Derivation of Monotone Likelihood Ratio Using Two Sided Uniformly Normal Distribution Techniques

D. A. Farinde

***Abstract*—**In this paper, two-sided uniformly normal distribution techniques were used in the derivation of monotone likelihood ratio. The approach mainly employed the parameters of the distribution for a class of all size α. The derivation technique is fast, direct and less burdensome when compared to some existing methods.

***Keywords*—**Neyman-Pearson Lemma, Normal distribution.

I. INTRODUCTION

IN statistical hypothesis testing, a uniformly most powerful (UMP) test is a hypothesis test which has the greatest power 1-β among all possible tests of a given size α. Neyman-Pearson Lemma further states that the likelihood-ration test is UMP for testing simple (point) hypothesis.

Let X denote a random vector taken from a parameterized family of probability density function (pdf) or probability mass function (pmf) given as f(x), which depends on the unknown deterministic parameter $\theta \in \Phi$. the parameter space Φ is partitioned into two disjoint sets Φ_0 *and* $\Phi 1$. Let H_0 denote the hypothesis that $\theta \in \Phi_0$ and H_1 denote the hypothesis that $\theta \in \Phi_1$.

The binary test for hypotheses is performed using a test function $\Phi(x)$.

$$\Phi(x) = \begin{cases} 1, if\ x \in R \\ 0, if\ x\ \in A \end{cases}$$

meaning that H_1 is in force if the measurement $X \in$ R and that H_0 is in force if the measurement $X \in$ A. $A \cup R$ is a disjoint covering of the measurement space.

Hence, a test function $\Phi(x)$ is uniformly most powerful of size α if for any other test function $\Phi(x)$ we have

$$Sup_{\theta \in \theta_0} E_\theta \Phi'(x) = \alpha' \leq \alpha = Sup_{\theta \in \theta_0} E_\theta \Phi(x)$$
$$E_\theta \Phi'(x) = 1 - \beta' \leq 1 - \beta \leq E_\theta \Phi(x)\ \forall \theta \in\ \Phi_1$$

To introduce the UMP to Normal distribution, we consider the standard normal distribution of the family of Normal distribution. Section II contains review of UMP test. In Section III, we explained hypothesis testing – uniformly most powerful tests. Section IV is devoted to the results in standard normal distribution.

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II. UNIFORMLY MOST POWERFUL (UMP) TEST

Reference [4] pioneered modern frequentist statistics as a model-based approach to statistical induction anchored on the notion of a statistical model.

As

$$\mathbf{G}\{g_\gamma(\mathrm{z}), z \in \mathbb{Z}, \theta \in \Theta \subseteq \mathfrak{R}^d. \dim \Theta < \mathbb{Z}\}$$

Fishers proposed to begin with pre-specified **G** as a hypothetical infinite population. He estimated the specification of **G** as a response to the question: what population is this a random sample? A mis-specified **G** would vitiate any procedure relying on $g_\gamma(\mathrm{z})$ or the likelihood function. Reference [5] argued for inductive inference spearheaded by his significant testing, and [8] argued for inductive behavior based on Neyman-Pearson testing. However, neither account gave a satisfactory answer to the canonical question. When do data provide evidence for a substantive claim hypothesis?

Over the last three decades, Fisher's specification problem has been recast in the form of model selection problems. The essential question, how could n infinite set of all possible models that could have given rise to data be narrowed down to a single statistical model **.** These models may be nested or non-nested. For non-nested case, see [2], [3], [11], [6], [1], and [9]. In the nested case, we consider a parametric family of densities and two hypotheses as H_0 and H_1. When the domain of density is dependent on parameter, the theories for hypothesis testing and model selection have not developed. For the testing problem of type

$$H_0: \theta \leq \theta_1\ or\ \theta \geq \theta_2 (\theta_1 < \theta_2)$$

Against

$$H_1: \theta_1 \leq \theta\ < \theta_2$$

When a class of size-α tests is considered, and the family is one-parameter exponential distributions, a uniformly most powerful (UMP) test to exist. However, under these conditions, a UMP test does not exist for H_0: $\theta_1 < \theta < \theta_2$ against H_1: $\theta < \theta_1$ $\theta > \theta_2$ *and* H_0: $\theta = \theta_0$ against $H_1: \theta \neq \theta_0$. Whereas, if the class of size-α tests is reduced to a class by taking only the unbiased test and the family of distributions is Polya type, we know that a UMP test does not exist for testing problems of the two latter types of hypothesis testing. When a UMP test does not exist, we may use the same approach used in the estimation problems. Imposing a restriction on the test to be considered and finding optimal test within the class of test under the restriction. Two such types of restrictions are unbiasedness and invariance. Under some distributional assumptions, let the power function of any test φ, $E_0\{\phi(T)\}$, is continous in θ. Let us consider a test of size-α unbiased

