

Design of Digital Differentiator to Optimize Relative Error

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Abstract—It is observed that the Weighted least-square (WLS) technique, including the modifications, results in equiripple error curve. The resultant error as a percent of the ideal value is highly non-uniformly distributed over the range of frequencies for which the differentiator is designed. The present paper proposes a modification to the technique so that the optimization procedure results in lower maximum relative error compared to the ideal values. Simulation results for first order as well as higher order differentiators are given to illustrate the excellent performance of the proposed method.

Keywords—Differentiator, equiripple, error distribution, relative error.

I. INTRODUCTION

DIGITAL differentiators (DDs) find application in many areas such as radar, biological signal processing, image processing etc. Higher order DDs find their application in all those areas where the calculation of geometric moments plays an important role, as in biological signal processing. The output of a digital differentiator (DD) system is a time derivative of the input to it. To obtain better results, the methods for the design of higher order DDs are being modified from time to time. Of the many methods available in the literature, the modified McClellan-Parks method by Rahenkamp and Vijaya kumar [1] was widely used and found well suited for the design of higher order FIR DDs. However, this method many times leads to results with large error. Pei and Shyu [2] extended the eigen filter method by formulating an error function in quadratic form, while Reddy et al [3] extended the Fourier series method. The method proposed by Reddy et al uses Fourier series in conjunction with accuracy constraints achieved through imposing constraints on magnitude and derivative at a particular frequency. The filter coefficients of DDs obtained by solving a system of linear equations with Choleksy decomposition technique forms the basis of the method proposed by Sunder and Ramachandran [4]. This is an extension of least-squares approach. Another method is due to ShianYang Tzeng et al [5] which is based on genetic algorithm (GA) approach. Analytical methods [6, 7] have been presented to simplify the optimization procedure using matrix properties of trigonometric function. In these

methods, simple closed form formulas to compute the eigenfilter related matrix elements are used by Pei and Shyu [6] while Mollova and Unbehauen [7] use simple analytic closed form relations for the least-square design of higher order DDs. Hopfield neural network [8] is successfully applied to solve various optimization problems by Tank and Hopfield, while Yue-Dar Jou [9] used neural network to solve the set of linear equations obtained while using the least-squares design of higher order DDs.

A weighted least-squares (WLS) approach for the design of linear-phase nonrecursive first-order DDs and Hilbert transforms is described by Sunder and Ramachandran [10]. In [11], Sondur et al discuss different issues of relevance in the design of equiripple FIR DDs using WLS technique. The authors also discuss the issues concerning convergence of WLS technique. It is observed that relative error at lower frequencies is very high. In the present paper, the WLS technique is modified and a method that results in significantly

II. WEIGHTED LEAST-SQUARES TECHNIQUE AND ITS MODIFICATION

A typical transfer function of a FIR filter of length N can be represented as

$$H(z) = \sum_{n=0}^{n=N-1} h(n)z^{-n} \quad (1)$$

The corresponding frequency response is characterized by

$$H(e^{j\omega}) = M(\omega)e^{j(\pi/2 - \omega(N-1)/2)} \quad (2)$$

where, $M(\omega)$ is the real valued amplitude response given by

$$M(\omega) = \sum_{n=n_0}^M b(n)s(\omega) \quad (3)$$

$b(n)$ is the filter coefficient, linearly related with $h(n)$ and M is function of filter length. According as length of filter is odd or even and nature of symmetry of the filter, the amplitude response can be classified into four fundamental types [9] as shown in TABLE I. An ideal k^{th} order DD has the following frequency response.

$$H(e^{j\omega}) = D(\omega)e^{jk\pi/2} \quad (4)$$

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where $D(\omega) = (\omega/2\pi)^k$ for $0 \leq \omega \leq \omega_p \leq \pi$. (5)

ω_p is the maximum pass-band frequency for which DD is designed. It is known that even order DD can be designed only by Type I and Type II filter. Further, a full-band, even order DD can be designed by only an odd length filter. Odd order DD can be designed only by Type III and Type IV filter and N is required to be even if the DD is full-band.

