International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:11, 2013

# Convergence and Comparison Theorems of the Modified Gauss-Seidel Method

Zhouji Chen

*Abstract*—In this paper, the modified Gauss-Seidel method with the new preconditioner for solving the linear system $Ax = b$, where $A$ is a nonsingular $M$-matrix with unit diagonal, is considered. The convergence property and the comparison theorems of the proposed method are established. Two examples are given to show the efficiency and effectiveness of the modified Gauss-Seidel method with the presented new preconditioner.

*Keywords*—Preconditioned linear system, $M$-matrix, Convergence, Comparison theorem.

## I. Introduction

IN this paper, we consider the Gauss-Seidel method for solving preconditioned linear system

$$PAx = Pb, \tag{1}$$

where $P$, called the left preconditioner [2], is nonsingular, $A = (a_{i,j})$ is an $n \times n$ nonsingular $M$-matrix, $x$ and $b$ are $n$-dimensional vectors. Throughout this paper, without loss of generality, we always assume that $A$ has unit diagonal elements, i.e., it has the form $A = I - L - U$, where $I$ is the identity matrix, $-L$ and $-U$ are strictly lower-triangular and strictly upper-triangular parts of $A$, respectively.

A variety of left preconditioners $P$ were proposed, see [3], [4], [5], [6], [12] and references therein. In 1991 A.D. Gunawardena *et al.* [3] proposed the preconditioner $P_s = I + S$, where $S$ is given by

$$S = (s_{i,j}) = \begin{cases} a_{i,i+1}, & i = 1, 2, \cdots, n-1; \\ 0, & \text{otherwise.} \end{cases}$$

In 2004, M. Morimoto *et al.* [6] have further extended the preconditioner $P_s$ as $P_{sm} = I + S + S_m$, where

$$S_m = ((s_m)_{i,j}) = \begin{cases} -a_{i,k_i}, & i = 1, \cdots, n-2,\ j > i+1; \\ 0, & \text{otherwise,} \end{cases}$$

with $k_i = \min\left\{j :\ \max_j |a_{i,j}|, i < n-1,\ j > i+1\right\}$.

Note that the preconditioners $P_s$ and $P_{sm}$ are constructed only by the elements from the upper triangular part of the matrix $A$, the preconditioning effect is not observed on the last row. To provide the preconditioning effect on the last row, many preconditioners are proposed, see for example [12] and references cited therein. Motivated by the same ideals, in this paper we propose the following preconditioner:

$$P_{s\max} = I + S + S_m + R_m, \tag{2}$$

where

$$R_m = ((r_m)_{i,j}) = \begin{cases} -a_{i,j}, & i = n,\ j = k_n; \\ 0, & \text{otherwise,} \end{cases}$$

Zhouji Chen is with the Department of Mathematics, Dingxi Teachers College, Dingxi 743000, China, (e-mail:zhjchentea@163.com).

with $k_n = \min\{j |\ |a_{n,j}| = \max\{|a_{n,l}|, l = 1, \cdots, n-1\}\}$.

In the following, we will discuss the convergence property of modified Gauss-Seidel (MGS) method with preconditioner (2), and then compare such MGS method with the classical Gauss-Seidel method and MGS method with preconditioners $P_{sm}$, respectively.

The remainder of the paper is organized as follows. Next section is the preliminaries. The convergence property of the proposed method and some comparison theorems are studied in Section III. Finally, in Section IV an example is given to confirm the theoretical analysis.

## II. Preliminaries

For the convenience of readers, in this section we give some of the notations, definitions and lemmas, which will be used in the sequel.

For $A = (a_{i,j})$, $B = (b_{i,j}) \in R^{n \times n}$, we write $A \geq B$ if $a_{i,j} \geq b_{i,j}$ holds for all $i, j = 1, 2, \cdots, n$. $A \geq O$, called non-negative, if $a_{i,j} \geq 0$ for all $i, j = 1, 2, \cdots, n$, here and in the sequel, $O$ is used to denote an $n \times n$ zero matrix. For the vectors $a$, $b \in R^n$, $a \geq b$ and $a \geq 0$ can be defined in the similar manner.