$$\mathcal{U}_\alpha = \left\{\phi \in \mathcal{D} | E_{\theta_0}\{\phi(X)\} = \alpha \text{ and } \frac{\partial}{\partial\theta} E_\theta\{\phi(X)\}|_{\theta=\theta_0} = 0\right\}$$

In this situation the two sided test is

$$\phi(T(x)) = \begin{cases} 1 & if\ T(x) < c_1\ or\ T(x) > c_2 \\ \gamma_i & if\ T(x)\ c_i; \quad i = 1, 2 \\ 0 & if\ c_1 < T(x) < c_2 \end{cases}$$

where $c_1, c_2, \gamma_1, \gamma_2$ are obtained by $E_{\theta_0}\{\phi(T)\}$ α and $E_{\theta_0}\{T\phi(T)\} = \alpha E_{\theta_0}\{T\}$. Such a test is therefore UMP size-α unbiased test in $\mathcal{U}_\alpha$for testing H_0 against H_1. Consider a family of distribution which it supports is dependent on its parameter. In such a situation the UMP test is known for uniform and double exponential distributions, see [7].

III. Hypothesis Testing – Uniformly Most Powerful Tests

We give the definition of a uniformly most powerful test in a general setting which includes one-sided and two-sided tests. We take the null hypothesis to be

$$H_0: \theta \in \Omega_0$$

And the alternative to be

$$H_1: \theta \in \Omega_0$$

We write the power function as Pow(θ, d) to make its dependence on the decision function explicit.

Definition: A decision function d* is a uniformly most powerful (UMP) decision function (or test) at significance level α_0 if

(1) Pow(θ, d*) ≤ α_0, $\forall \theta \in \Omega_0$
(2) For every decision function d which satisfies (1), we have Pow(θ, d) ≤ Pow(θ, d*), , $\forall \theta \in \Omega_1$.

Do UMP tests ever exist? If the alternative hypothesis is one-sided then they do for certain distributions and statistics. We proceed by defining the needed property on the population distribution and the statistic.

Definition: Let T = t(X_1, X_2, . . . , X_n) be a statistic. Let $f(x_1, x_2, \ldots, x_n|\theta)$ be the joint density of the random sample. We say that $f(x_1, x_2, \ldots, x_n|\theta)$ has a monotone likelihood ratio in the statistic T if for all $\theta_1 = \theta_2$ the ratio

$$\frac{f(x_1, \ldots x_n|\theta_2)}{f(x_1, \ldots x_n|\theta_1)}$$

depends on $x_1, x_2 \ldots x_n$ only through t($x_1, \ldots x_n$) and the ratio is an increasing function of $f(x_1, \ldots x_n)$.

Example: Consider a Bernoulli distribution for the population, i.e. we are looking at a population proportion. So each X_i = 0, 1, and p = P(X = x). The joint density is

$$f(x_1, \ldots x_n|p) = p^{n\bar{x}}(1-p)^{n-n\bar{x}}$$

where

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Let $p_1 < p_2$we have

$$\frac{f(x_1, \ldots, x_n|p_2)}{f(x_1, \ldots, x_n|p_1)} = \left[\frac{p_2(1-p_1)}{p_1(1-p_2)}\right]^{n\bar{x}} \left[\frac{1-p_2}{1-p_1}\right]^n$$

So the ratio depends on the sample only through the sample mean $\bar{x}$and it is an increasing function of $\bar{x}$. (It is an easy algebra exercise to check that if

$$p_2 > p_1 \text{ then } p_2(1-p_1)/(p_1(1-p_2)) > 1.)$$

Example: Now consider a normal population with unknown mean μ and known varianceσ^2. So, the joint density is

$$f(x_1, \ldots, x_n|\mu) = \frac{1}{(2\pi)^{\pi/2}\sigma^n} exp\left(-\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right)$$

Now let $\mu_1 < \mu_2$. A little algebra shows

$$\frac{f(x_1, \ldots, x_n|\mu_2)}{f(x_1, \ldots, x_n|\mu_1)} = exp\left(\frac{n}{\sigma^2}\bar{x}(\mu_2 - \mu_1) + \frac{(\mu_1^2 - \mu_2^2)n}{2\sigma^2}\right)$$

So the ratio depends on $x_1, x_2, \ldots, x_n$ only through$\bar{x}$, and the ratio is an increasing function of$\bar{x}$.

