(k+1) - u_1(k) = r_1(k) - a_1(k)e^{u_1(k)} - a_2(k)e^{u_2(k)}, \\
u_2(k+1) - u_2(k) = r_2(k) - \sum_{l=0}^{m} \frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} \\
\qquad - \frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}}, \\
u_3(k+1) - u_3(k) = \sum_{l=0}^{m} \frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}} - d_2(k),
\end{cases}
\tag{3}
$$

and we only need to establish the existence of $\omega-$periodic solutions for system (3).

To apply Lemma 2.2, we define

$$X = Z = \{(u_1(k), u_2(k), u_3(k))^T \in \mathbf{R}^3, u_i(k+\omega) = u_i(k)\},$$

$$\|(u_1, u_2, u_3)^T\| = \left(\sum_{i=1}^{3} \max_{k \in I_\omega} |u_i(k)|^2 \right)^{\frac{1}{2}}.$$

Denote $u(k) = (u_1(k), u_2(k), u_3(k))^T$, then X and Z are both Banach spaces when they are endowed with the above norm $\|\cdot\|$.

Let

$$
Nu = N \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} r_1(k) - a_1(k)e^{u_1(k)} - a_2(k)e^{u_2(k)} \\ r_2(k) - \sum_{l=0}^{m} \frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} - \frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}} \\ \sum_{l=0}^{m} \frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}} - d_2(k) \end{bmatrix},
$$

$$Lu = u(k+1) - u(k),$$

$$Pu = Qu = \frac{1}{\omega} \sum_{k=0}^{\omega-1} u(k).$$

Obviously, $\ker L = \mathbb{R}^3$, $\mathrm{Im}\, L = \{(u_1, u_2, u_3)^T \in Z : \bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0, t \in \mathbb{T}\}$, $\dim \ker L = 3 = \mathrm{codim}\, \mathrm{Im}\, L$.

Since $\mathrm{Im}\, L$ is closed in Z, then L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projections such that $\mathrm{Im}\, P = \ker L$ and $\mathrm{Im}\, L = \ker Q =$

$\mathrm{Im}(I - Q)$. Furthermore, the generalized inverse (of L) $K_P : \mathrm{Im}\, L \to \ker P \cap \mathrm{Dom}\, L$ exists and is given by

$$Kp(u) = \sum_{s=1}^{k-1} u(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} u(s)(\omega - s).$$

Clearly, QN and $K_P(I - Q)N$ are continuous. According to the Arzela-Ascoli theorem, it is not difficulty to prove that $K_P(I-Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$. In addition, $QN(\bar{\Omega})$ is bounded. Therefore, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now, we shall search an appropriate open bounded subset Ω for the application of the continuation theorem, Lemma 2.2. For the operator equation $Lu = \lambda Nu$, where $\lambda \in (0,1)$, we have

$$
\begin{cases}
u_1(k+1) - u_1(k) = \lambda \left[r_1(k) - a_1(k)e^{u_1(k)} - a_2(k)e^{u_2(k)} \right], \\
u_2(k+1) - u_2(k) = \lambda \left[r_2(k) - \sum_{l=0}^{m} \frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} \right. \\
\qquad \left. - \frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}} \right], \\
u_3(k+1) - u_3(k) = \lambda \left[\sum_{l=0}^{m} \frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}} - d_2(k) \right].
\end{cases}
\tag{4}
$$

Assume that $(u_1(k), u_2(k), u_3(k))^T \in X$ is a solution of (4) for some $\lambda \in (0,1)$. Summing on both sides of system (5) over I_ω with respect to k, we can derive

$$
\begin{cases}
\bar{r}_1 \omega = \sum_{k=0}^{\omega-1} a_1(k)e^{u_1(k)} + \sum_{k=0}^{\omega-1} a_2(k)e^{u_2(k)}, \\
\bar{r}_2 \omega = \sum_{k=0}^{\omega-1} \sum_{l=0}^{m} \frac{d_{1l}(k)e^{u_2(k-l)}}{e^{u_1(k-l)}} + \sum_{k=0}^{\omega-1} \frac{b(k)e^{u_3(k)}}{\delta(k)+e^{u_2(k)}}, \\
\bar{d}_2 \omega = \sum_{k=0}^{\omega-1} \sum_{l=0}^{m} \frac{h_l(k)e^{u_2(k-l)}}{\delta_l(k)+e^{u_2(k-l)}}.
\end{cases}
\tag{5}
$$