TABLE I DIFFERENT TYPES OF FIR FILTER

Type	$s(\omega)$	M	n_0	Nature
I	$\cos n\omega$	(N-1)/2	0	Symmetric
II	$\cos(n-1/2)\omega$	N/2	1	Symmetric
III	$\sin n\omega$	(N-1)/2	1	Anti-symmetric
IV	$\sin(n-1/2)\omega$	N/2	1	Anti-symmetric

According to [10] and [11], the optimal coefficients are obtained by minimizing the weighted mean squared error. The weighted mean squared error is expressed as

$$E_{mse} = \sum_{i=1}^k W(\omega_i) E_a^2(\omega_i) \quad (6)$$

where $E_a(\omega) = D(\omega) - M(\omega)$ (7)

$E_a(\omega)$ is the error function. $W(\omega)$ is a frequency dependant weighting function and k is the number of points at which the error function is sampled. Minimization of the mean square error using the WLS technique leads to the following system of linear equations [10], [11].

$$Qb = d \quad (8)$$

where $Q = \sum_{l=1}^k W(\omega_l) S(\omega_l) S^T(\omega_l)$ (9)

and $d = \sum_{l=1}^k W(\omega_l) D(\omega_l) S(\omega_l)$ (10)

The error function is not known before hand. In [10] Cholesky decomposition iterative method is used to solve the system of equations and uses the envelope of the error function to update the weighing function. Though this results in faster convergence, it is observed that the modified method used in [12] results in lower maximum error. This is reflected in the details given in TABLE II. The method in [12] has been used in the present paper also and is summarized in the following

Modified WLS algorithm:

Step 1) Initialize $W_0(\omega_l)$; $W_0(\omega_l)=1$ can be assumed initially.

Step 2) Compute Q and d and solve the system of equations

$$Qb = d$$

Step 3) Valuate the error function $E_a(\omega_l)$ and $|E_a(\omega_l)|$

Step 4) Carry out different iterations while updating the

TABLE II COMPARISON OF DIFFERENT METHODS FOR THIRD-ORDER DD OF LENGTH 27

Method	Peak amplitude error
Analytical	1.021e-3
Sunder	1.022e-3
Eigenfilter	1.010e-3
Minimax	2.967e-4
Proposed	2.964e-4

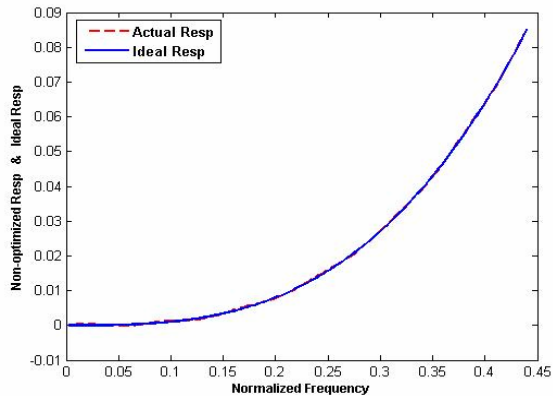
weighing function as

$$W_{k+1}(\omega_l) = \beta_k^\theta(\omega_l) W_k(\omega_l) \quad (11)$$

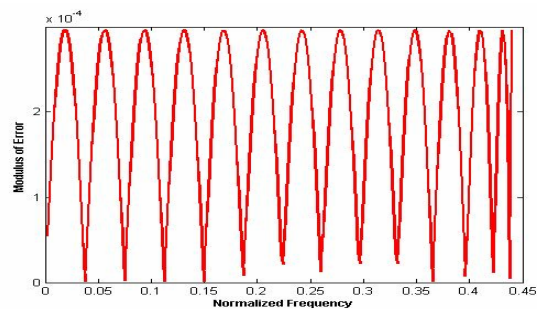
where $\beta_k(\omega_l) = |E_a(\omega_l)|$. (12)

Normalize the weights by dividing all the values by the largest value and go to step 2. Based on a large number of observations, a θ value of 1.88 is recommended.

Response of a third-order DD of length 27 is shown in Fig.1(a). The equiripple error variation obtained by using the above procedure is shown in Fig.2(b). A DD of order 3 and length 27 has been considered to enable comparison with the example of Type 3 cited in [9]. Fig. 2 shows distribution of modulus of error expressed as per cent of desired response. The distribution of modulus of error is shown in (a), (b), (c) and (d) of Fig. 2.



