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Dynamics and Feedback Control for a New Hyperchaotic System

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Abstract—In this paper, stability and Hopf bifurcation analysis of a novel hyperchaotic system are investigated. Four feedback control strategies, the linear feedback control method, enhancing feedback control method, speed feedback control method and delayed feedback control method, are used to control the hyperchaotic attractor to unstable equilibrium. Moreover numerical simulations are given to verify the theoretical results.

Keywords—Feedback control, Hopf bifurcation, hyperchaotic system, stability.

I. Introduction

In 1963, the first chaotic system was introduced by Lorenz [1]. Over time, many interesting chaotic and hyperchaotic systems were constructed in [2]–[5], due to the potential applications in many fields. For these systems, dynamics and control have been extensively investigated, such as bifurcation behavior [6]–[9], various feedback control strategies [10]–[15], chaos synchronization [5], [16], [17], and so on.

Recently, Zheng et al. constructed the following 4D autonomous hyperchaotic system in [5] by adding an additional state to an existing chaotic system:

$$\begin{cases} \dot{x} = a(y-x) + w, \\ \dot{y} = bx + cy + xz + w, \\ \dot{z} = -x^2 - hz, \\ \dot{w} = -fy, \end{cases} \tag{1}$$

where a, b, c, h, f are parameters to be determined. It has been proved that system (1) shows hyperchaotic behavior with two positive Lyapunov exponents when $a = 20, b = 14, c = 10.6, h = 2.8$ and $f = 4$. The full state hybrid projective synchronization of this new hyperchaotic system is also studied by using adaptive control.

In the following, the stability of equilibrium and existence of local Hopf bifurcation is discussed. Some feedback control techniques are employed to control hyperchaos to equilibrium. By the way, the corresponding numerical examples are given to illustrate the theoretical analysis.

II. Local Bifurcation

In this section, we mainly study the stability of equilibrium by analyzing the distribution of characteristic roots of linear system. Through direct computation and with the help of Routh–Hurwitz criteria, it is easy to obtain following lemmas about the existence and stability of zero equilibrium.

Lemma 1 System (1) has three equilibria $O(0,0,0,0)$, $E_1(\sqrt{(a+b)h}, 0, a\sqrt{(a+b)h}, -(a+b))$ and $E_2(-\sqrt{(a+b)h}, 0, -a\sqrt{(a+b)h}, -(a+b))$.

Lemma 2 For system (1), the equilibrium $O(0,0,0,0)$ is asymptotically stable if and only if $a > c$, $h > 0$ and $(a-c)(f-ac-ab) > f(a+b) > 0$.

Proof Linearizing system (1), the Jacobian matrix at $O(0,0,0,0)$ is in the form of

$$J = \begin{bmatrix} -a & a & 0 & 1 \\ b & c & 0 & 1 \\ 0 & 0 & -h & 0 \\ 0 & -f & 0 & 0 \end{bmatrix}$$

and the characteristic equation is

$$(\lambda+h)(\lambda^3 + (a-c)\lambda^2 + (f-ac-ab)\lambda + f(a+b)) = 0. \tag{2}$$

According to the Routh–Hurwitz criterion, the real parts of all the characteristic roots are negative if and only if $a > c$, $h > 0$ and $(a-c)(f-ac-ab) > f(a+b) > 0$, thus the equilibrium is stable.

Now, we shall study the existence of the local Hopf bifurcation regarding f as the bifurcation parameter.

Theorem 1 If $a > c$, $h > 0$ and $(a-c)(f-ac-ab) > f(a+b) > 0$, then system (1) undergoes a Hopf bifurcation at the equilibrium $O(0,0,0,0)$ when f passes through the critical value $f_0 = -a(a-c)$.

Proof Let $\lambda = \mathrm{i}\omega (\omega > 0)$ be a root of (2). Substituting it into (2) and separating real and imaginary parts, we have

$$\begin{cases} \omega^2 = f - ac - ab, \\ \omega^2(a-c) = f(a+b), \end{cases}$$

then

$$\omega = \omega_0 = \sqrt{f-ac-ab}, \quad f = f_0 = -a(a-c).$$

Differentiating both sides of equation (2) with respect to f, we can obtain

$$3\lambda^2\frac{\mathrm{d}\lambda}{\mathrm{d}f} + 2\lambda(a-c)\frac{\mathrm{d}\lambda}{\mathrm{d}f} + (f-ac-ab)\frac{\mathrm{d}\lambda}{\mathrm{d}f} + (a+b) = 0$$

and

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}f}\right)^{-1} = -\frac{3\lambda^2 + 2\lambda(a-c) + (f-ac-ab)}{a+b},$$

then

$$\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}f}\right)^{-1}_{\lambda=\mathrm{i}\omega_0} = \frac{2\omega_0^2}{a+b} > 0.$$

