International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:3, 2013

Cubic B-spline collocation method for numerical solution of the Benjamin-Bona-Mahony-Burgers equation

M. Zarebnia and R. Parvaz

Abstract—In this paper, numerical solutions of the nonlinear Benjamin-Bona-Mahony-Burgers (BBMB) equation are obtained by a method based on collocation of cubic B-splines. Applying the Von-Neumann stability analysis, the proposed method is shown to be unconditionally stable. The method is applied on some test examples, and the numerical results have been compared with the exact solutions. The L_∞ and L_2 in the solutions show the efficiency of the method computationally.

Keywords—Benjamin-Bona-Mahony-Burgers equation; Cubic B-spline; Collocation method; Finite difference.

I. INTRODUCTION

IN this paper we consider the solution of the BBMB equation

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + uu_x = 0, \quad x \in [a,b], \; t \in [0,T], \tag{1}$$

with the initial condition

$$u(x,0) = f(x), \quad x \in [a,b], \tag{2}$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \tag{3}$$

where α and β are constants.
BBMB equations play a dominant role in many branches of science and engineering [1]. For $\alpha = 0$, Eq. (1) is called the Benjamin-Bona-Mahony (BBM) equation. In the past several years, many different methods have been used to estimate the solution of the BBMB equation and the BBM equation, for example, see [2-6].
The paper is organized as follows. In Section 2, cubic B-spline collocation method is explained. In Section 3, we develop an algorithm for the numerical solution of the BBMB equation. Section 4, is devoted to stability analysis of the method. In Section 5, examples are presented. A summary is given at the end of the paper in Section 6. Note that we have computed the numerical results by Mathematica-7 programming.

II. CUBIC B-SPLINE COLLOCATION METHOD

The interval $[a,b]$ is partitioned in to a mesh of uniform length $h = x_{i+1} - x_i$ by the knots $x_i, i = 0, 1, \ldots, N$ such that $a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N = b$. Our numerical treatment for BBMB equation using the collocation method with cubic B-spline is to find an approximate solution $U_N(x,t)$ to the exact solution $u(x,t)$ in the form

$$U_N(x,t) = \sum_{i=-3}^{N-1} c_i(t) B_i(x), \tag{4}$$

where $c_i(t)$ are time-dependent quantities to be determined from the boundary conditions and collocation form of the differential equations. Also $B_i(x)$ are the cubic B-spline basis functions at knots, given by [7,8]

$$B_i(x) = \frac{1}{6h^3}\begin{cases} (x-x_i)^3, & x \in [x_i, x_{i+1}), \\ h^3 + 3h^2(x-x_{i+1})^2 + 3h(x-x_{i+1}) \\ -3(x-x_{i+1}), & x \in [x_{i+1}, x_{i+2}), \\ h^3 + 3h^2(x_{i+3}-x)^2 + 3h(x_{i+3}-x) \\ -3(x_{i+3}-x), & x \in [x_{i+2}, x_{i+3}), \\ (x_{i+4}-x)^3, & x \in [x_{i+3}, x_{i+4}). \\ 0, & otherwise. \end{cases} \tag{5}$$

The values of $B_i(x)$ and its derivatives may be tabulated as in Table 1. The values of U and its space derivatives at the knots x_i can be obtained as

$$U_i = \frac{1}{6}(c_{i-3} + 4c_{i-2} + c_{i-1}), \tag{6}$$

$$U_i' = \frac{1}{2h}(c_{i-1} - c_{i-3}), \tag{7}$$

$$U_i'' = \frac{1}{h^2}(c_{i-3} - 2c_{i-2} + c_{i-1}). \tag{8}$$

TABLE I
B_i ,$B_i^{'}$ AND $B_i^{''}$ AT THE NODE POINTS.

x	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
$B_i(x)$	0	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$	0
$B_i'(x)$	0	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0
$B_i''(x)$	0	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0

III. CONSTRUCTION OF THE METHOD

To apply the proposed method, discretizing the time derivative in the usual finite difference way. Using the finite difference method, we can write

$$\frac{u^{n+1}-u^n}{\Delta t} - \frac{u_{xx}^{n+1}-u_{xx}^n}{\Delta t} - \alpha\frac{u_{xx}^{n+1}+u_{xx}^n}{2} +$$