(a) Ideal and actual responses



(b) Equiripple error distribution

Fig. 1 Third-order DD of length 27, $\omega_p=0.88\pi$

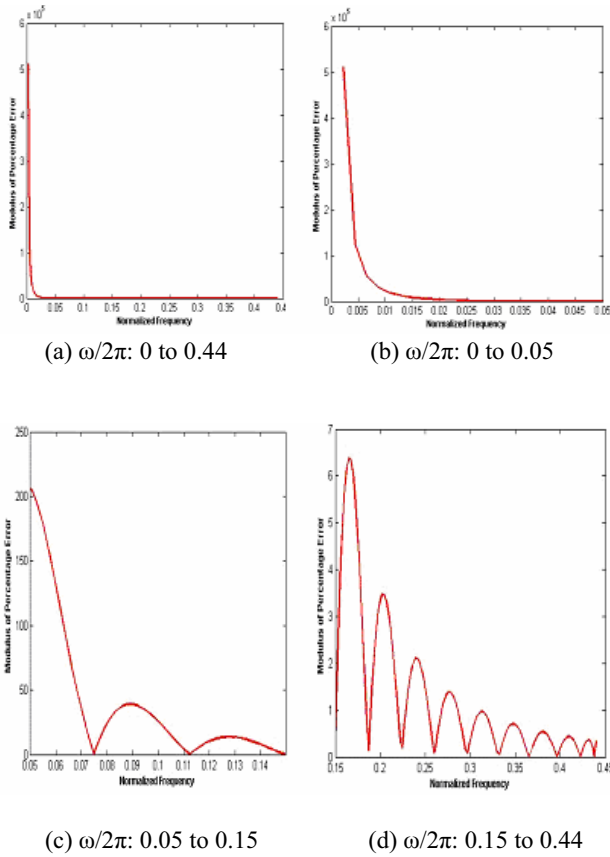


Fig. 2 Modulus of error expressed as per cent of ideal response for the third-order DD of length 27, $\omega_p=0.88\pi$

It is observed that the per cent error is very large for lower values of normalized frequencies. The large relative error at lower values of the normalized frequency is also expressed in Table III.

TABLE III
VARIATION OF MODULUS OF PER CENT ERROR IN THIRD-ORDER DD OF LENGTH 27, $\omega_p=0.88\pi$

Normalized Frequency	Per cent Error
0.0022	5.117e5
0.0066	5.43e4
0.0264	1291
0.0484	208.1
0.1012	22.98
0.1276	13.74
0.1672	6.333
0.2024	3.48
0.242	2.091
0.2772	1.389
0.2838	1.148
0.3124	0.9658

III. PROPOSED TECHNIQUE TO OPTIMIZE RELATIVE ERROR

An instrument based on such design can not measure differentials with reasonable accuracy at lower frequencies. Therefore WLS algorithm needs to be modified so that the relative error (Per cent error) defined below is optimized.

$$Percent\ error = \frac{Ea(\omega)}{D(\omega)} \times 100 \tag{13}$$

The modified WLS algorithm proposed in this paper is as follows.

Relative error optimization algorithm:

Step 1) Initialize $W_0(\omega_l)$; $W_0(\omega_l)=1$ can be assumed initially.

Step 2) Compute Q and d and solve the system of equations

$$Qb = d$$

Step 3) Valuate the relative error function $RE_a(\omega_l)$ and $|RE_a(\omega_l)|$ using

$$RE_a(\omega_l) = \frac{D(\omega_l) - M(\omega_l)}{D(\omega_l)} \tag{14}$$

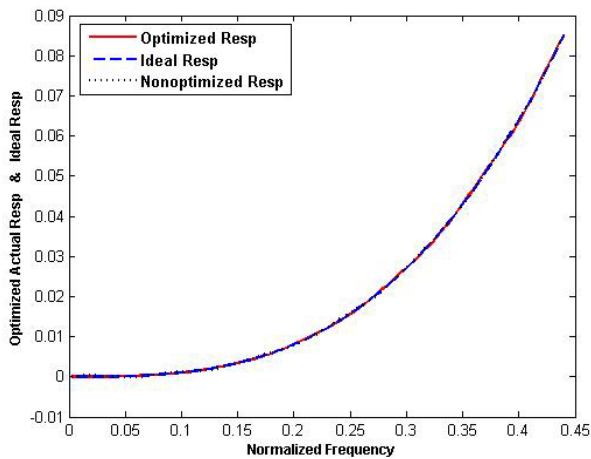
Step 4) Carry out different iterations while updating the weighing function as

