Convergence of a one-step iteration scheme for quasi-asymptotically nonexpansive mappings

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Abstract—In this paper, we use a one-step iteration scheme to approximate common fixed points of two quasi-asymptotically nonexpansive mappings. We prove weak and strong convergence theorems in a uniformly convex Banach space. Our results generalize the corresponding results of Yao and Chen [15] to a wider class of mappings while extend those of Khan, Abbas and Khan [4] to an improved one-step iteration scheme without any condition and improve upon many others in the literature.

Keywords—One-step Iteration Scheme, Asymptotically Quasi-nonexpansive Mapping, Common Fixed Point, Condition (A'), Weak and Strong Convergence

I. INTRODUCTION

THROUGHOUT this paper, $\mathbb{N}$ denotes the set of all positive integers. Let E be a real Banach space and C a nonempty subset of E. Let $S, T : C \longrightarrow C$ be two mappings. We denote by $F(T)$ the set of fixed points of T and by $F := F(T) \cap F(S)$ the set of common fixed points of S and T. A mapping S of C into itself is said to be asymptotically nonexpansive if for a sequence $\{k_n\} \subseteq [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$, $\|S^n x - S^n y\| \leq k_n \|x - y\|$ holds for all $x, y \in C$ and for all $n \in \mathbb{N}$. S is called uniformly k-Lipschitzian if for some $k > 0$, $\|S^n x - S^n y\| \leq k \|x - y\|$ for all $x, y \in C$ and $n \in \mathbb{N}$.

If $F(T) \neq \phi$, then T is called asymptotically quasi-nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$, we have $\|T^n x - p\| \leq k_n \|x - p\|$ for all $x \in C$, $p \in F(T)$ and $n \in \mathbb{N}$. Clearly, an asymptotically nonexpansive mapping must be uniformly k-Lipschitzian as well as asymptotically quasi-nonexpansive but the converse may not hold. In recent years, Mann and Ishikawa iteration schemes have been studied extensively by many authors to solve one-parameter nonlinear operator equations as well as variational inequalities in Hilbert and Banach spaces. Liu [6] introduced an iteration scheme with error terms. Later on, Xu [13] improved these methods by giving more satisfactory error terms. Both methods constitute generalizations of Mann and Ishikawa (one mapping) iteration schemes. The case of two mappings in iteration schemes has also remained under study since Das and Debata[2] gave and studied a two mappings scheme. Also see, for example, [5] and [11]. Note that two mappings case, that is , approximating common fixed points, has its own importance as it has a direct link with the minimization problem, see for example [10]. It has been recently noted that once the results without error terms have been proved, it is not difficult to prove them with error terms.

To approximate the common fixed points of two mappings, the following Ishikawa type two-step iteration scheme has been widely used (see, for example, [5], [8], [11], [14] and references cited therein):

$$x_1 \in C, \ x_{n+1} = (1-a_n)x_n + a_n S[(1-b_n)x_n + b_n T x_n], \ (1)$$

for all $n \in \mathbb{N}$, where $\{a_n\}$ and $\{b_n\}$ are in $[0, 1]$ satisfying certain conditions. Note that when $S = T$, (1) reduces to Ishikawa iteration scheme:

$$x_1 \in C, \ x_{n+1} = (1-a_n)x_n + a_n T^n[(1-b_n)x_n + b_n T^n x_n], \tag{2}$$

for all $n \in \mathbb{N}$, where $\{a_n\}$ and $\{b_n\}$ are in $[0, 1]$ satisfying certain conditions.

In this paper, we take up the problem of approximating common fixed points of quasi-asymptotically nonexpansive mappings S and T through weak and strong convergence of the sequence defined by:

$$x_1 \in C, \ x_{n+1} = a_n x_n + b_n T^n x_n + c_n S^n x_n \tag{3}$$

for all $n \in \mathbb{N}$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0, 1)$ with $a_n + b_n + c_n = 1$. Basically, this scheme was considered along-with error terms by Yao and Chen [15] for two single-valued nonexpansive mappings, and by Abbas et al. [1] (without error terms) for multivalued mappings. Note that when either $S = T$ or one of S and T is I, (3) reduces to Mann iteration scheme:

$$x_1 \in C, \ x_{n+1} = (1 - a_n)x_n + a_n T^n x_n, \tag{4}$$

for all $n \in \mathbb{N}$, where $\{a_n\}$ in $(0, 1)$ satisfies certain conditions. Similarly, when $T = I$, (1) also reduces to Mann iteration scheme. However, (1) and (3) are independent. Moreover, (1) is a two-step iteration scheme while (3) uses only one step. Using the above one-step iteration scheme, we investigate the approximation of common fixed points of asymptotically quasi-nonexpansive mappings through weak and strong convergence theorems. We will prove our first strong convergence theorem in general Banach spaces and then apply it to get convergence results in uniformly convex Banach spaces. Later, we remark that these results are also true with errors in the sense of Xu [13] (or Liu [6]). Our results proved in this paper generalize the corresponding results of Yao and Chen [15] to a wider class of mappings while those of Khan, Abbas and Khan [4] to an improved one-step iteration scheme without any condition and improve upon many others in the literature.