Theorem 1: Suppose $f(x_1, \ldots, x_n|\theta)$has a monotone likelihood ratio in the statistic$T = t(X_1, \ldots, X_n)$. Consider hypothesis testing with alternative hypothesis $H_a: \theta > \theta_1$ and null hypothesis$H_0: \theta \le \theta_0$ or $H_0: \theta = \theta_0$. Let α_0, c be constants such that$P(T \ge c) = \alpha_0$. Then the test that rejects the null hypothesis if $T \ge c$ is a UMP test at significant level, α_0.

Example: We consider the example of a normal population with known variance and unknown mean. We saw that the likelihood ratio is monotone in the sample mean. So if we reject the null hypothesis when $\bar{X}_n \ge c$, this will be a UMP test with significance level $\alpha = P(\bar{X}_n \ge c|\mu_0)$. Give a desired significance level α_0 , we choose c so this equation holds. Then the theorem tells us we have a UMP test. So for every $\mu > \mu_0$, our test makes $Pow(\mu)$ as large as possible.

Example: We consider the example of a Bernoulli distribution for the population (population proportion). To be concrete, suppose the null hypothesis is $p \le 0.1$ and the alternative is > 0.1 . We have a random sample of size n = 20. Let $\bar{X}$ be the sample proportion. By what we already done, the test that rejects the null hypothesis when $\bar{X} \geq c$ will be a UMP test. We want to choose c so that $(\bar{X} \geq c) = \alpha_0$. However, $\bar{X}$ is a discrete RV (it can only be 0/20, 1/20, 2/20, , 19/20), so this is not possible. Suppose we want a significant level of 0.005. Using software (or a table of the binomial distribution) we find that $P(\bar{x} \geq 6/20|p = 0.1) = 0.0113$ and $P(\bar{x} \geq 7/20|p = 0.1) = 0.0024$. So we must take c = 7/20. Then the test that rejects the null if $\bar{x} \geq 7/20$ is a UMP test at significance level 0.005.

What about two-sided test alternatives? It can be shown that there is no UMP test in the setting.

IV. Uniform Most Powerful (UMP) Test

Let $X = (X_1, \dots, X_n)$ be an independent random sample. A test φ for testing $H_0: \theta \in \phi_0$ against $H_a: \theta \in \phi_1$ is said to be a uniformly most powerful test of size α if it is of size- α and it has no smaller power than that of any other test α, in class of level α tests i.e.

$$Sup_{\theta \in \theta_0} E_\theta\{\Phi(x)\} = 0$$

$$Sup_{\theta \in \theta_0} E_\theta \Phi'(x) = \alpha' \leq \alpha = Sup_{\theta \in \theta_0} E_\theta \Phi(x)$$

and for every $\theta \in \phi_0$,

$$E_\theta\{\Phi'(x)\} > E_\theta\{\Phi(x)\}$$

The known theorem provided a UMP test of size α for one-sided testing problem $H_0: \theta < \theta_0$ against $H_1: \theta > \theta_0$, whenever the p.d.f of $\theta \in \Phi$has monotone likelihood ratio. The theorem holds for such distributions provided they have monotone likelihood ratio in (x). For testing H_0 against H_1 and test of the form

$$\phi(x) = \begin{cases} 1 & \phi(x) > K \\ \gamma_i & \phi(x) = K \\ 0 & \phi(x) < K \end{cases}$$

has non-decreasing power function and is UMP of its size provided its size is positive.

Reference [10] explained that for every $\alpha, 0 < \alpha < 1$, and every $\theta_0 \in \phi$, there exist numbers $-\infty < K < +\infty$ *and* $0 \leq \gamma \leq 1$ and such that the test given above is UMP size α test of H_0 against H_1.

Examples of Uniformly Most Powerful Test

If the same result of MPT test is obtained for UMP by changing the composite range $H_1: \theta > 0$ to specified range and then consider alternative hypothesis $H_0: \theta = \theta_0$then the result obtained is said to be UMPT. i.e when H_0 is simple and H_1 is composite (one-sided) then a UMPT exist.