$$W_{k+1}(\omega_l) = \beta_k^\theta(\omega_l)W_k(\omega_l) \tag{15}$$

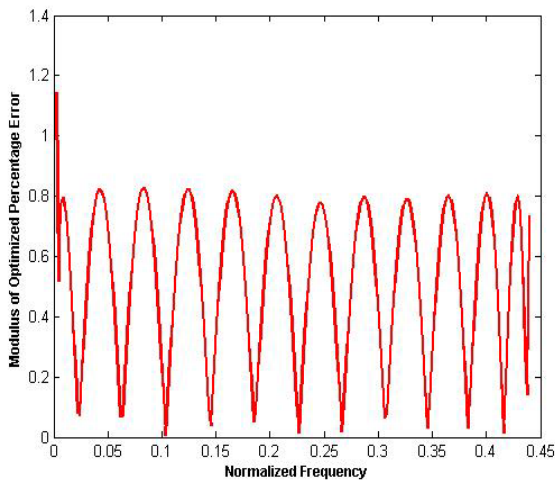
$$\text{where } \beta_k(\omega_l) = |RE_a(\omega_l)|. \tag{16}$$

Normalize the weights by dividing all the values by the largest value and go to step 2. A θ value of 1.88 is recommended.

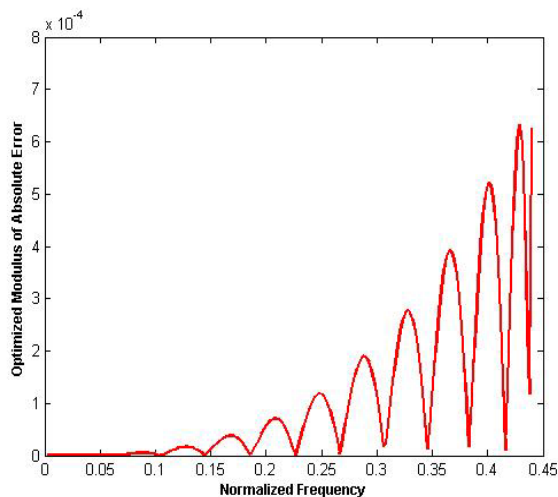
A simulation result obtained using the above algorithm is referred to in this paper as ‘Optimized’. The result of simulation of optimized DD of third-order and length 27 is shown in Fig.3. Fig. 3(a) shows the three responses – ideal, optimized and nonoptimized. The ideal and nonoptimized responses are the same as ideal and actual responses already shown in Fig.2(a). The optimized response is the response corresponding to the minimum value of the largest relative error obtained by the use of Relative error optimization algorithm. From Fig.2(b) it is noticed that the relative error (per cent error) has an almost equiripple distribution. The maximum per cent error is 1.142 where as the most part of the frequency-range has equiripple error distribution with maximum value 0.8257. Fig.3(c) shows the modulus of error distribution of the optimized DD. It would be noticed that the absolute error at higher frequencies is higher and the error at lower frequencies is lower. From Fig.2 it is observed that for normalized frequencies in the range 0 to 0.2838, the per cent error is 1.148 or higher. In this range of frequencies, the nonoptimized DD has its modulus of error significantly higher compared to that of the optimized DD. The optimized DD has a modulus of per cent error lower than 1.148 over the entire range of frequencies.



(a) Ideal, optimized and nonoptimized responses



(b) Optimized modulus of per cent error



(c) Optimized modulus of absolute error

Fig. 3 Optimized Third-order DD of length 27, $\omega_p=0.88\pi$

IV. ADDITIONAL EXAMPLE AND DISCUSSION

The efficacy of the Relative error optimization algorithm has been demonstrated through another example. In this example a first-order, full-band DD of length 20 has been considered. Fig.4(a) shows that the ideal response, nonoptimized response and optimized response are almost coincident. Fig.4(b) gives distribution of error in the nonoptimized response of the DD. Fig.4(c) shows that the relative error is large at lower frequencies although it is very small for frequencies close to the band edge. Fig.5 gives details distribution of error in the optimized DD. Fig.5(a) shows the distribution of per cent error and Fig.5(b) that of the absolute error. The maximum per cent error of the optimized DD is 1.087. The variation of per cent error in the nonoptimized DD is given in Table IV.

