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Control of A Cart-Ball System Using State-Feedback Controller

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Abstract**—**A cart-ball system is a challenging system from the control engineering point of view. This is due to the nonlinearities, multivariable, and non-minimum phase behavior present in this system. This paper is concerned with the problem of modeling and control of such system. The objective of control strategy is to place the cart at a desired position while balancing the ball on the top of the arc-shaped track fixed on the cart. A State-Feedback Controller (SFC) with a pole-placement method will be designed in order to control the system. At first, the mathematical model of a cart-ball system in the state-space form is developed. Then, the linearization of a model will be established in order to design a SFC. The integral control strategy will be performed as to control the cart position of a system. Simulation work is then performed using MATLAB/SIMULINK software in order to study the performance of SFC when applied to the system.
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Keywords**—**Cart-Ball System, Integral Control, Pole-Placement Method, State-Feedback Controller.

I. Introduction

A CART-BALL SYSTEM is basically an inverted pendulum problem, much used as a benchmark problem. The control objective is to balance the ball on the top of the arc and at the same time place the cart at the desired position. This kind of problem has a long history with various approaches have been tried [1-3]. The system consists of a cart and a ball on top of it. The cart is free to move to the left or right on a straight bounded track and the ball can swing in the vertical plane. The purpose of this paper is to present the performance of the State-Feedback Controller (SFC) using pole placement method when applied to the system in terms of steady-state error, overshoot percentage, disturbance rejection and etc.

II. Mathematical Modelling

Fig. 1 shows the free body diagram of a cart-ball system. It can be seen that the disturbance, F_W is applied horizontally to the ball. It is assumed that the direction towards the right is considered to be positive and direction to the left is considered negative. It is also assumed that all frictions forces are negligible and thus neglected.

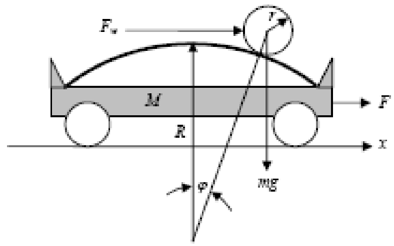

Fig 1.: The free body diagram of the system

Referring to Fig. 1, the force balance in the x-direction gives the mass times acceleration of the cart plus the mass time of the x-direction acceleration of the point mass must equal to the external force of the system. Thus, it can be written as,

$$M\frac{d^2x}{dt^2} + m\frac{d^2}{dt^2}[y + (R + r)\sin\varphi = F + F_w \quad (1)$$

Equation (1) is expended as follows,

$$(M+m)\ddot{y} - m(R+r)(\sin\varphi)\dot{\varphi}^2 + m(R+r)(\cos\varphi)\ddot{\varphi} = F + F_w \quad (2)$$

The torque in the clockwise direction caused by the horizontal wind disturbance is given by,

$$F_w\cos\varphi(R + r) = Td \quad (3)$$

Hence, the torque balance equation becomes

$$F_x\cos\varphi(R + r) - (F_y\sin\varphi)(R + r) = mg\sin\varphi(R + r) + F_w\cos\varphi(R + r) \quad (4)$$

Equation (4) can be re-arranged as follows:

$$m\ddot{y}\cos\varphi + (R+r)m\ddot{\varphi} = mg\sin\varphi + F_w\cos\varphi \quad (5)$$

Whereby, (2) and (5) represent the dynamic of the system which include the horizontal wind disturbance, F_w. These equations can be represented in the state-space form by executing the following steps:

i. Solve the torque balance expression for $m(R + r)\ddot{\varphi}$ and place into the force balance equation, giving,

$$m(R+r)\ddot{\varphi} = mg\sin\varphi + F_w\cos\varphi - m\ddot{y}\cos\varphi \quad (6)$$

and

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$$[M + m - m\cos^2\varphi]\ddot{y} = F + m(R + r)\sin\varphi\cos\varphi + F_w \sin^2\varphi \quad (7)$$

ii. Solve the toque balance equation for $\ddot{x}$ and put into the force balance equation, producing

$$\ddot{x} = \frac{mg\sin\varphi + F_w\cos\varphi - m(R + r)\ddot{\varphi}}{m\cos\varphi} \quad (8)$$

And

$$(M + m)\frac{mg\sin\varphi + F_w\cos\varphi - m(R + r)\ddot{\varphi}}{m\cos\varphi} - m(R + r)\sin\varphi\dot{\varphi}^2 + m(R + r)\cos\varphi\ddot{\varphi} = F - F_w \quad (9)$$

Multiplying both side by $\cos\varphi$, gives

$$[m(R+r)\cos^2\varphi - (M+m)(R+r)]\ddot{\varphi} = F\cos\varphi - (M+m)g\sin\varphi + m(R+r)\cos\varphi\sin\varphi\dot{\varphi}^2 - \left(\frac{M+m}{m}\right)F_w\cos\varphi + F_w\cos\varphi \quad (10)$$

iii. Define the state vector x as

$$x_1 = \varphi; x_2 = \dot{\varphi}; x_3 = x; x_4 = \dot{x} \quad (11)$$

By using (8), (10) and (11) the state-space equation can be written as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{F\cos x_1 - (M+m)g\sin x_1 + m(R+r)(\cos x_1 \sin x_1)x_2^2 - \frac{M}{m}F_w\cos x_1}{m(R+r)\cos^2 x_1 - (M+m)(R+r)}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{F + m(R+r)(\sin x_1)x_2^2 - mg\cos x_1\sin x_1 + F_w\sin^2 x_1}{M + m - m\cos^2 x_1} \quad (12)$$

Equation (12) represents the mathematical model of a cart-ball system with disturbance and it is highly nonlinear and will be used in performing the simulation in this work.