II. PRELIMINARIES

Let E be a Banach space and let C be a nonempty bounded convex subset of E. We recall that E is said to satisfy Opial's condition [7] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies

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that $\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying this condition are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition. A mapping $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

The following result of Yao and Noor will be helpful.

Lemma 1. [16]Let E be a uniformly convex Banach space. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$ and $0 < \liminf_{n\to\infty} a_n < \liminf_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n + b_n) < 1$. Suppose that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are three sequences in E. Then the conditions: $\limsup_{n\to\infty}\|x_n\| \leq d$, $\limsup_{n\to\infty}\|y_n\| \leq d$, $\limsup_{n\to\infty}\|z_n\| \leq d$ and $\lim_{n\to\infty}\|a_n x_n + b_n y_n + c_n z_n\| = d$ imply that $\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|y_n - z_n\| = \lim_{n\to\infty}\|z_n - x_n\| = 0$, where $d \geq 0$ is some constant.

III. Weak and Strong Convergence Theorems

We first prove a lemma which, in fact, constitutes a considerably big part of the proofs of both weak and strong convergence theorems.

Lemma 2. Let E be a uniformly convex Banach space and let C be its bounded, closed and convex subset. Let S and T be two uniformly k-Lipschitzian asymptotically quasi-nonexpansive mappings of C into itself with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Define a sequence $\{x_n\}$ in C as:

$$x_1 \in C, \quad x_{n+1} = a_n x_n + b_n T^n x_n + c_n S^n x_n$$

for all $n \in \mathbb{N}$, where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are three sequences in $(0, 1)$ satisfying $a_n + b_n + c_n = 1$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$. If $F \neq \phi$, then $(i) \lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in F$ and $(ii) \lim_{n\to\infty}\|x_n - Sx_n\| = 0 = \lim_{n\to\infty}\|x_n - Tx_n\|$.

Proof. Let $p \in F$. Then by definition

$$\|x_{n+1} - p\| = \|a_n(x_n - p) + b_n(T^n x_n - p) + c_n(S^n x_n - p)\|$$
$$\leq (a_n + k_n b_n + k_n c_n)\|x_n - p\|.$$

Setting $v_n = a_n + k_n b_n + k_n c_n$, the above inequality takes the shape $\|x_{n+1} - p\| \leq v_n \|x_n - p\|$ for all $n \in \mathbb{N}$. By induction, $\|x_{n+m} - p\| \leq \left(\prod_{i=n}^{n+m-1} v_i\right)\|x_n - p\|$ for all $m, n \in \mathbb{N}$. Also note that $v_n - 1 = a_n + k_n b_n + k_n c_n - a_n - b_n - c_n = (b_n + c_n)(k_n - 1)$ so that $\sum_{n=1}^{\infty}(v_n - 1) < \infty$, we obtain $\lim_{n\to\infty}\prod_{i=n}^{\infty} v_i = 1$ and hence $\lim_{n\to\infty}\|x_n - p\|$ exists. Let $\lim_{n\to\infty}\|x_n - p\| = c$ where $c \geq 0$ is a real number. If $c = 0$, the result is evident. So we assume $c > 0$. Now

$$c = \lim_{n\to\infty}\|a_n(x_n - p) + b_n(T^n x_n - p) + c_n(S^n x_n - p)\|. \tag{5}$$

Since $\|T^n x_n - p\| \leq k_n\|x_n - p\|$ and $\|S^n x_n - p\| \leq k_n\|x_n - p\|$, therefore

$$\limsup_{n\to\infty}\|T^n x_n - p\| \leq c \tag{6}$$

and

$$\limsup_{n\to\infty}\|S^n x_n - p\| \leq c. \tag{7}$$

Applying Lemma 1 on (5), (6) and (7), we obtain

$$\lim_{n\to\infty}\|x_n - T^n x_n\| = \lim_{n\to\infty}\|T^n x_n - S^n x_n\| = \lim_{n\to\infty}\|x_n - S^n x_n\| = 0.$$

Next,

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|a_n x_n + b_n T^n x_n + c_n S^n x_n \\ &\quad - (a_n + b_n + c_n)x_n\| \\ &\leq b_n\|x_n - T^n x_n\| + c_n\|x_n - S^n x_n\| \\ &\leq b\|x_n - T^n x_n\| + c\|x_n - S^n x_n\|.\end{aligned}$$

This gives

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{8}$$

For simplicity, set $Q_n = \|x_n - T^n x_n\|$ and $R_n = \|x_n - S^n x_n\|$ for all $n \in \mathbb{N}$. Now

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - Tx_{n+1}\| \\ &\leq Q_{n+1} + k\|x_{n+1} - T^n x_{n+1}\| \\ &\leq Q_{n+1} + k(\|x_{n+1} - x_n\| + \|x_n - T^n x_n\| \\ &\quad + \|T^n x_n - T^n x_{n+1}\|) \\ &\leq Q_n + (k + k^2)\|x_{n+1} - x_n\| + kQ_n.\end{aligned}$$

Using $\lim_{n\to\infty} Q_n = 0$ and (8), we get

$$\limsup_{n\to\infty}\|x_{n+1} - Tx_{n+1}\| \leq 0.$$

Hence

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

It can be shown similarly that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$.