M. Zarebnia is with the Department of Mathematics, University of Mohaghegh Ardabili, P. 0. Box 179, Ardabil, Iran, Tel: (+98) 451-5520457; Fax: (+98) 451-5520458; e-mail: zarebnia@uma.ac.ir.
R. Parvaz is with the Department of Mathematics, University of Mohaghegh Ardabili, P. 0. Box 179, Ardabil, Iran, email: rizap2010@gmail.com.

$$\beta\frac{u_x^{n+1}+u_x^n}{2}+\frac{(uu_x)^{n+1}+(uu_x)^n}{2}=0. \tag{9}$$

The nonlinear term in Eq. (9) can be approximated by using the following formula [9]:

$$(uu_x)^{n+1}=u^{n+1}u_x^n+u^nu_x^{n+1}-(uu_x)^n. \tag{10}$$

Substituting the approximate solution U for u and putting the values of the nodal values U and its derivatives using Eqs. (6)- (8) at the knots in Eq. (9) yield the following difference equation with the variables c_i , $i=-3,\dots,N-1$,

$$\acute{a}c_{i-3}^{n+1}+\acute{b}c_{i-2}^{n+1}+\acute{c}c_{i-1}^{n+1}=\acute{d}c_{i-3}^{n}+\acute{e}c_{i-2}^{n}+\acute{f}c_{i-1}^{n}, i=0,1,\dots,N, \tag{11}$$

where

$$\begin{cases} \acute{a}=\frac{x}{6}-\frac{y}{2h}+\frac{z}{h^2},\\ \acute{b}=\frac{4x}{6}-\frac{2z}{h^2},\\ \acute{c}=\frac{x}{6}+\frac{y}{2h}+\frac{z}{h^2},\\ \acute{d}=1+\frac{w}{h^2}-\frac{v}{2h},\\ \acute{e}=4-\frac{2w}{h^2},\\ \acute{f}=1+\frac{w}{h^2}+\frac{v}{2h}, \end{cases} \tag{12}$$

with $x=1+\frac{\Delta tu_x^n}{2}$, $y=\frac{\beta\Delta t}{2}+\frac{\Delta tu^n}{2}$, $z=-1-\frac{\alpha\Delta t}{2}$, $w=-1+\frac{\alpha\Delta t}{2}$, $v=-\frac{\beta\Delta t}{2}$.

The system (11) consists of $N+1$ linear equations in $N+3$ unknowns $\{c_{-3},c_{-2},\dots,c_{N-2},c_{N-1}\}$. To obtain a unique solution for $C=\{c_{-3},\dots,c_{N-1}\}$, we must use the boundary conditions. From the boundary conditions and Table 1, we can write

$$\frac{1}{6}(c_{-3}^{n+1}+4c_{-2}^{n+1}+c_{-1}^{n+1})=0, \tag{13}$$

$$\frac{1}{6}(c_{N-3}^{n+1}+4c_{N-2}^{n+1}+c_{N-1}^{n+1})=0. \tag{14}$$

Associating (13) and (14) with (11) we obtain a $(N+3)\times(N+3)$ system of equations in the following form

$$AC=Q, \tag{15}$$

where

$$A=\begin{pmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & \dots & 0\\ \acute{a} & \acute{b} & \acute{c} & \dots & 0\\ \vdots & \ddots & \ddots & \vdots & \\ 0 & \dots & \acute{a} & \acute{b} & \acute{c}\\ 0 & \dots & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{pmatrix}, \tag{16}$$

$$C=(c_{-3}^{n+1},c_{-2}^{n+1},\dots,c_{N-2}^{n+1},c_{N-1}^{n+1})^T, \tag{17}$$

Q=

$$\begin{cases} \Big(0,u(x_0,t)+(-1+\alpha\frac{\Delta t}{2})u_{xx}(x_0,t)-\beta\frac{\Delta t}{2}u_x(x_0,t),\dots,\\ u(x_N,t)+(-1+\alpha\frac{\Delta t}{2})u_{xx}(x_N,t)-\beta\frac{\Delta t}{2}u_x(x_N,t),0\Big)^T, \\ \qquad\qquad if\ \ t=\Delta t,\\ \Big(0,\Psi_0^n,\dots,\Psi_N^n,0\Big)^T, \qquad if\ \ t>\Delta t, \end{cases} \tag{18}$$