**Definition 1.** [11] An $n \times n$ matrix $A$ is a $L$-matrix if $a_{i,i} > 0,\ i = 1, \cdots, n$ and $a_{i,j} \leq 0$ for all $i, j = 1, \cdots, n,\ i \neq j$. A nonsingular $L$-matrix $A$ is a nonsingular $M$-matrix if $A^{-1} \geq O$.

**Lemma 1.** [9] Let $A$ be a non-negative $n \times n$ nonzero matrix. Then

(a). $\rho(A)$, the spectral radius of $A$, is an eigenvalue of $A$;

(b). There exists a positive eigenvector corresponding to $\rho(A)$;

(c). $\rho(A)$ is a simple eigenvalue of $A$;

(d). $\rho(A)$ does not decrease when any entry of $A$ is increased.

**Definition 2.** Let $A$ be a real matrix. Then

$$A = M - N$$

is called a splitting of $A$ if $M$ is a nonsingular matrix. The splitting is called

(a). regular if $M^{-1} \geq O$ and $N \geq O$ [9];

(b). weak regular if $M^{-1} \geq O$ and $M^{-1}N \geq O$ [1];

(c). nonnegative if $M^{-1}N \geq O$ [8];

(d). $M$-splitting if $M$ is a nonsingular $M$-matrix and $N \geq O$ [7].

**Definition 3.** We call $A = M - N$ the Gauss-Seidel splitting of $A$, if $M = I - L$ is nonsingular and $N = U$. In addition, the splitting is called (a). Gauss-Seidel convergent if $\rho(M^{-1}N) < 1$;

(b). Gauss-Seidel regular if $M^{-1} = (I - L)^{-1} \geq O$ and $N = U \geq O$.

**Lemma 2.** [4] Let $A = M - N$ be an $M$-splitting of $A$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is a nonsingular $M$ matrix.

**Lemma 3.** [1] Let $A$ be a nonnegative matrix. Then

(a). If $\alpha x \leq Ax$ for some nonnegative vector $x$, $x \neq 0$, then $\alpha \leq \rho(A)$.

(b). If $Ax \leq \beta x$ for some positive vector $x$, then $\rho(A) \leq \beta$. Moreover, if $A$ is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector $x$, then

$$\alpha \leq \rho(A) \leq \beta$$

and $x$ is a positive vector.

**Lemma 4.** [10] Let $A$ be a nonsingular $M$ matrix, and let $A = M_1 - N_1 = M_2 - N_2$ be two convergent splittings, the first one weak regular and the second one regular. If $M_1^{-1} \geq M_2^{-1}$, then

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1.$$

## III. Convergence property and comparison theorems

In this section, the convergence property theorem of the MGS method with the preconditioner $P_{s\max}$ is given, the effectiveness of the preconditioner $P_{s\max}$ for the MGS method is confirmed by establishing some comparison theorems.

We split the preconditioned matrices $A_s = P_s A$, $A_{sm} = P_{sm}A$ and $A_{s\max} = P_{s\max}A$ as

$$\begin{aligned} A_s &= M_s - N_s, \\ A_{sm} &= M_{sm} - N_{sm}, \\ A_{s\max} &= M_{s\max} - N_{s\max}, \end{aligned}$$

where

$$\begin{aligned} M_s &= I - D - L - E, \\ M_{sm} &= I - D - D' - L - E - E', \\ M_{s\max} &= I - D - D' - L - E - E' \\ &\quad -D'' - E'' + R_m \end{aligned} \tag{3}$$

and

$$\begin{aligned} N_s &= U - S + SU, \\ N_{sm} &= U - S - S_m + SU + S_mU + F', \\ N_{s\max} &= U - S - S_m + SU + S_mU + F'. \end{aligned} \tag{4}$$