Following is our weak convergence theorem.

Theorem 1. Let E be a uniformly convex Banach space satisfying Opial's condition and let C, S, T and $\{x_n\}$ be as taken in Lemma 2. If $F \neq \phi$ and $I - T$ and $I - S$ are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of S and T.

Proof. Let $p \in F$. Then $\lim_{n\to\infty}\|x_n - p\|$ exists as proved in Lemma 2. We prove that $\{x_n\}$ has a unique weak subsequential limit in F. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 2, $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed at zero, therefore we obtain $Tu = u$. Similarly, $Su = u$. Thus $u \in F$. Again in the same fashion, we can prove that $v \in F$. Next, we prove the uniqueness. To this end, if u and v are distinct, then by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - u\| &= \lim_{n_i\to\infty}\|x_{n_i} - u\| \\ &< \lim_{n_i\to\infty}\|x_{n_i} - v\| \\ &= \lim_{n\to\infty}\|x_n - v\| \\ &= \lim_{n_j\to\infty}\|x_{n_j} - v\| \\ &< \lim_{n_j\to\infty}\|x_{n_j} - u\| \\ &= \lim_{n\to\infty}\|x_n - u\|,\end{aligned}$$

which is a contradiction nd hence the proof.

Corollary 1. Let E be a uniformly convex Banach space satisfying the Opial's condition and C, T be as taken in Lemma 2 and $\{x_n\}$ as in (4). If $F(T) \neq \phi$ and $I - T$ is demiclosed at zero, then $\{x_n\}$ converges weakly to a fixed point of T.

We now turn to strong convergence theorems. Our first result in this direction is in a general real Banach space as follows:

Theorem 2. Let E be a real Banach space and C, $\{x_n\}$ S, T be as taken in Lemma 2. If $F \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n \to \infty} d(x_n, F) = 0$. As proved in Lemma 2, we have

$$\|x_{n+1} - p\| \leq v_n \|x_n - p\|.$$

This gives

$$d(x_{n+1}, F) \leq v_n d(x_n, F),$$

so that $\lim_{n \to \infty} d(x_n, F)$ exists. But by hypothesis $\liminf_{n \to \infty} d(x_n, F) = 0$, therefore we must have $\lim_{n \to \infty} d(x_n, F) = 0$. Following the method of Tan and Xu [12], we can prove that $\{x_n\}$ is a Cauchy sequence in a closed subset C of a Banach space E. Thus it must converge in C. Let $\lim_{n \to \infty} x_n = q \in C$. Now $\lim_{n \to \infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. It is well-known that F is closed and so $q \in F$.

Two mappings $S, T : C \to C$, where C is a subset of a normed space E, are said to satisfy Condition (A') [3] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either $\|x - Sx\| \geq f(d(x, F))$ or $\|x - Tx\| \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Note that the above Condition (A') reduces to Condition (A) of Senter and Dotson [9] when $S = T$. Our next theorem is an application of Theorem 2 and makes use of the Condition (A').

Theorem 3. Let $E, C, S, T, \{x_n\}$ be as taken in Lemma 2. Let S, T satisfy Condition (A'). If $F \neq \phi$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof. By Lemma III, $\lim_{n \to \infty} \|x_n - x^*\|$ exists for all $x^* \in F$. Let it be c for some $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose $c > 0$. Now $\|x_{n+1} - x^*\| \leq v_n \|x_n - x^*\|$ gives that $d(x_{n+1}, F) \leq v_n d(x_n, F)$ and so $\lim_{n \to \infty} d(x_n, F)$ exists. Moreover, by Lemma 2, $\lim_{n \to \infty} \|x_n - Tx_n\| = 0 = \lim_{n \to \infty} \|x_n - Sx_n\|$. By the Condition (A'), we get

$$\lim_{n \to \infty} f(d(x_n, F)) \leq \lim_{n \to \infty} \|x_n - Tx_n\| = 0$$

or

$$\lim_{n \to \infty} f(d(x_n, F)) \leq \lim_{n \to \infty} \|x_n - Sx_n\| = 0.$$

In both the cases,

$$\lim_{n \to \infty} f(d(x_n, F)) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, therefore we have

$$\lim_{n \to \infty} d(x_n, F) = 0.$$

Now all the conditions of Theorem 2 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a point of F.

Corollary 2. Let E, C, T be as taken in Lemma 2. Let T satisfy Condition (A'). If $F(T) \neq \phi$, then $\{x_n\}$ defined by (4) converges strongly to a fixed point of T.

Remarks. (i)All the results proved in this paper can easily be extended to those with errors in the sense of Xu [13] (or Liu [6]).

(ii) Theorems $1 - 3$ also set analogues of the corresponding results in [3] using one-step iteration scheme for two asymptotically quasi-nonexpansive mappings on a bounded domain.

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