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Control of Pendulum on a Cart with State Dependent Riccati Equations

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***Abstract*—**State Dependent Riccati Equation (SDRE) approach is a modification of the well studied LQR method. It has the capability of being applied to control nonlinear systems. In this paper the technique has been applied to control the single inverted pendulum (SIP) which represents a rich class of nonlinear underactuated systems. SIP modeling is based on Euler-Lagrange equations. A procedure is developed for judicious selection of weighting parameters and constraint handling. The controller designed by SDRE technique here gives better results than existing controllers designed by energy based techniques.

***Keywords*—**State Dependent Riccati Equation (SDRE), Single Inverted Pendulum (SIP), Linear Quadratic Regulator (LQR).

I. Introduction

THE simple inverted pendulum (SIP) stabilization and tracking is a problem for which a number of nonlinear design methods have been used. The problem remains that some of these techniques are too complex and may not provide simple solutions to controlling transients of the system. The main purpose of this paper is to use ideas from time tested techniques of a classical method like LQR [1], [2] and build on them to obtain a control strategy for nonlinear systems that satisfies practical constraints on the system.

In this paper, nonlinear stabilization problem is addressed as the stabilization of SIP by minimizing an accumulative cost functional quadratic in states and controls and then it is extended to give solution for cart position tracking problem. For linear systems, this leads to linear feedback control, which is found by solving a Riccati equation, and thus referred to as linear quadratic regulator (LQR). But SIP is a nonlinear system with underactuation degree one and its linearization is far from adequate for control design purposes. The technique used here involves manipulating the system dynamic equations into a pseudo-linear state-dependent coefficient (SDC) form, in which system matrices are given explicitly as a function of the current state. Treating the system matrices as constant, the approximate solution of the nonlinear state-dependent Riccati equation is obtained for the reformulated pseudo-linear dynamical system in discrete time steps. The solution is then used to calculate a feedback control law that is optimized around the system state estimated at each time step. This technique, referred to as State-Dependent Riccati Equation (SDRE) [3] control, is thus an extension to the LQR as it solves the LQR problem at each time step. The SDRE technique is a systematic way of designing nonlinear feedback controllers which approximates the solution of the infinite horizon optimal control and has the capability of being implemented in real-time for a broad class of problems [4], [5].

The rest of the paper is organized in the following way: In section II, Lagrangian modeling of SIP is done. Then, in section III, SDRE technique is explained briefly and corresponding SDC formulation is shown. The procedure for selection of weighting matrices is also explained in this section. It is followed by simulations and results in section IV. Lastly, conclusions and future scope of study are given in section V.

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II. Modeling

A. Single Inverted Pendulum Modeling

The model of single inverted pendulum is based on [6].

$$ml^2 \ddot{x}_1 + ml\cos x_1 \ddot{x}_2 - mgl\sin x_1 = 0$$
$$(M+m)\ddot{x}_2 + ml\cos x_1 \ddot{x}_1 - ml\sin x_1 \dot{x}_1^2 = u \quad (1)$$

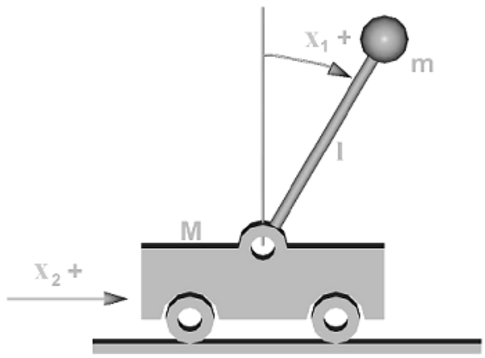

Fig. 1 Planar Pendulum on a cart

The Euler-Lagrange equation is of the form

$$D(x)\ddot{x} + C(x,\dot{x})\dot{x} + G(x) = Hu \quad (2)$$

where all these matrices have the values as

$$D = \begin{bmatrix} ml^2 & ml\cos x_1 \\ ml\cos x_1 & M+m \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$$C = \begin{bmatrix} 0 & 0 \\ -ml\sin x_1 \dot{x}_1 & 0 \end{bmatrix}, G = \begin{bmatrix} mgl\sin x_1 \\ 0 \end{bmatrix}$$

$$G_{SD} = \begin{bmatrix} \frac{mgl\sin(x_1)}{x_1} & 0 \\ 0 & 0 \end{bmatrix}$$

The last matrix is somewhat like G and is useful when SDC form is obtained

B. A Slightly Different Way of Modeling of Input

In [6], the form of input is changed, based on the work given in [7] by exploiting the fact that the relation between input and $\ddot{x}_2$ is invertible. So the new form is

$$\ddot{x}_1 = a\sin x_1 - b\cos x_1 u$$
$$\ddot{x}_2 = u \tag{3}$$

It gives the equation (2) matrices as

$$G = \begin{bmatrix} \sin(x_1) \\ 0 \end{bmatrix}, G_{SD} = \begin{bmatrix} \frac{\sin(x_1)}{x_1} & 0 \\ 0 & 0 \end{bmatrix}$$

D is identity matrix and C is zero matrix.