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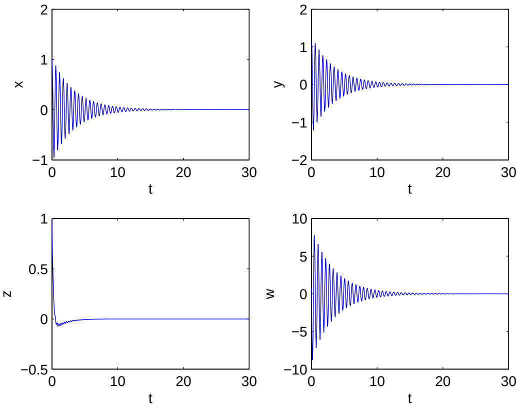

Fig. 1. Stable states of system (1) with $(a, b, c, h, f) = (-10, 25, -20, 5, 75)$.

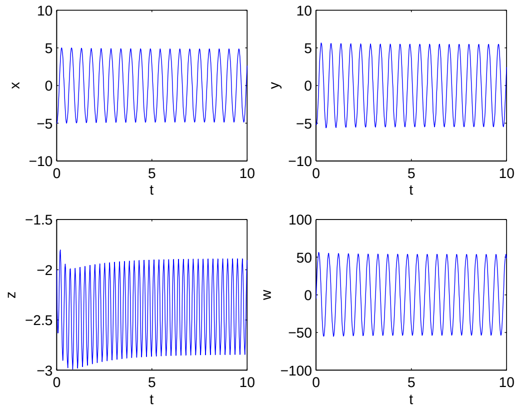

Fig. 2. Periodic states of system (1) with $(a, b, c, h, f) = (-10, 25, -20, 5, 120)$.

Thus, the conditions for Hopf bifurcation are satisfied and the equilibrium $O(0, 0, 0, 0)$ is stable when $f < f_0$, periodic solutions exist when $f > f_0$.

If $a = -10, b = 25, c = -20, h = 5$, then the critical value $f_0 = 100$. From Fig.1–2, the equilibrium $O(0, 0, 0, 0)$ is asymptotically stable when $f = 75 < f_0$ and periodic solution bifurcates from the equilibrium when $f = 120 > f_0$.

III. Feedback Control Methods

In this section, we will use four different feedback control methods to suppress hyperchaotic attractor to equilibrium $O(0, 0, 0, 0)$. Assume that the controlled hyperchaotic system is given by

$$\begin{cases} \dot{x} = a(y - x) + w + u_1, \\ \dot{y} = bx + cy + xz + w + u_2, \\ \dot{z} = -x^2 - hz + u_3, \\ \dot{w} = -fy + u_4, \end{cases} \tag{3}$$

where $u_i (i = 1, 2, 3, 4)$ are external control inputs which will be suitably derived from the trajectory of the hyperchaotic system, specified by (x, y, z, w) to the equilibrium $O(0, 0, 0, 0)$ of uncontrolled system.

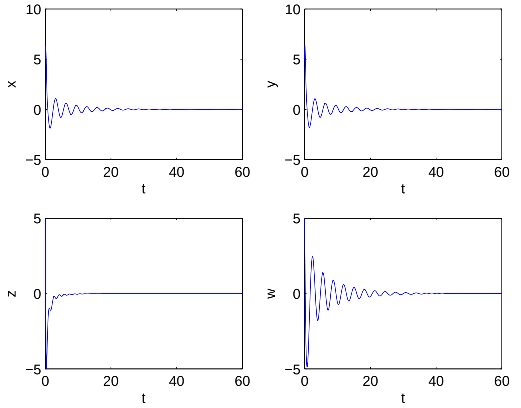

Fig. 3. The states of system (4) with $(a, b, c, h, f, k) = (20, 14, 10.6, 2.8, 4, 25)$.