On the other hand, if H_0 is simple generally no UMPT exists.

Example 1

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from a distribution with density

$$F(x|\theta) = \begin{cases} \theta e^{-\theta x}, x > 0 \\ 0, otherwise \end{cases}$$

For testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ where θ_0 is specified, what is the UMPT?

Solution

In this case H_0 is simple and H_1 is composite. Let $H_1: \theta = \theta_1 > \theta_0$ be the simple hypothesis. Then MPT for $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ is given by

$$\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x}} \leq k$$

Since

$$L(\theta|\underline{x}) = \theta e^{-\theta x_1} \theta e^{-\theta x_2}, \dots, \theta e^{-\theta x_n}$$

$$L(\theta|\underline{x}) = \theta^n e^{-\sum \theta x_i}$$

Therefore

$$\frac{L(\theta_0|\underline{x})}{L(\theta_1|\underline{x})} = \frac{\theta_0^n e^{-\sum \theta_0 x_i}}{\theta_1^n e^{-\sum \theta_1 x_i}} \leq k$$

$$\frac{L(\theta_0|\underline{x})}{L(\theta_1|\underline{x})} = \frac{\theta_0^n}{\theta_1^n} e^{-\sum \theta_0 x_i + \sum \theta_1 x_i} \leq k$$

$$\frac{L(\theta_0|\underline{x})}{L(\theta_1|\underline{x})} = \frac{\theta_0^n}{\theta_1^n} e^{\sum x_i(\theta_1 - \theta_0)} \leq k$$

Taking the natural log, we have

$$\ln\left\{\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x})}\right\} = \ln\left\{\frac{\theta_0^n}{\theta_1^n} e^{\sum x_i(\theta_1 - \theta_0)}\right\} \leq \ln k$$

$$= \ln\left(\frac{\theta_0}{\theta_1}\right)^n + \ln\left(e^{\sum x_i(\theta_1 - \theta_0)}\right) \leq \ln k$$

$$= \ln\left(\frac{\theta_0}{\theta_1}\right)^n + \sum x_i(\theta_1 - \theta_0)\ln(e) \leq \ln k$$

$$= \sum x_i(\theta_1 - \theta_0) \leq \ln k - \ln\left(\frac{\theta_0}{\theta_1}\right)^n$$

$$= (\theta_1 - \theta_0)\sum x_i \leq \ln \frac{k}{\left(\frac{\theta_0}{\theta_1}\right)^n} = \ln \frac{k\theta_1^n}{\theta_0^n}$$

Let $\ln \frac{k\theta_1^n}{\theta_0^n} = k_1$ since $\ln \frac{k\theta_1^n}{\theta_0^n}$ is a constant, then we have

$$= (\theta_1 - \theta_0)\sum x_i \leq k_1$$

$$= \sum x_i \leq \frac{k_1}{(\theta_1 - \theta_0)}$$

Also, let $\frac{k_1}{(\theta_1 - \theta_0)} = k_2$ since $\frac{k_1}{(\theta_1 - \theta_0)}$ is also a constant, then we have

$$= \sum x_i \leq k_2$$

Divide through by n, we have

$$=\frac{1}{n}\sum x_i \leq \frac{k_2}{n} = c_1$$
$$=\frac{1}{n}\sum x_i \leq \frac{k_2}{n} = c$$

Therefore,

$$\bar{x} \leq c$$

Since the same MPT will be obtained for each simple hypothesis $H_1: \theta = \theta_1 > \theta_0, \bar{x} \leq c$ is the UMPT.

Example 2

Let $X_1, X_2, \ldots, X_n$ be a random sample of from $N(\theta, 1)$, find the UMPT for testing

$$H_0: \theta = 0 \; against \; H_1: \theta > 0.$$

Solution

$H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$, we want to find a UMPT. Here H_0 is simple and H_1 is composite. Consider a specific alternative hypothesis $H_1: \theta = \theta_0 > 0$. Then an application of Neyman-Pearson lemma to test $H_0: \theta = 0$ against $H_1: \theta = \theta_0$ gives

$$L(\theta|\underline{x}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_1-\mu)^2} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_2-\mu)^2} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_3-\mu)^2} \times \ldots \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_n-\mu)^2}$$

$$L(\theta|\underline{x}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

$$\frac{L(\theta=0|\underline{x})}{L(\theta=\theta_0|\underline{x})} = \frac{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-0)^2}}{\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2}} \leq k$$