This form is given so that the results can be compared with those obtained in [6].

III. State Dependent Riccati Equation Method

A. The Technique

The SDRE approach involves manipulating the dynamic equations

$$\dot{x} = f\ (x,\ u) \tag{4}$$

into a pseudo-linear state-dependent coefficient (SDC) form in which system matrices are explicit functions of the current state:

$$\dot{x} = A(x)x + B(x)u \tag{5}$$

A standard LQR problem (Riccati equation) can then be solved at each time step to design the state feedback control law on-line. For digital implementation, above equation is approximately discretized at each time step into

$$x_{k+1} = \phi(x_k)x_k + \Gamma(x_k)u_k \tag{6}$$

And the SDRE regulator is then specified similar to LQR in discrete form.

$$u_k = -R^{-1}\Gamma^T(x_k)P(x_k)x_k = -K(x_k)x_k \tag{7}$$

where $P(x_k)$ is the steady state solution of is the steady state solution of the difference Riccati equation, obtained by solving the discrete-time algebraic Riccati equation

$$\phi^T[P - P\Gamma(R + \Gamma^T P\Gamma)^{-1}\Gamma^T P]\phi - P + Q = 0 \tag{8}$$

using state-dependent matrices $\phi(x_k)$ and $\Gamma(x_k)$, which are treated as being constant at each time step. Here Q is positive semi-definite matrix and R is positive definite matrix. Both have only real values as their members.

The approach can be seen as nonlinear extension of LQR. For tracking the reference signal r, input is taken as

$$u_k = -R^{-1}\Gamma^T(x_k)P(x_k)(x_k - r) = -K(x_k)(x_k - r) \tag{9}$$

B. SDC Formulation for use in SDRE Technique

The State Dependent Coefficient (SDC) form that can be obtained from matrices of Euler Lagrange equation is as given below

$$\dot{x} = \begin{pmatrix} 0 & I \\ -D^{-1}G_{SD} & -D^{-1}C \end{pmatrix} x + \begin{pmatrix} 0 \\ D^{-1}H \end{pmatrix} u \tag{10}$$

It helps to bring the dynamic equation to the form of equation (5), required to apply the SDRE technique [5].

C. Choosing Q and R

The choice of Q and R is very crucial to the stabilization and performance of the system. For simplicity we will restrict ourselves to using only constant values for entries to these matrices, even though results of [4] suggest that taking exponential as values of entries may improve the results. These matrices are taken as diagonal matrices to further simplify our selection. The tuning of these matrices is mostly as in LQR [1], [2] but with an added consideration especially valid in case of underactuated systems. The method of tuning these matrices, as followed during simulation is as given below:

1) For values in Q: The larger the values in Q, the larger the gain matrix, more the control input, the faster the time taken to reduce perturbations. So to enforce some constraint on a state, the corresponding entry in Q should be altered. The overshoot and settling time trade-offs have to be done here.

2) For values in R: Increase in values in R causes decrease in values of feedback gain. So sluggishness increases. But it can be used to out advantage. For example, in case of square wave there are sudden changes of state that may increase the control input beyond limits. To prevent this from happening it is better to make the system a bit sluggish by increasing R values. Overshoot can be controlled using R also.

3) For underactuated signals: In case of underactuated systems, penalizing the states (by increasing corresponding values in Q) that are not fully actuated will increase the control effort but its effect on the state will be very less. So preference should be given to control through fully actuated states.

The strategy used for tuning these matrices during simulations was to design the controller for the worst case scenario (e.g. while tracking square wave, where there are abrupt changes in states that may cause maximum control input limit violation). Then the desired waveform was given as reference, and performance improvement was done first through actuated states and then with underactuated states, till the limit (in our case maximum control input, or maximum cart velocity or track length) is again reached. Damping of velocity terms may require penalizing velocity in cost functional terms also.

IV. Results and Simulations

The technique was tested by simulation using MATLAB. The technique was tested in comparison with IDA-PBC method to find relative merits and demerits. Then the practical constraint handling capability of the technique was investigated.