A. *Linear feedback control*

First, we use the simple linear feedback control approach with $u_1 = -kx$, $u_2 = u_3 = u_4 = 0$ and get the following controlled system:

$$\begin{cases} \dot{x} = a(y - x) + w, \\ \dot{y} = bx + cy + xz + w - ky, \\ \dot{z} = -x^2 - hz, \\ \dot{w} = -fy, \end{cases} \tag{4}$$

where k is the feedback coefficient. The characteristic equation of linear part for (4) at equilibrium $O(0, 0, 0, 0)$ is

$$(\lambda + h)[\lambda^3 + (a + k - c)\lambda^2 + (f + ak - ac - ab)\lambda + f(a + b)] = 0.$$

By Routh–Hurwitz criterion, the equilibrium $O(0, 0, 0, 0)$ of controlled system (4) is stable if and only if $a + k - c > 0$, $h > 0$ and $(a + k - c)(f + ak - ac - ab) > f(a + b) > 0$, which can be shown in Fig.3.

B. *Enhancing linear feedback control*

It is difficult for a complex system to be controlled by only one feedback variable, and in such cases the feedback gain is always very large. So we use the enhancing feedback control method and the controlled system is

$$\begin{cases} \dot{x} = a(y - x) + w - kx, \\ \dot{y} = bx + cy + xz + w - ky, \\ \dot{z} = -x^2 - hz, \\ \dot{w} = -fy, \end{cases} \tag{5}$$

then the corresponding characteristic equation at $O(0, 0, 0, 0)$ is

$$(\lambda + h)[\lambda^3 + (a + 2k - c)\lambda^2 + (ak + k^2 - ac - ck - ab + f)\lambda + f(a + b + k)] = 0.$$

Thus, the zero equilibrium of (5) is asymptotically stable if and only if $a + 2k - c > 0$, $h > 0$ and $(a + 2k - c)(ak +$

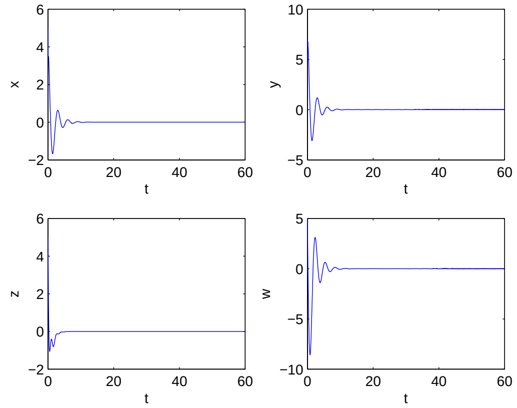

Fig. 4. The states of system (5) with $(a, b, c, h, f, k) = (20, 14, 10.6, 2.8, 4, 19)$.

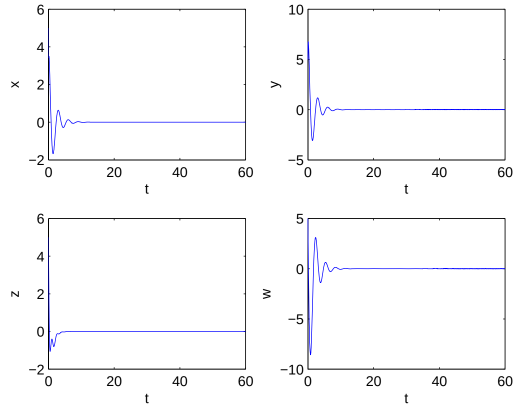

Fig. 5. The states of system (6) with $(a, b, c, h, f, k) = (20, 14, 10.6, 2.8, 4, 15)$.

$k^2 + f - ac - ck - ab) > f(a+b+k) > 0$, which means that the system with $(a, b, c, h, f) = (20, 14, 10.6, 2.8, 4)$ is stable when the feedback coefficient $k > 17.9864$ (see Fig.4).

C. Speed feedback control

If we multiply the derivative of independent variable with coefficient, we call it speed feedback control [12]. Let the controlled system be

$$\begin{cases} \dot{x} = a(y-x) + w, \\ \dot{y} = bx + cy + xz + w, \\ \dot{z} = -x^2 - hz, \\ \dot{w} = -fy - k(bx + cy + xz + w), \end{cases} \tag{6}$$

the Jacobian matrix of (6) at $O(0,0,0,0)$ is

$$J = \begin{bmatrix} -a & a & 0 & 1 \\ b & c & 0 & 1 \\ 0 & 0 & -h & 0 \\ -kb & -f-ck & 0 & -k \end{bmatrix}$$

and the characteristic equation is

$$(\lambda + h)[\lambda^3 + (a-c)\lambda + (ak + f - ac - ab + bk)\lambda \\ +f(a+b)] = 0.$$

Obviously, the equilibrium O(0,0,0,0) of (6) is stable if and only if $a - c > 0$, $h > 0$ and $(a-c)(ak+f-ac-ab+kb) > f(a+b) > 0$. In this case, the system is stable only with the feedback coefficient $k > 14.7785$ as shown in Fig.5.