In (3) and (4), $D$, $E$ are the diagonal and strictly lower triangular part of $SL$, $D'$, $E'$ and $F'$ are the diagonal, strictly lower and strictly upper triangular parts of $S_mL$, $D''$ and $E''$ are the diagonal and strictly lower triangular parts of $R_m(L + U)$, respectively. Moreover, if we make the assumption (A):

$$\begin{cases} 0 < a_{i,i+1}a_{i+1,i} + a_{i,k_i}a_{k_i,i} < 1, & i = 1, \cdots, n-2; \\ 0 < a_{i,i+1}a_{i+1,i} < 1, & i = n-1; \\ 0 < a_{n,k_n}a_{k_n,n} < 1, \end{cases}$$

then $M_s$ and $M_{sm}$ are nonsingular [6]. It is easy to see that the matrix $M_{s\max}$ is nonsingular, and the MGS iteration matrix $T_{s\max} = M_{s\max}^{-1}N_{s\max}$ for $A_{s\max}$ is well defined.

### A. Convergence property

In what follows, we show that the splitting $A_{s\max} = M_{s\max} - N_{s\max}$ is the regular and Gauss-Seidel convergent splitting when $A$ is an $M$-matrix.

**Theorem 1.** Let $A$ be a nonsingular $M$-matrix with unit diagonal elements. If the assumptions (A) is satisfied, then $A_{s\max} = M_{s\max} - N_{s\max}$ is the regular and Gauss-Seidel convergent splitting.

**Proof.** Note that the assumption (A) is satisfied, the diagonal elements of $A_{s\max}$ are positive and $M_{s\max}^{-1}$ is well defined. It is known that (see [1]) an $L$ matrix $A$ is a nonsingular $M$ matrix if and only if there exist a positive vector $y$ such that $Ay > 0$. By taking such $y$, the fact that $I + S + S_m + R_m \geq O$ implies $A_{s\max}y = (I + S + S_m + R_m)Ay > 0$. Consequently, the $L$-matrix $A_{s\max}$ is a nonsingular $M$-matrix, which means that $A_{s\max}^{-1} \geq 0$.

From $L \geq R_{\max} \geq O$, we known that $L + E + E' + E'' \geq O$. Under the assumption (A), we have $D + D' + D'' < I$, so that $(I - D - D' - D'') \geq O$. Hence

$$\begin{aligned} & M_{s\max}^{-1} \\ = \; & [(I - D - D' - D'') - (L + E + E' + E'')]^{-1} \\ = \; & [I - (I - D - D' - D'')^{-1}(L + E + E' + E'')]^{-1} \\ & \cdot (I - D - D' - D'')^{-1} \\ = \; & \{I + (I - D - D' - D'')^{-1}(L + E + E' + E'') \\ & + [(I - D - D' - D'')^{-1}(L + E + E' + E'')]^2 + \cdots \\ & + [(I - D - D' - D'')^{-1}(L + E + E' + E'')]^{n-1}\} \\ & \cdot (I - D - D' - D'')^{-1} \\ \geq \; & O, \end{aligned}$$

here we use the fact that for the strictly lower triangular matrix $L + E + E' + E''$, the equality $(L + E + E' + E'')^n = O$ holds.

On the other hand, it is easy to see that $N_{s\max} = U - S - S_m + SU + S_mU + F' \geq O$ since $U \geq S - S_m$ and $SU + S_mU + F' \geq O$. Therefore, it follows from Definition 2, 3 and Lemma 2 that $A_{s\max} = M_{s\max} - N_{s\max}$ is the regular and Gauss-Seidel convergent splitting.

**Remark 1.** Similar to the Theorem 1, under the assumption (A), one can easily get that $A_{sm} = M_{sm} - N_{sm}$ is the regular and Gauss-Seidel convergent splitting when $A$ is a nonsingular $M$-matrix with unit diagonal elements.

### B. Comparison theorems

In this subsection, by establishing some comparison theorems, we confirm that the MGS method with the preconditioner $P_{s\max}$ is superior to the classical Gauss-Seidel method and the MGS methods with the preconditioners $P_{sm}$.