A. Comparison with IDA-PBC Designed Controllers

The technique was tested with the data given in [6]. The model used is as given in section II (*B*). Full state feedback was considered. Parameters *a* and *b* were both taken to be one. Initial conditions were taken as

Cart displacement=-0.1 m, Pendulum angle=π/2-0.2 rad

Cart velocity=0, Pendulum angular velocity=0.1 rad/s

The objective of control in this problem is to regulate the cart position at a displacement of 20 metres measured from the zero position and stabilize the system there (far away from the initial displacement).

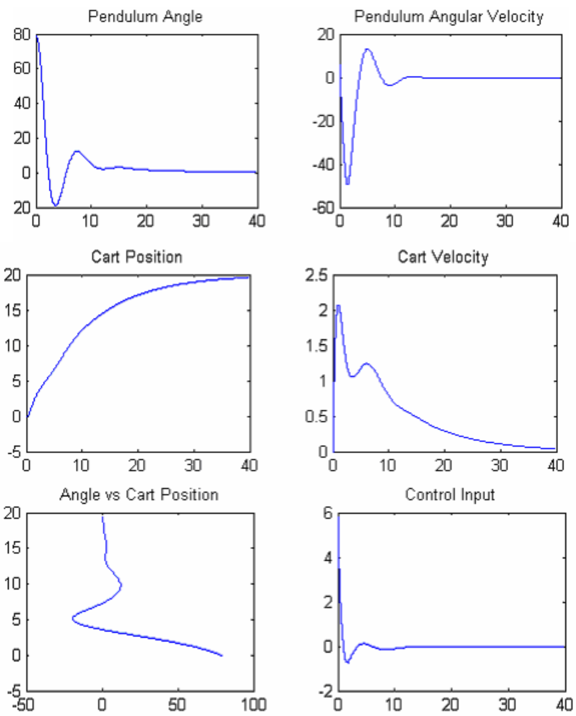

Fig. 2 Results for Problem explained in Section IV *A* when simulation for is run for 40 seconds. (Angles displayed in degrees, time in seconds and displacement in metres)

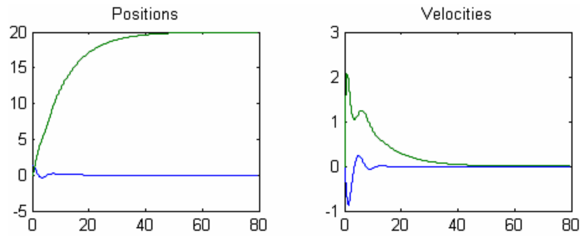

Fig. 3 Results for Problem explained in Section IV *A* when simulation for is run for 80 seconds to show steady state conditions. (Angles displayed in radians, distance in metres and time in seconds)

The selection of *Q* and *R* in this case is not very rigorous but still the results obtained are better than those provided in [6]. Mainly *Q* was selected by penalizing each state to get the desired effect and *R* was used to bring control input down. The criterion described in section III (*C*) (*3*) was not used as control effort is not a constraining factor here. Even though the sluggishness is increased by increase in *R* the results given in [6] never challenged the results obtained by SDRE. The results obtained are given in Figs. 2 and 3. In comparison with results in [6], for the same problem we can see that the overshoots are much less than those obtained by design though IDA-PBC. Overshoot in case of cart position is particularly high in [6]. Still the system controlled by SDRE technique settles down much earlier. Clearly, the control effort required is much less here in comparison.

The performance of the controller obtained by SDRE techniques is improved as it is easier to tune *Q* and *R* matrices as per requirement than the implementation of energy shaping concepts.

B. Constraint Handling Capabilities of SDRE

For this example, data from Quanser Consulting's IP02 setup was used, which is as given below

Mass of Cart = 0.94kg, Mass of Pendulum = 0.230kg, Pendulum length= 0.6096m

Instead of giving results having absurdly high values of parameters just to show large region of attraction we have tried to provide results realizable using the above setup and so have incorporated stringent input, cart velocity and track length constraints.

The constraints imposed are

Track Length= 0.914m, Maximum Cart Velocity= 3.989m/s, Maximum Input= 0.6294N.

The last constraint is very stringent. The objective in this problem is to stabilize the system while cart position tracks a reference signal. An angular displacement of 20 degrees (in the direction adverse to tracking) was given to pendulum initially.