with

$$\Psi_i^n=\acute{d}c_{i-3}^n+\acute{e}c_{i-2}^n+\acute{f}c_{i-1}^n.$$

IV. Stability analysis

In this section, we present the stability of the cubic B-spline approximation (11) using the Von-Numann method [10,11]. According to the Von-Neumann method, we have

$$c_i^n=\xi^n exp(\lambda khi), \qquad \lambda^2=-1, \tag{19}$$

where k is the mode number and h is the element size. To apply this method, we have linearized the nonlinear term uu_x by consider u as a constant in term (9). We obtain the equation:

$$\begin{aligned} &\bar{a}\xi^{n+1}exp(\lambda kh(i-3))+\bar{b}\xi^{n+1}exp(\lambda kh(i-2))+\\ &\bar{c}\xi^{n+1}exp(\lambda kh(i-1))=\bar{d}\xi^{n}exp(\lambda kh(i-3))+\\ &\bar{e}\xi^{n}exp(\lambda kh(i-2))+\bar{f}\xi^{n}exp(\lambda kh(i-1)), \end{aligned} \tag{20}$$

where

$$\begin{cases} \bar{a}=\frac{1}{6}-\frac{\acute{y}}{2h}+\frac{\acute{x}}{h^2},\\ \bar{b}=\frac{4}{6}-\frac{2\acute{x}}{h^2},\\ \bar{c}=\frac{1}{6}+\frac{\acute{y}}{2h}+\frac{\acute{x}}{h^2},\\ \bar{d}=\frac{1}{6}+\frac{\acute{y}}{2h}+\frac{\acute{z}}{h^2},,\\ \bar{e}=\frac{4}{6}-\frac{2\acute{z}}{h^2},,\\ \bar{f}=\frac{1}{6}-\frac{\acute{y}}{2h}+\frac{\acute{z}}{h^2},, \end{cases} \tag{21}$$

with $\acute{x}=-1-\frac{\alpha\Delta t}{2}$, $\acute{y}=\frac{\beta\Delta t}{2}+\frac{\Delta tu^n}{2}$, $\acute{z}=-1+\frac{\alpha\Delta t}{2}$.

Dividing both sides of (20) by exp($(i-2)\lambda$kh), we can write:

$$\begin{aligned} &\xi^{n+1}\Big(\bar{a}exp(\lambda kh)+\bar{b}+\bar{c}exp(-\lambda kh)\Big)=\\ &\xi^{n}\Big(\bar{d}exp(\lambda kh)+\bar{e}+\bar{f}exp(-\lambda kh)\Big), \end{aligned} \tag{22}$$

Eq. (22) can be rewritten in a simple form as:

$$\xi=\frac{X-\lambda Y}{X_1+\lambda Y}, \tag{23}$$

where

$$X = (\tfrac{1}{6} + \tfrac{\acute{z}}{h^2})\cos(kh) + (\tfrac{1}{3} - \tfrac{\acute{z}}{h^2}),$$

$$X_1 = (\tfrac{1}{6} + \tfrac{\acute{x}}{h^2})\cos(kh) + (\tfrac{1}{3} - \tfrac{\acute{x}}{h^2}),$$

$$Y = (\tfrac{\acute{y}}{2h})\sin(kh).$$

X and X_1 can be rewritten in the form:

$$X_1 = (\tfrac{1}{6} - \tfrac{1}{h^2})\cos(kh) + (\tfrac{1}{3} + \tfrac{1}{h^2}) + \tfrac{\alpha\Delta t}{2h^2}(1 - \cos(kh)),$$

$$X = (\tfrac{1}{6} - \tfrac{1}{h^2})\cos(kh) + (\tfrac{1}{3} + \tfrac{1}{h^2}) - \tfrac{\alpha\Delta t}{2h^2}(1 - \cos(kh)).$$

We note that $X \leq X_1$, so $|\xi|^2 = \xi\bar{\xi} = \frac{X^2+Y^2}{X_1^2+Y^2} \leq 1$. Therefore, the linearized numerical scheme for the BBMB equation is unconditionally stable.

V. Numerical examples

We now obtain the numerical solutions of the BBMB equation for two problems. To show the efficiency of the present method for our problem in comparison with the exact solution, we report L_∞ and L_2 using formulae

$$L_\infty = \max_i \mid U(x_i,t) - u(x_i,t) \mid,$$

$$L_2 = (h\sum_i \mid U(x_i,t) - u(x_i,t) \mid^2)^{\frac{1}{2}},$$

where U is numerical solution and u denotes analytical solution.