Since $(u_1(k), u_2(k), u_3(k))^T \in X$, there exist $\xi_i, \eta_i \in I_\omega, i = 1, 2, 3$, such that

$$u_i(\xi_i) = \min_{k \in I_\omega}\{u_i(k)\}, \quad u_i(\eta_i) = \max_{k \in I_\omega}\{u_i(k)\}. \tag{6}$$

In view of (5) and (6), we have

$$\sum_{k=0}^{\omega-1} |u_1(k+1) - u(k)| < 2\bar{r}_1 \omega,$$

$$\sum_{k=0}^{\omega-1} |u_2(k+1) - u(k)| < 2\bar{r}_2 \omega,$$

$$\sum_{k=0}^{\omega-1} |u_3(k+1) - u(k)| < 2\bar{d}_2 \omega.$$

From (6) and the first equation of (5), we have

$$u_1(\xi_1) < \ln \frac{\bar{r}_1}{\bar{a}_1}$$

and

$$u_2(\xi_2) < \ln \frac{\bar{r}_1}{\bar{a}_a},$$

then

$$
\begin{aligned}
u_1(k) &\leq u_1(\xi_1) + \frac{1}{2} \sum_{k=0}^{\omega-1} |u_1(k+1) - u_1(k)| \\
&< \ln \frac{\bar{r}_1}{\bar{a}_1} + \bar{r}_1 \omega := M_1,
\end{aligned}
$$

$$u_2(k) \leq u_2(\xi_2) + \frac{1}{2}\sum_{k=0}^{\omega-1}|u_2(k+1)-u_2(k)|$$
$$< \ln\frac{\bar{r}_1}{\bar{a}_2}+\bar{r}_2\omega := M_2.$$

From (6) and the second equation of (5), we have

$$u_3(\xi_3) < \ln\frac{\bar{r}_2\omega}{\sum_{k=0}^{\omega-1}\frac{b(k)}{\delta(k)+e^{M_2}}},$$

and

$$u_3(k) \leq u_3(\xi_3) + \frac{1}{2}\sum_{k=0}^{\omega-1}|u_3(k+1)-u_3(k)|$$
$$< \ln\frac{\bar{r}_2\omega}{\sum_{k=0}^{\omega-1}\frac{b(k)}{\delta(k)+e^{M_2}}}+\bar{d}_2\omega := M_3.$$

From (6) and the third equation of (5), we have

$$u_2(\eta_2) > \ln\frac{\bar{d}_2\omega}{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{h_l(k)}{\delta_l(k)}}$$

and

$$u_2(k) \geq u_2(\eta_2) - \frac{1}{2}\sum_{k=0}^{\omega-1}|u_2(k+1)-u_2(k)|$$
$$> \ln\frac{\bar{d}_2\omega}{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}\frac{h_l(k)}{\delta_l(k)}}-\bar{r}_2\omega := L_2.$$

Similarly, we can also get

$$u_1(k) \geq u_1(\eta_1) - \frac{1}{2}\sum_{k=0}^{\omega-1}|u_1(k+1)-u_1(k)|$$
$$> \ln\frac{\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}d_{1l}(k)e^{L_2}}{\bar{r}_2\omega}-\bar{r}_1\omega := L_1.$$

By the assumption of theorem, we can obtain

$$u_3(k) \geq u_3(\eta_3) - \frac{1}{2}\sum_{k=0}^{\omega-1}|u_3(k+1)-u_3(k)|$$
$$> \ln\frac{\bar{r}_2\omega e^{L_1}-e^{M_2}\sum_{k=0}^{\omega-1}\sum_{l=0}^{m}d_{1l}(k)}{\sum_{k=0}^{\omega-1}\frac{b(k)}{\delta(k)}}-\bar{d}_2\omega := L_3.$$