$$\frac{L(\theta=0|\underline{x})}{L(\theta=\theta_0|\underline{x})} = \frac{e^{-\frac{1}{2\sigma^2}\sum(x_i)^2}}{e^{-\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2}} \leq k$$

$$\frac{L(\theta=0|\underline{x})}{L(\theta=\theta_0|\underline{x})} = e^{\frac{1}{2\sigma^2}\sum(x_i-\theta_0)^2 - \frac{1}{2\sigma^2}\sum(x_i)^2} \leq k$$

$$\frac{L(\theta=0|\underline{x})}{L(\theta=\theta_0|\underline{x})} = e^{\frac{1}{2\sigma^2}\sum(\theta^2 - 2x_i\theta_0)} \leq k$$

$$\ln\left\{\frac{L(\theta=0|\underline{x})}{L(\theta=\theta_0|\underline{x})}\right\} = \ln\left(e^{\frac{1}{2\sigma^2}\sum(\theta^2 - 2x_i\theta_0)}\right) \leq lnk$$

$$\frac{1}{2\sigma^2}\sum(\theta^2 - 2x_i\theta_0)\ln e \leq lnk = k_1$$

$$\sum(\theta^2 - 2x_i\theta_0)\ln e \leq 2\sigma^2 k_1$$

$$-2\theta_0\sum x_i \leq 2\sigma^2 k_1 - n\theta^2 = k_2$$

$$\sum x_i \geq \frac{k_2}{-2\theta_0} = c_1$$

$$\sum x_i \geq c_1$$

$$\frac{1}{n}\sum x_i \geq \frac{c_1}{n} = c$$

Therefore

$$\bar{x} \geq c$$

where c is determined such that $P(\bar{x} \geq c|\theta = 0)$ and not by $\theta_0 \; in \; H_1$ (hence independent of θ_0 and the critical region will be the same if we had selected another value of $\theta = \theta_1 > 0$. Therefore the test given by $\bar{x} \geq c$ is a UMPT.

Example 3

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \sigma^2)$. Find the UMPT for testing$H_0: \sigma^2 = 1 \; versus \; H_1: \sigma^2 > 1$.

Solution

Consider a particular simple alternative hypothesis $H_1 := \sigma^2 = \sigma_0^2 > 1$. Then the MPT for testing H_0 against is given by

$$L(\sigma^2|\underline{x}) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_1-\mu)^2} \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_2-\mu)^2} \times \ldots \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_n-\mu)^2}$$

$$L(\theta|\underline{x}) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2}$$

$$\frac{L(\sigma^2=1|\underline{x})}{L(\sigma^2=\sigma_0^2|\underline{x})} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x_i-0)^2}}{\left(\frac{1}{\sigma_0\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum(x_i-0)^2}} \leq k$$

$$\frac{L(\sigma^2=1|\underline{x})}{L(\sigma^2=\sigma_0^2|\underline{x})} = \sigma_0^n e^{-\frac{1}{2}\sum(x_i)^2 + \frac{1}{2\sigma_0^2}\sum(x_i)^2} \leq k$$

$$\ln\left\{\frac{L(\sigma^2=1|\underline{x})}{L(\sigma^2=\sigma_0^2|\underline{x})}\right\} = \ln\left(\sigma_0^n e^{-\frac{1}{2}\sum(x_i)^2 + \frac{1}{2\sigma_0^2}\sum(x_i)^2}\right) \leq lnk$$
$$= \ln(\sigma_0^n) + \ln\left(e^{-\frac{1}{2}\sum(x_i)^2 + \frac{1}{2\sigma_0^2}\sum(x_i)^2}\right) \leq lnk$$

$$\ln(\sigma_0^n) + -\frac{1}{2}\sum(x_i)^2 + \frac{1}{2\sigma_0^2}\sum(x_i)^2 \leq lnk$$

$$-\frac{1}{2}\sum(x_i)^2 + \frac{1}{2\sigma_0^2}\sum(x_i)^2 \leq lnk - \ln(\sigma_0^n) = \ln\left(\frac{k}{\sigma_0^n}\right) = k_1$$