Example 1. Consider the BBMB equation with $\alpha = 0$ and $\beta = 1$ in the interval $[-40, 60]$, with the exact solutio $u(x,t) = 3c\text{sech}^2(k(x - vt - x_0))$. We have taken $c = 0.03$, $v = 1$, $x_0 = 0$ and $k = \frac{c}{4v(c+1)}$. The initial condition is taken from the exact solution. Table 2 gives a comparison between the $L_\infty$and L_2 found by our method in different times and different values of N with $\Delta t = 0.1$. Also Table 3 gives comparison of absolute errors found by present method with $\Delta t = 0.01$ and $N = 300$.

TABLE II
NUMERICAL RESULTS FOR EXAMPLE 1.

Method	$Time$	N	$L_2 \times 10^3$	$L_\infty \times 10^3$
present method	1	100	0.902611	0.328588
present method	10	100	8.14052	1.76978
present method	20	100	16.2506	3.53644
present method	1	300	0.507465	0.328588
present method	10	300	4.71001	1.77131
present method	20	300	9.40151	3.54203
method in [12]	20	1000	14.45	3.996

TABLE III
COMPARISON OF ABSOLUTE ERRORS FOR EXAMPLE 1 WITH $\Delta t = 0.01$ AND $N = 300$.

$x \backslash t$	0.5	1	1.5
-30	2.12681×10^{-4}	1.99925×10^{-4}	5.42065×10^{-5}
-20	1.0672×10^{-3}	1.00969×10^{-3}	2.72896×10^{-3}
-10	3.57975×10^{-3}	3.52499×10^{-3}	1.00584×10^{-3}
0	7.45797×10^{-4}	1.20873×10^{-3}	5.59997×10^{-4}
10	3.9186×10^{-3}	4.06787×10^{-3}	1.25094×10^{-3}
20	1.36982×10^{-3}	1.50618×10^{-3}	5.09114×10^{-4}
30	2.81202×10^{-4}	3.12619×10^{-4}	1.07576×10^{-5}

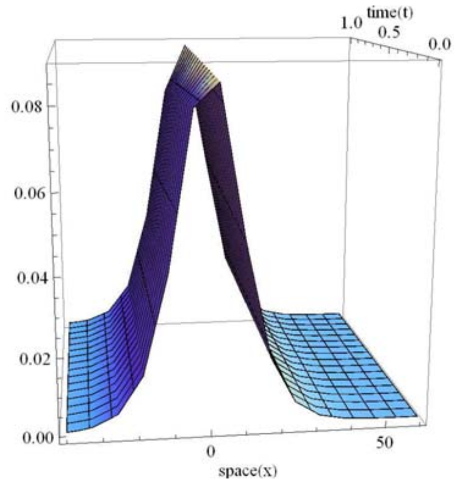

Fig. 1. Three-dimensional plot for Example 1.

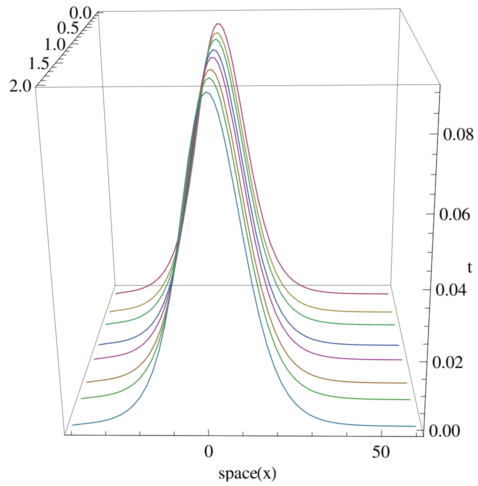

Fig. 2. Approximate solution graphs of Example 1 for $x \in [-40, 60]$ with $\Delta t = 0.01$ and $N = 300$.

Example 2. As a last study we consider here a numerical solution of the BBMB in the interval $[-12, 12]$ with $\alpha = 1$, $\beta = 1$ and initial condition $u(x, 0)$ =sech$^2(\frac{x}{4})$. Tables 4 and 5 give numerical results with $\Delta t = 0.01$ and $N = 200$. Also Fig 3 shows approximate solution graphs.

TABLE IV
NUMERICAL RESULTS FOR EXAMPLE 2 WITH $\Delta t = 0.01$ AND $N = 200$.