Comparing $\rho(T)$ with $\rho(T_{s\max})$, we have the following comparison theorem:

**Theorem 2.** Let $A$ be a nonsingular $M$-matrix with unit diagonal elements. Then under the assumption (A), we have

$$\rho(T_{s\max}) \leq \rho(T) < 1.$$

**Proof.** From Theorem 1, we know that $A_{s\max} = P_{s\max}A = M_{s\max} - N_{s\max}$ is the Gauss-Seidel convergent splitting, i.e., $\rho(T_{s\max}) < 1$. Since $A$ is a nonsingular $M$-matrix, the classic Gauss-Seidel splitting $A = (I - L) - U$ of $A$ is clearly regular and convergent, that is to say $\rho(T) < 1$. Therefore, in what follows, we only need to show the inequality $\rho(T_{s\max}) \leq \rho(T)$ holds.

Firstly, consider the following splitting of $A$

$$\begin{aligned} A &= (I + S + S_m + R_m)^{-1} M_{s\max} \\ &\quad -(I + S + S_m + R_m)^{-1} N_{s\max}. \end{aligned}$$

If we let $M_1 = (I + S + S_m + R_m)^{-1} M_{s\max}$ and $N_1 = (I + S + S_m + R_m)^{-1} N_{s\max}$, then we have $T_{s\max} = M_1^{-1} N_1$.

Secondly, note that

$$\begin{aligned} & M_1^{-1} \\ = \;& (I - D - D' - L - E - E' - D'' - E'')^{-1} \\ & \cdot (I + S + S_m + R_m) \\ \geq \;& (I - D - D' - L - E - E' - D'' - E'')^{-1} \\ = \;& [I - (I - D - D' - D'')^{-1}(L + E + E' + E'')]^{-1} \\ & \cdot (I - D - D' - D'')^{-1} \\ \geq \;& [I - (I - D - D' - D'')^{-1}(L + E + E' + E'')]^{-1} \\ \geq \;& (I - L)^{-1}, \end{aligned}$$

$N_{s\max} \geq O$ and $U \geq O$, then it follows from Lemma 4 that $\rho(M_1^{-1} N_1) \leq \rho((I - L)^{-1} U)$, i.e., $\rho(T_{s\max}) \leq \rho(T)$.

Hence $\rho(T_{s\max}) \leq \rho(T) < 1$.

Next we compare $P_{s\max}$ with $P_{sm}$.

**Theorem 3.** Let $A$ be a nonsingular $M$-matrix with unit diagonal. Then under the assumption (A), we have

$$\rho(T_{s\max}) \leq \rho(T_{sm}) < 1.$$

**Proof.** From Theorem 1 and Remark 1, we can see that $\rho(T_{s\max}) < 1$ and $\rho(T_{sm}) < 1$. Now we are in the position to prove the following inequality

$$\rho(T_{s\max}) \leq \rho(T_{sm}).$$

As $A$ is a nonsingular $M$-matrix, then under the assumption of theorem, we known that there exists a positive eigenvector $x$ such that $T_{sm}x = \rho(T_{sm})x$ and $\rho(T_{sm}) > 0$. It follows from (3) and (4) that $N_{s\max} = N_{sm}$ and $M_{s\max} - M_{sm} = R_m A$, thus

$$M_{sm}^{-1} - M_{s\max}^{-1} = M_{s\max}^{-1} R_m A M_{sm}^{-1}. \tag{5}$$

From (5), one obtains

$$T_{sm} - T_{s\max} = M_{s\max}^{-1} R_m A T_{sm}. \tag{6}$$

Multiplying by $x$ on both sides of (6) gives

$$\rho(T_{sm})x - T_{s\max}x = \rho(T_{sm}) M_{s\max}^{-1} R_m A x,$$

Since $\rho(T_{sm}) > 0$, one can obtain that (cf. [5])

$$Ax \geq 0.$$

Hence,

$$T_{s\max}x \leq \rho(T_{sm})x,$$

it follows from Lemma 3 that

$$\rho(T_{s\max}) \leq \rho(T_{sm}).$$

The proof is completed.