D. Delayed feedback control

The effects of time delay on nonlinear systems have long been investigated [18]–[20]. For simplicity, we consider the following controlled system by only adding time delay feedback to the third equation of system (1),

$$\begin{cases} \dot{x} = a(y-x) + w, \\ \dot{y} = bx + cy + xz + w, \\ \dot{z} = -x^2 - hz + K(z(t) - z(t-\tau)), \\ \dot{w} = -fy, \end{cases} \tag{7}$$

where $\tau \geq 0$ is time delay and K indicates the strength of the feedback, and then we shall study the stability and local Hopf bifurcation at equilibrium $O(0,0,0,0)$. The characteristic equation is given by

$$(\lambda + h - K + Ke^{-\lambda\tau})[\lambda^3 + (a-c)\lambda + (f - ac - ab)\lambda \\ +f(a+b)] = 0.$$

It is only need to consider the distribution of the root of the following transcendental equation:

$$\lambda + h - K + Ke^{-\lambda\tau} = 0. \tag{8}$$

Clearly, $\mathrm{i}\omega_0 (\omega_0 > 0)$ is a root of (8) if and only if ω_0 satisfies

$$\mathrm{i}\omega_0 + h - K + ke^{-\mathrm{i}\omega_0\tau} = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} \omega_0 - K \sin\omega_0\tau = 0, \\ K\cos\omega_0\tau = K - h, \end{cases}$$

then we have $\omega_0 = \sqrt{2Kh - h^2}$ and $\tau_j = \frac{1}{\omega_0}[\arcsin\frac{\omega_0}{K} + 2j\pi]$, where $j = 0, 1, 2, \ldots$.

Suppose $\lambda(\tau) = \alpha(\tau) + \mathrm{i}\omega(\tau)$ denotes the root of (8) and differentiate both sides of (8) with respect to τ, it is easy to obtain

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} - Ke^{-\lambda\tau}\left(\lambda + \tau\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right) = 0.$$

It follows that

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\tau} = \frac{K\lambda}{e^{\lambda\tau} - K\tau},$$

moreover,

$$\left.\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right|_{\lambda=\mathrm{i}\omega_0,\tau=\tau_j} = \frac{\mathrm{i}K\omega_0}{\cos\omega_0\tau + \mathrm{i}\sin\omega_0\tau - K\tau}$$

and

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{\lambda=\mathrm{i}\omega_0,\tau=\tau_j} = \frac{\omega_0^2}{(\cos\omega_0\tau - K\tau)^2 + (\sin\omega_0\tau)^2} > 0.$$

Due to the results from [21] and [22], we have the following theorem about stability and bifurcation for system (7) and can control the hyperchaotic attractor effectively.

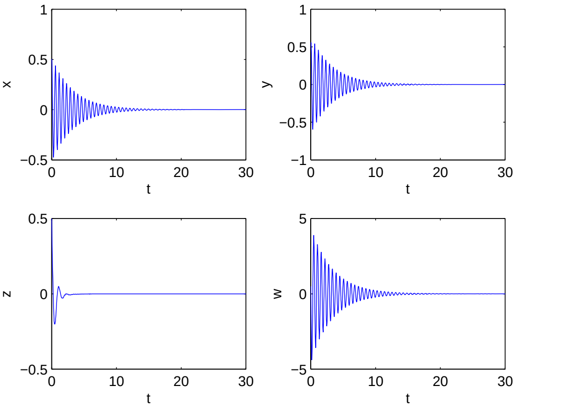

Fig. 6. The states of system (7) with $(a, b, c, h, f, K, \tau) = (-10, 25, -20, 5, 75, 3, 0.3)$.

Theorem 2 Suppose that $a > c$, $h > 0$, $f < -a(a-c)$ and $(a-c)(f-ac-ab) > f(a+b) > 0$ are satisfied. The equilibrium $O(0,0,0,0)$ of (7) is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. System (7) undergoes a Hopf bifurcation at $O(0,0,0,0)$ when $\tau = \tau_j$, which means that periodic solutions may bifurcate from the equilibrium when $\tau > \tau_0$.

For instance, if $(a, b, c, h, f, K) = (-10, 25, -20, 5, 75, 3)$, then system (7) is stable when the bifurcation parameter $\tau < \tau_0 = 0.376137$, see Fig.6.

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