$x \backslash t$	0.2	0.5	0.7
-12	3.33333×10^{-11}	-2.23333×10^{-10}	-3.33333×10^{-11}
-10	0.0229513	0.0198217	0.0179501
-5	0.256278	0.224742	0.206391
0	0.978102	0.933352	0.897596
5	0.319376	0.380993	0.42342
10	0.0304198	0.0397963	0.0472631
12	2×10^{-10}	6.66666×10^{-11}	-2.66667×10^{-10}

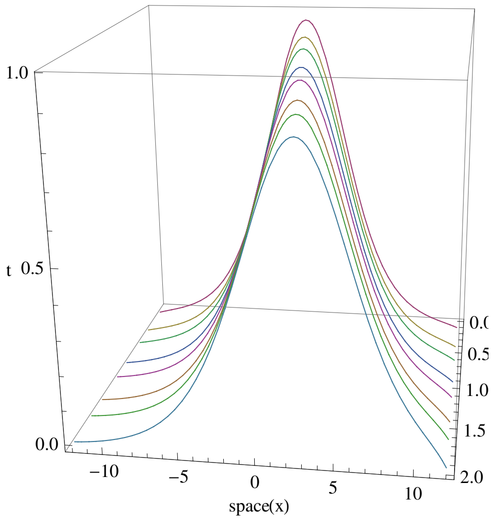

Fig. 3. Approximate solution graphs of Example 2 for $x \in [-12, 12]$ with $\Delta t = 0.01$ and $N = 200$.

TABLE V
NUMERICAL RESULTS FOR EXAMPLE 2 WITH $\Delta t = 0.01$ AND $N = 200$.

$x\backslash t$	1	1.5	2
-12	1.33333×10^{-11}	-1.26667×10^{-10}	8.66667×10^{-11}
-10	0.0154352	0.0119344	0.00916508
-5	0.182231	0.149239	0.123215
0	0.83834	0.733537	0.631526
5	0.487532	0.589817	0.676809
10	0.0605017	0.088659	0.12528
12	-1.13333×10^{-9}	-3.33333×10^{-10}	-1.01048×10^{-16}

VI. Conclusion

The cubic B-spline collocation method is used to solve the Benjamin-Bona-Mahony-Burgers(BBMB) equation. The stability analysis of the method is shown to be unconditionally stable. The numerical results given in the previous section demonstrate the good accuracy and stability of the proposed scheme in this research.

References

[1] G. Stephenson, Partial Differential Equations for Scientists and Engineers, Imperial College Press, 1996.
[2] B. Wang, Random attractors for the stochastic BenjaminBonaMahony equation on unbounded domains, J. Differential Equations 246 (2009) 2506-2537.
[3] K. Al-Khaled, S. Momani, A. Alawneh, Approximate wave solutions for generalized Benjamin-Bona-Mahony-Burgers equations, Applied Mathematics and Computation, 171 (2005) 281-292.
[4] M.A. Raupp, Galerkin methods applied to the Benjamin-Bona-Mahony equation, Bol Soc Brazil Mat 6 (1975), 65-77.
[5] J. Avrin, J.A. Goldstein, Global existence for the Benjamin-Bona-Mahony equation in arbitrary dimensions, Nonlinear Anal. 9 (1985) 861-865.
[6] A.O. Celebi, V.K. Kalantarov, M. Polat, Attractors for the generalized Benjamin-Bona-Mahony equation, J. Differential Equations 157 (1999) 439-451.
[7] J. Stoer, R. Bulirsch, Introduction to Numerical Analysis, third edition ,Springer-Verlg, 2002.
[8] D. Kincad, W. Cheny, Numerical analysis, Brooks/COLE, 1991.
[9] S.G. Rubin ,R.A. Graves, Cubic Spline Sppproximation for Problems in Fluid Mechanics, NASA TR R-436, Washington, DC; 1975.
[10] G.D. Smith, Numerical Solution of Patial Differential Method, Second Edition, Oxford University Press, 1978.
[11] R.D Richtmyer, K.W. Morton, Difference Methods for Initial-Value Problems, Inter science Publishers (John Wiley), New York, (1967).
[12] L. R. T. Gardner, G. A. Gardner and I. Dag, A B-spline finite element method for the regularised long wave equation, Commun. Numer. Meth. Eng. 12 (1995) 795-804.