$$-\frac{1}{2}\sum(x_i)^2 + \frac{1}{2\sigma_0^2}\sum(x_i)^2 \leq k_1$$

$$-\sum(x_i)^2 + \frac{1}{\sigma_0^2}\sum(x_i)^2 \leq 2k_1$$

$$-\sigma_0^2\sum(x_i)^2 + \sum(x_i)^2 \leq 2\sigma_0^2 k_1$$

$$\sum {x_i}^2(\sigma_0^2 - 1) \geq -2\sigma_0^2 k_1$$

$$\sum {x_i}^2 \geq \frac{-2\sigma_0^2 k_1}{(\sigma_0^2 - 1)} = c$$

$$\sum {x_i}^2 \geq c$$

where

$$c = \frac{-2\sigma_0^2 k_1}{(\sigma_0^2 - 1)}$$

Observe that as long as $\sigma_0^2 > 1$ the MPT will remain the same for each simple alternative hypothesis $H_1: \sigma^2 = \sum x_i^2 \geq c$ where c is determined once. α, the probability of type 1 error, is specified, and independent of σ_0^2. Thus, $P(\sum x_i^2 \geq c | H_0 \; is \; true) = \alpha$. Since the critical region is independent of σ_0^2, the test obtained here is UMPT. If we were testing$H_0: \sigma^2 = 1$ against $H_1: \sigma^2 < 1$, it can be verified that the corresponding UMPT is $\sum x_i^2 \leq c$.

Example 4

Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x|\theta)$ where

$$f(x|\theta) = \left\{\frac{1}{\Gamma(\theta)}\right\} e^{-x} x^{\theta-1}, \quad x > 0, \theta > 0$$

What is the UMPT for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$?

Solution

Let us consider the alternative $H_1: \theta = \theta_1 > \theta_0$. Then MPT for $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$ is given by

$$L(\theta|\underline{x}) = \frac{1}{\Gamma(\theta)} e^{-x_1} {x_1}^{\theta-1} \times \frac{1}{\Gamma(\theta)} e^{-x_2} {x_2}^{\theta-1} \times \ldots \times \frac{1}{\Gamma(\theta)} e^{-x_n} {x_n}^{\theta-1}$$

$$L(\theta|\underline{x}) = \left(\frac{1}{\Gamma(\theta)}\right)^n e^{-\sum x_i} \prod_{i=1}^{n} x_i^{\theta-1}$$

$$\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x})} \leq k$$

$$\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x})} = \frac{\left(\frac{1}{\Gamma(\theta_0)}\right)^n e^{-\sum x_i} \prod_{i=1}^{n} x_i^{\theta_0-1}}{\left(\frac{1}{\Gamma(\theta_1)}\right)^n e^{-\sum x_i} \prod_{i=1}^{n} x_i^{\theta_1-1}} \leq k$$

$$\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x})} = \left(\frac{\Gamma(\theta_1)}{\Gamma(\theta_0)}\right)^n e^{-\sum x_i + \sum x_i} \frac{\prod_{i=1}^{n} x_i^{\theta_0-1}}{\prod_{i=1}^{n} x_i^{\theta_1-1}} \leq k$$

$$\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x})} = \frac{\prod_{i=1}^{n} x_i^{\theta_0-1}}{\prod_{i=1}^{n} x_i^{\theta_1-1}} \leq k$$

$$\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x})} = \prod_{i=1}^{n} x_i^{\theta_0-1-(\theta_1-1)} \leq k$$

$$\ln\left\{\frac{L(\theta = \theta_0|\underline{x})}{L(\theta = \theta_1|\underline{x})}\right\} = \ln\left(\prod_{i=1}^{n} x_i^{\theta_0-1-(\theta_1-1)}\right) \leq lnk$$

$$= \ln\left(\prod_{i=1}^{n} x_i^{\theta_0-\theta_1}\right) \leq lnk$$

$$(\theta_0 - \theta_1)\ln\left(\prod_{i=1}^{n} x_i\right) \leq lnk$$

$$\ln\left(\prod_{i=1}^{n} x_i\right) \leq \frac{lnk}{(\theta_0 - \theta_1)}$$

$$\ln\left(\sum x_i\right) \leq \frac{lnk}{(\theta_0 - \theta_1)} = k$$

$$\sum x_i \leq e^k = k_1$$

$$\sum x_i \leq k_1$$

$$\frac{1}{n}\sum x_i \leq \frac{k_1}{n} = c$$

Therefore,

$$\bar{x} \leq c, \text{Since } \theta_0 - \theta_1 > 0$$

Since the same MPT will be obtained for each simple hypothesis

$$H_1: \theta = \theta_1 > \theta_0,$$

then

$$\bar{x} \leq c \text{ is the UMPT.}$$

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