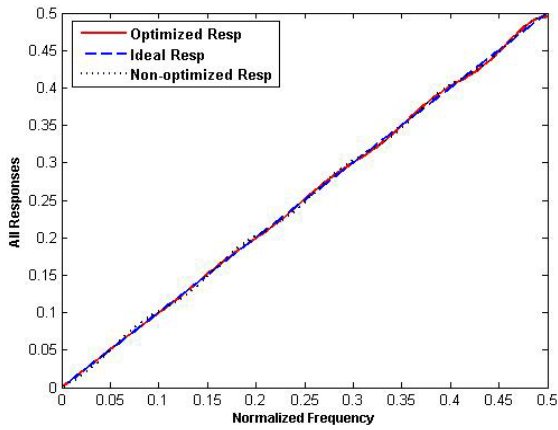
TABLE IV
VARIATION OF MODULUS OF PER CENT ERROR IN FIRST-ORDER,
FULL-BAND DD OF LENGTH 20

Normalized frequency	Per cent relative error
0.0025	27.76
0.0250	18.49
0.0500	6.056
0.1275	3.579
0.1800	2.548
0.2325	1.984
0.2850	1.629
0.3375	1.384
0.3875	1.205
0.395 or higher	1.092 or lower

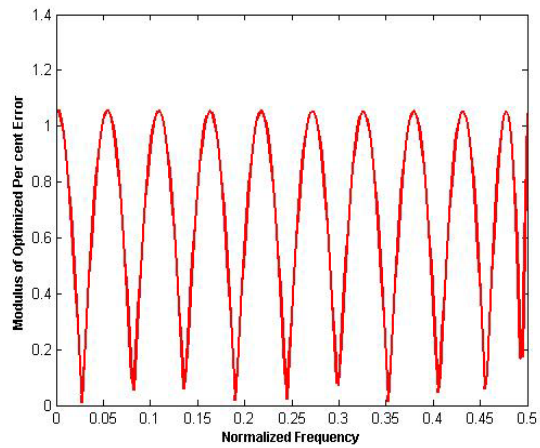
It is seen that the per cent error of the optimized DD is lower than 1.092 for the entire range of frequencies, whereas it is lower than 1.092 in the nonoptimized DD for normalized frequencies only in the range 0.395 to 0.5. The following are some of the observations made with regard to the use of Relative error optimization algorithm and Modified WLS algorithm:

(i) The WLS algorithm results in lower value of maximum error. But this is quite large compared to the ideal values at lower frequencies. Thus the estimate of the differential has large error. If such an algorithm is used in instruments to measure differentials of varying quantities, the measured value may be very much off the mark. The discrepancy is more pronounced in case of DDs used for higher order differentiation.

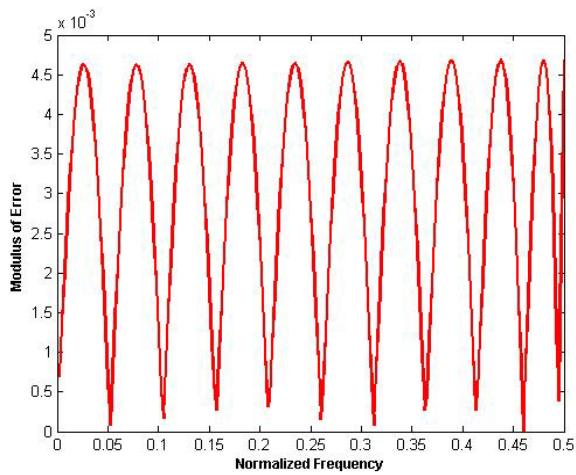
(ii) The Relative error optimization algorithm distributes the absolute error such that the error at lower frequencies is lower and that at higher frequencies is higher. The algorithm results in a wider range of frequencies at which the relative error is smaller than any prescribed value, compared to the range obtained from WLS algorithm.



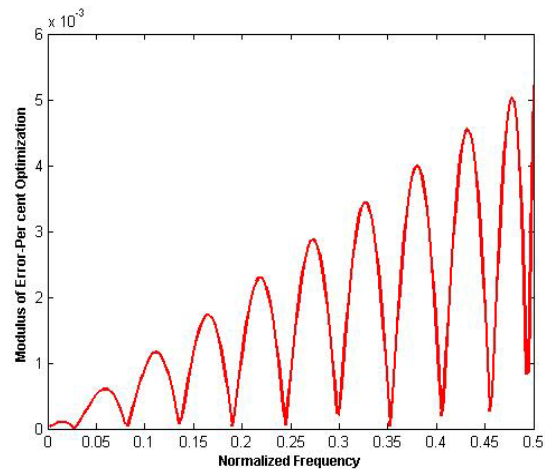
(a) Ideal, optimized and nonoptimized responses