From above, we have $\max_{k\in I_\omega}|u_i(k)| \leq \max\{|M_i|,|L_i|\} := R_i, i=1,2,3$ and R_i is independent of λ. Let $R = R_1 + R_2 + R_3 + R_0$, where R_0 is taken sufficiently large such that every solution $\|(x^*,y^*,z^*)^T\|$ of the algebraic equations

$$\begin{cases} \bar{r}_1 - \bar{a}_1 e^x - \bar{a}_2 e^y = 0, \\ \bar{r}_2\omega - \omega\overline{\sum_{l=0}^{m}d_{1l}}e^{y-x} - \sum_{k=0}^{\omega-1}\frac{b(k)e^z}{\delta(k)+e^y} = 0, \\ \bar{d}_2\omega - \sum_{k=0}^{\omega-1}\frac{h_l(k)e^y}{\delta(k)+e^y} = 0 \end{cases}$$

satisfies $\|(x^*,y^*,z^*)^T\| < R$. Now, we define $\Omega = \{(u_1,u_2,u_3)^T \in X\}, \|(u_1,u_2,u_3)^T < R$. Then it is clear that Ω verifies the requirement (a) of Lemma 2.2. If $(u_1,u_2,u_3)^T \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}^3$, then $(u_1,u_2,u_3)^T$ is a constant vector in $\mathbb{R}^3$ with $\|(u_1,u_2,u_3)^T\| = |u_1| + |u_2| + |u_3| = R$, so we have $QNu \neq 0$.

By the invariance property of homotopy, direct calculation produces $\deg(JQN, \Omega \cap \ker L, 0) = 1 \neq 0$. Now, we have proved that Ω satisfies all conditions of Lemma 2.2. Thus, system (2) has at least one positive $\omega-$periodic solution. This completes the proof.

REFERENCES

[1] Haifeng Huo, Wantong Li, J J Nieto. Periodic solutions of delayed predator-prey model with the Beddington-DeAngelis functional response. Chaos, Solitons and Fractals, 33(2007), 505–512.
[2] Changyong Xu, Meijuan Wang. Permanence for a delayed discrete three–level food–chain model with Beddington–DeAngelis functional response. Applied Mathematics and Computation, 187(2007), 1109–1119.
[3] A. Maiti, A.K. Pal, G.P. Samanta. Effect of time–delay on a food chain model. Applied Mathematics and Computation, 200(2008), 189–203.
[4] Chengjun Sun, Michel Loreau. Dynamics of a three–species food chain model with adaptive traits. Chaos, Solitons and Fractals, 41(2009), 2812–2819.
[5] Zhijun Zeng. Dynamics of a non–autonomous ratio–dependent food chain model. Applied Mathematics and Computation, 215(2009), 1274–1287.
[6] Raid Kamel Naji, Ranjit Kumar Upadhyay, Vikas Rai. Dynamical consequences of predator interference in a tri–trophic model food chain. Nonlinear Analysis: Real World Applications, 11(2010), 809–818.
[7] Fengyan Wang, Guoping Pang. Chaos and Hopf bifurcation of a hybrid ratio–dependent three species food chain. Chaos, Solitons and Fractals, (36)2008, 1366–1376.
[8] Pingzhou Liu, K. Gopalsamy. Global stability and chaos in a population model with piecewise constant arguments. Applied Mathematics and Computation, 101(1999), 63–88.
[9] Bingbing Zhang, Meng Fan. A remark on the application of coincidence degree to periodicity of dynamic equtions on time scales. J. Northeast Normal University(Natural Science Edition), 39(2007), 1–3.(in Chinese)
[10] R E Gaines, J L Mawhin. Coincidence Degree and Nonlinear Differential Equations. Lecture Notes in Mathematics, Berlin: Springer–Verlag, 1977.