**Remark 2.** From Theorem 2 and 3, we know that the spectral radius of the MGS iteration matrix with preconditioner $P_{s\max}$ is smaller than that of the classical Gauss-Seidel iteration matrix and MGS iteration matrices with preconditioners $P_{sm}$ under some conditions. This confirm that the preconditioner $P_{s\max}$ presented in this paper really improves the spectral radius of the Gauss-Seidel method..

## IV. Example

In this part, we give an example to illustrate the theory developed in Section 3.

**Example 1.** When the central difference scheme on a uniform grid with $N \times N$ interior nodes ($N^2 = n$) is applied to the discretization of the two-dimension convection-diffusion equation

$$-\triangle u + \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = f$$

in the unit squire $\Omega$ with Dirichlet boundary conditions, we obtain a system of linear equations (1) of the coefficient matrix

$$A = I \otimes C + D \otimes I,$$

where $\otimes$ denotes the Kronecker product,

$$C = \text{tridiag}(-\frac{2+h}{8},\ 1,\ -\frac{2-h}{8})$$

and

$$D = \text{tridiag}(-\frac{1+h}{4},\ 0,\ -\frac{1-h}{8})$$

are $N \times N$ tridiagonal matrices, and the step size is $h = \frac{1}{N}$.

The spectral radii of the classical Gauss-Seidel (i.e., $P = I$) iteration matrix and the MGS iteration matrices with the preconditioners $P_s$, $P_{sm}$ and $P_{s\max}$ for different problem size $n$ are listed in Table I.

TABLE I
COMPARISON OF THE SPECTRAL RADII FOR EXAMPLE 1

| | n=16 | n=64 | n=256 | n=1024 |
|---|---|---|---|---|
| $P = I$ | 0.6288 | 0.8744 | 0.9639 | 0.9904 |
| $P = P_s$ | 0.4676 | 0.8002 | 0.9406 | 0.9840 |
| $P = P_{sm}$ | 0.3363 | 0.7255 | 0.9155 | 0.9770 |
| $P = P_{s\max}$ | 0.3354 | 0.7252 | 0.9154 | 0.9769 |

From Table I, the numerical results for Example 1, we can see that $\rho(T_{s\max}) < 1$ for all cases. Moreover, the spectral radus of the MGS iteration matrix with the preconditioner $P_{s\max}$, i.e., $\rho(T_{s\max})$, is the smallest one among $\rho(T)$, $\rho(T_s)$, $\rho(T_{sm})$ and $\rho(T_{s\max})$. The results verify our theoretical analysis in Section 3.

## References

[1] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.
[2] K. Chen, Matrix Preconditioning Techniques and Applications, Cambridge University Press, Cambridge, 2005.
[3] A.D. Gunawardena, S.K. Jain, L. Snyder, Modified iterative methods for consistent linear systems, *Linear Algebra Appl.* 1991; **154–156**:123–143.
[4] W. Li, Comparison results for solving preconditioned linear systems, *J. Comput. Appl. Math.* 2005; **176**:319–329.

[5] W. Li, A note on the preconditioned Gauss-Seidel (GS) method for linear systems, *J. Comput. Appl. Math.* 2005; **182**:81–90.
[6] M. Morimoto, K. Harada, M. Sakakihara, H. Sawami, The Gauss-Seidel iterative method with the preconditioning matrix $(I + S + S_m)$, *Japan J. Indust. Appl. Math.* 2004; **21**:25–34.
[7] H. Schneider, Theorems on M-splittings of a singular M-matrix which depend on graph structure, *Linear Algebra Appl.* 1984; **58**:407–424.
[8] Y.Z. Song, Comparisons of nonnegative splttings of matrices, *Linear Algebra Appl.* 1991; **154–156**: 433–455.
[9] R.S. Varga, Matrix Iterative Analysis, 2nd edition, Springer, 2000.
[10] Z.I. Woźniki, Nonnegative splitting theory, *Japan J. Industrial Appl. Math.* 1994; **11**: 289–342.
[11] D.M. Young, Iterative solution of large linear systems, Academic Press, New York, 1971.
[12] B. Zheng, S.-X. Miao, Two new modified Gauss-Seidel methods for linear system with M-matrices, *J. Comput. Appl. Math.* 2009; **233**: 922–930.