The model used is given in section II (*A*). The process of tuning *Q* and *R* is demonstrated in this section. For simplicity only the entries in *Q* matrix corresponding to position terms were used. Values of *Q*=diagonal(7, 10, 0, 0), and *R*=5 were used in case of all the reference signals first and then these were changed to improve the tracking. These are very conservative values and have to be kept small so that the control in-

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put remains within limits. The results may seem to be sluggish but it is only as Q and R were designed here to keep control input within limits in the worst case of sudden change in reference waveform (i.e. square wave) and applied to continuous waveforms (like sinusoidal wave) also. Relaxation of this control input constraint improves lag times considerably.

The tracking of Square wave with above values is shown in Fig. 4. In all the figures used for this example, angles are displayed in degrees and other quantities in SI units. Fig. 5 displays the tracking of Sinusoidal wave for the same values of Q and R as used for square wave reference signal. Here the control input is well within limits, so the performance can be improved by using higher values of Q as diagonal (10, 45, 0, 0). In addition the waveform is continuous so the sluggishness introduced in earlier case can be removed by making R=2.

To check the robustness of the controllers, the least robust controller designed here was picked. The controller was calculated for original mass for sinusoidal wave and applied to the system whose parameters have been perturbed. Robustness was checked by with respect all constraints. If we observe robustness using the loss of tracking and stability as the criteria for failure and neglect constraints, the magnitude of perturbations that can be handled were be much higher.

To check the robustness for change of cart mass we decreased its mass to 5% of its original value. It should also

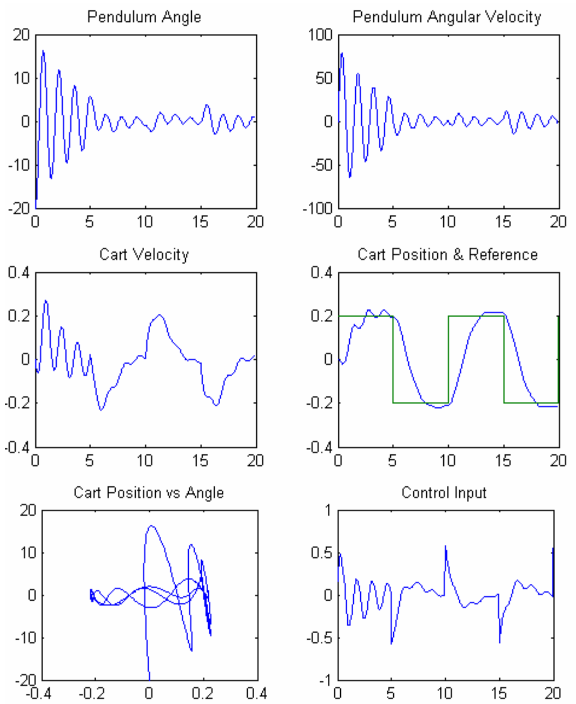

Fig. 4 Tracking of a Square Wave using Q=diagonal(7, 10, 0, 0), and R=5

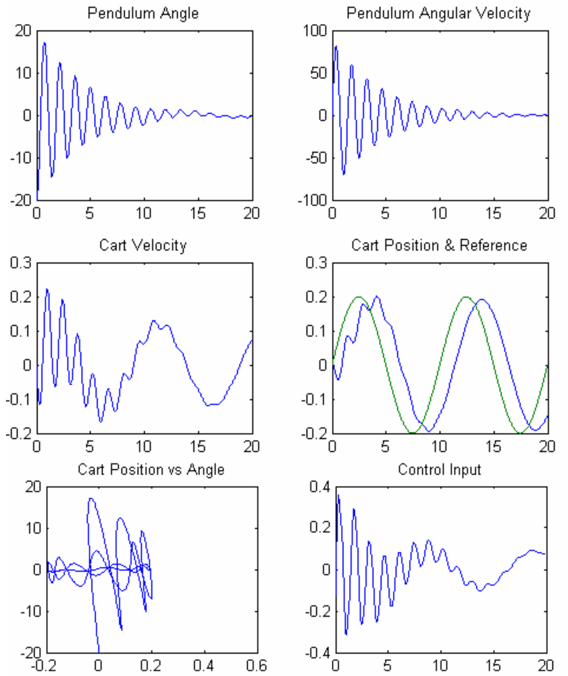

Fig. 5 Tracking of a Sinusoidal Wave using Q=diagonal (7, 10, 0, 0), and R=5

be noted that the controller used is most sensitive to perturbations due to the low value of R. The system is stable and tracks well even with this perturbation. In some respects the perturbations improved the performance. That is due to the fact that Q and R were not tuned to perfection as we just concentrated on weights corresponding to position terms. Increasing the weight