III. LINEARIZATION

As the requirement of the Pole Placement method which must use a linearized model in designing the controller, thus the linearization should be accomplished to convert from nonlinear model of (12) to linear model. Thus, to linearized (12), the following approximations are to be considered [3],

$$\cos\varphi \cong 1, \sin\varphi \cong \varphi, \cos^2\varphi \cong 1, \sin^2\varphi \cong 0 \quad (13)$$

Now, equations (6) and (10) becomes

$$M\ddot{y} = F - mg\varphi$$

$$M(R + r)\ddot{\varphi} = F - (M + m)g\varphi - \left(\frac{M + m}{m}F_w\right) + F_w \quad (14)$$

In the matrix form it can be written as,

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ a_2 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix} F + \begin{bmatrix} 0 \\ d \\ 0 \\ 0 \end{bmatrix} F_w \quad (15)$$

where,

$$a_1 = \frac{M + m}{M(R + r)}$$

$$a_2 = -Mg/M$$

$$b_1 = -1/M(R + r)$$

$$b_2 = 1/M$$

$$d = -1/(m(R + r))$$

Equation (15) is a linearized model of a cart-ball system. Using the physical data tabulated in Table 1, then, (15) can be rewritten as follows,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.55 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.98 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ -1.91 \\ 0 \\ 1 \end{bmatrix} F + \begin{bmatrix} 0 \\ -5.25 \\ 0 \\ 0 \end{bmatrix} F_w \quad (16)$$

Table 1: Physical data of a cart-ball system

Parameter	Symbol	Rating
Cart radius of the arc	R	0.50m
Cart weight	M	1.0kg
Cart position	y	-
Cart driving force	F	-
Ball rolling radius	r	0.025m
Ball angular deviation	$\varphi = \theta$	Max 0.60 rad
Ball weight	m	0.675kg
Ball horizontal force	H	-
Ball vertical force	V	-
Gravity	g	$9.81 ms^{-2}$

IV. STATE-FEEDBACK CONTROLLER DESIGN

This part presents the SFC design using the pole placement method. Through this method, all closed loop poles can be chosen to be at stable location to guarantee the stability of the system. The general closed loop state equation using the state-feedback controller is given by

$$\dot{x}(t) = (A - BK)x(t) + Br(t)$$
$$y(t) = Cx(t) \quad (17)$$

Suppose the SFC is to be designed such that the cart can be moved anywhere along the track with overshoot of 20% and settling time of 5 (s). This specification can be achieved by placing the dominant poles at $s_{1,2} = -0.8 \pm j1.5617$. The other two poles are chosen at s_3=-8 and s_4 = -10. These value are 10 times greater than the dominant poles, s_1 and s_2, therefore the system will not be affected by the extra poles [4].

In this work, the SFC is applied to the system with and without the integral control. First, the simulation is performed without the integral control, and the calculated value of gains, *K* are found to be at,

$k_1 = -76.444; k_2 = -15.449; k_3 = -13.195; k_4 = -9.826$ The steady-state error is calculated using (18)

$$e(\infty) = 1 + C(A - BK)^{-1} = 1.0758m \quad (18)$$

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Clearly seen from (18), the steady-state error is considerably big, means that the cart will does not stop at desired position. In order to eliminate the steady-state error in this system, the Integral Control is introduced. The general state-space representation for the SFC with Integral Control is [4],

$$\begin{bmatrix}\dot{x}(t)\\ \dot{x}_N(t)\end{bmatrix}=\begin{bmatrix}(A-BK) & BK_e\\ -C & 0\end{bmatrix}\begin{bmatrix}x(t)\\ x_N(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}r(t)$$
$$y(t)=\begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x(t)\\ x_N(t)\end{bmatrix} \qquad (19)$$

After manipulating (19) and use the data in Table 1, yield,

$$\dot{x}=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 20.53 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -0.98 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\end{bmatrix}x+\begin{bmatrix}0\\ -1.91\\ 0\\ 1\\ 0\end{bmatrix}F+\begin{bmatrix}0\\ -5.25\\ 0\\ 0\\ 0\end{bmatrix}f_w+\begin{bmatrix}0\\ 0\\ 0\\ 0\\ 1\end{bmatrix}r(t)$$
$$y=\begin{bmatrix}0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix}x \qquad (20)$$

Since the Integral Control increase the system type, therefore an extra pole is needed and has been selected at s_5=-15 to ensure no interruption to the system occurs. Then, the characteristic equation becomes,

$$s^5+34.60s^4+405.879s^3+1861.612s^2+2997.65s+3694.8=0 \qquad (21)$$

The corresponding gains, K are found at,

$$k_1=-308.1744;\ k_2=-76.0894;\ k_3=-160.5866$$
$$k_4=-110.3322;\ k_5=197.9297$$

It can be shown that by using (18), the steady-state error is zero i.e, the cart stops exactly at the desired location.