(a) Optimized modulus of per cent error

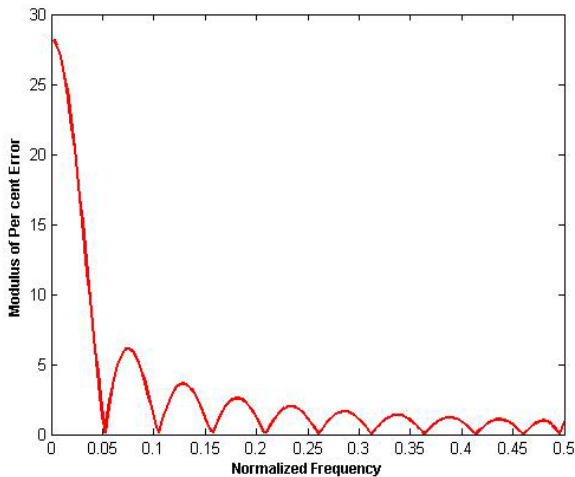


(b) Equiripple error distribution



(b) Optimized modulus of absolute error

Fig. 5 Optimized First-order, full-band DD of length 20



(c) Modulus of error expressed as per cent of ideal response

Fig. 4 Nonoptimized First-order, full-band DD of length 20

REFERENCES

- [1] C.A.Rahenkamp and B.V.K.Vijay Kumar, "Modifications to the McClellan, Parks and Rabiner computer Program for Designing Higher order Differentiating FIR Filter", *IEEE Transactions Acoust Speech Signal Processing*, vol Assp-34, p 1671, December 1986.
- [2] S.C.Pei and J.J.Shyu, "Eigen filter Design of Higher order Digital Differentiators", *IEEE Transactions Acoust Speech Signal Processing*, vol 37, p 505, April 1989.
- [3] M.R.R.Reddy, B.Kumar and S.C.Dutta Roy, "Design Efficient Second and Higher Order FIR Digital Differentiators for Low Frequencies", *Signal Processing*, vol 20, p 219, July 1990.
- [4] S.Sunder and R.P.Ramachandran, "Least-squares Design of Higher Order Nonrecursive Differentiators", *IEEE Transactions on Signal Processing*, vol 42, No. 4, p 956, April 1994.
- [5] Shian-Tang, Tzeng, Hung-Ching Lu, "Genetic Algorithm Approach for Designing Higher order Digital Differentiators," *Signal Processing*, vol 79, p 175, 1999.
- [6] S.C.Pei and J.J.Shyu, "Analytic closed-formed matrix for designing higher order digital differentiators using eigen-approach," *IEEE Transactions on Signal Processing*, vol 44, No. 3, pp 698-701, March 1996.
- [7] G.S.Mollova and R.Unbehauen, "Analytical design of higher order differentiators using least-squares technique," *Electron. Lett.*, vol 37, No. 22, pp 1098-1099, 2001.
- [8] D.W.Tank and J.Hopfield, "Simple neural optimization networks: an A/D converter, signal decision circuit, and a linear programming

- circuit," *IEEE Trans.Circuits Syst.*, vol. CAS-33, No. 4, pp 533-541, April 1986.
- [9] Yue-Dar Jou, "Neural Least-Squares Design of Higher Order Digital Differentiators," *IEEE Signal Processing Letters*, vol 12, No. 1, pp9-12, Jan 2005.
- [10] S.Sunder and V.Ramachandran, "Design of Equiripple Nonrecursive Digital Differentiators and Hilbert Transformers Using a Weighted Least-squares Technique", *IEEE Transactions on Signal Processing*, vol 42, No. 9, p 2504, Sep 1994.
- [11] V.V.Sondur, V.B.Sondur, D.H.Rao, M.V.Latte, N.H.Ayachit, "Issues in the Design of Equiripple FIR Higher Order Digital Differentiators using Weighted Least-squares Technique," in: Proceedings of the 2007 IEEE International Conference on Signal Processing and Communications, Nov 24-27, 2007, Dubai, UAE, pp 189-192'
- [12] V.V.Sondur, V.B.Sondur, N.H.Ayachit, "Design of FIR Digital Differentiators using Modified Weighted Least-squares Technique," communicated to *IEEE Signal Processing Letters*.