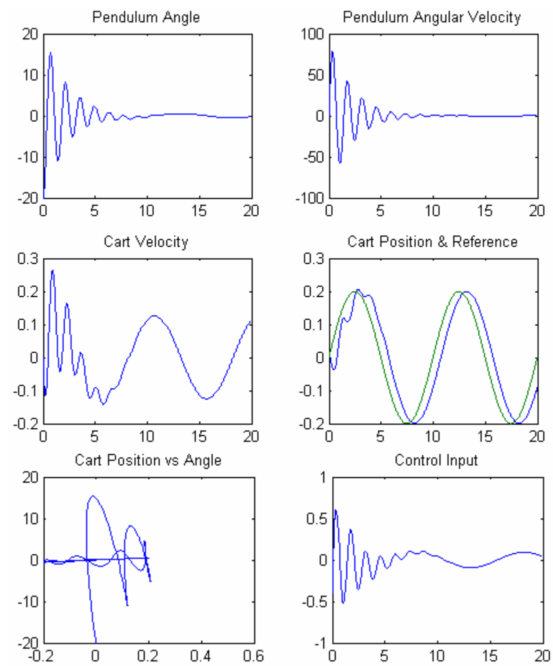

Fig. 6 Tracking of a Sinusoidal Wave using Q=diagonal (10, 45, 0, 0), and R=2

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of cart has a more dramatic effect as we are working near the limits of maximum input. Doubling the weight will increase oscillations and control input requirements as affects tracking. But system still remains stable. Increasing the weight of pendulum by about 10% causes violation of maximum input constraint. But the tracking is alright even for higher values of perturbations, though oscillations are somewhat increased. For the decrease in pendulum mass (to even less than 5% of the original), the system remains stable and control input also remains bounded within limits, though the pendulum may keep on oscillating and control effort increases. With increase in pendulum length by 15% it is observed that the maximum input constraint is violated. For 10% increase in length, the cart velocity increase and control effort increase is noticeable. Shrinking the pendulum length (even to 10% of the original) does not cause any limit violation and stabilization. But oscillations increase.

The perturbations that are permissible are somewhat lesser in case of the controller designed to track continuous signals like sine wave. For example, if we use controller designed for saw tooth waveform tracking and apply it for tracking sine wave, we can increase pendulum length by 40% because of more sluggishness.

After that, all the constraints were relaxed, just to check the working of the controller for increased amplitude of the reference signal and harsher initial conditions. R=1, same Q as above were used. Initial angular displacement of 89 degrees adverse to tracking was given. The graphs were obtained as in Fig. 7. No special tuning was done for this case. Still the results are good.

The introduction of integral control is one option that can be used to reduce steady state errors, but it is not required here.

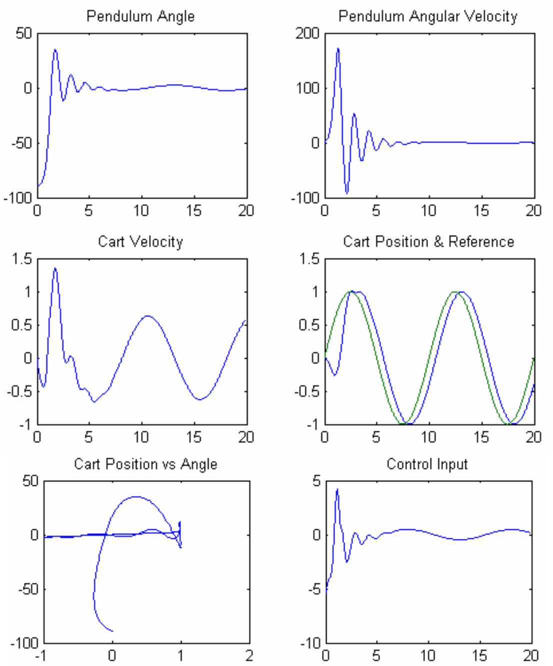

Fig. 7 Tracking of a Sinusoidal Wave using Q=diagonal (10, 45, 0, 0), and R=1 when all the constraints are removed

V. Conclusion

This paper demonstrates the systematic procedure that can be used for designing SDRE based controllers for underactuated class of systems. Apart from showing better results than energy based controllers this paper illustrates that these controllers can be designed within physical constraints imposed on the system. The constraint handling and transient condition control have been incorporated in the design procedure.

The system response can be further improved by using exponential entries for Q matrix instead of constant matrix. The future scope of study in this field is in the direction of using the geometric structure of the systems in the design of these controllers and developing this theory by drawing parallels with energy based control techniques like in [8] so as to get better system understanding.

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