V. SIMULATION RESULT

The plant described by (12) is absolutely unstable since there is one pole at the right hand side of the s-plane. Fig. 2 shows the unforced response of a system when theta (angle of a ball from vertical position) is set to be at 0.1 (rad) and no disturbance is applied to the system. The simulation utilizes equation (12) and (16) to represent both nonlinear and linear models of the system. The linear and nonlinear response gives almost the same result. It shows that the approximation used in the linearization method does not affect much the system.

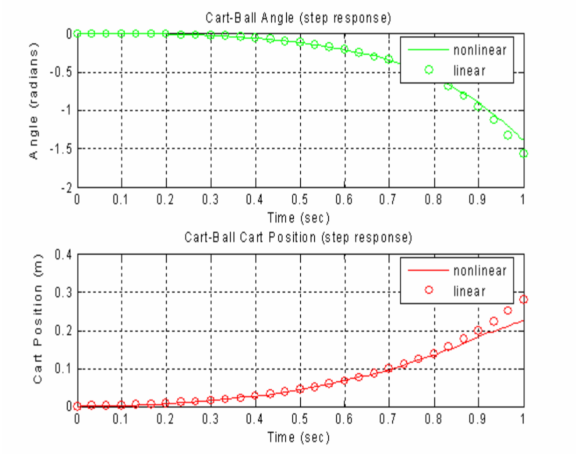

Fig.2: Unforced response of the system.

A. SFC Without Integral Control

Fig. 3(a) and 3(b) show the outputs of the system with no disturbance apply to the system and cart is required to move 1(m) from the original position.

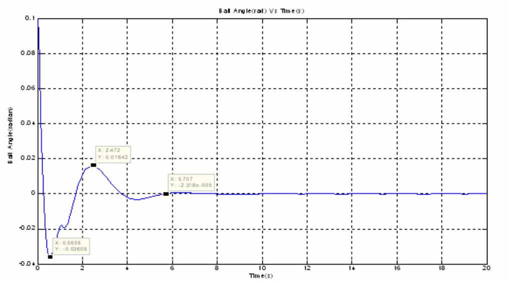

Fig. 3(a) Ball angle output.

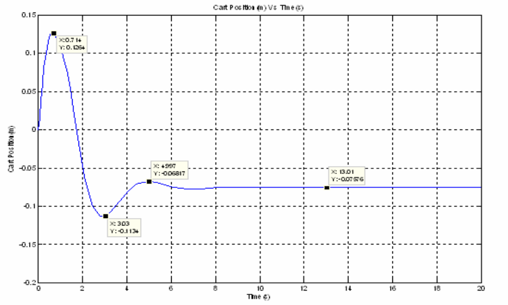

Figure 3(b) Cart position output

It can be seen that the ball returns to 0 degree after 3.8 (s) with the steady-state error of 1.0758 (m) which is similar to the result calculated in *Part IV*.

B. SFC With Integral Control

Fig. 5 depicts the output of a system with all specifications are same as in *Part V(A)*.

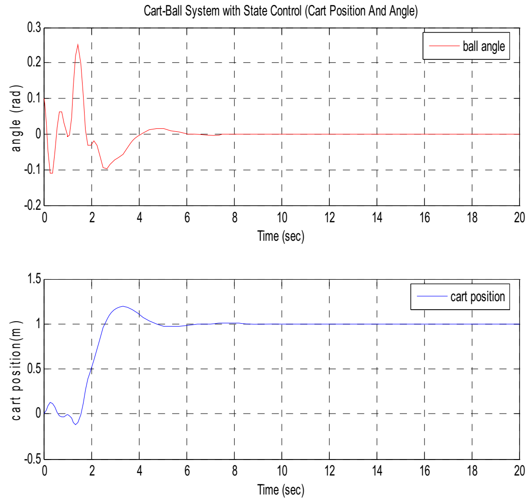

Fig. 4 Ball angle output and Cart position output

From Fig. 4, the ball returns to 0 (degree) after 6 (s) and the steady-state error is zero (after 6 sec) which is similar to the result calculated in *Part IV*. Therefore, using the Integral Control it can be sure that the cart will stop at any selected desired position.

The overshoot of the system of a cart is 19.7% which is within the objective limit determined earlier. However, the settling time is 6 (s) which is 1 (s) over the objective value, error is 20%.

C. Disturbance Rejection Analysis

Different values of disturbance are used as to determine the maximum value of disturbance force that can be handled by the State-Feedback Controller. A pulse signal is used to represent disturbance force which is starts at 7 (s) and ends at 8 (s) as shown in Fig. 5.

In the first part simulation, the disturbance value is set to be at 0.1 (N), and effect to the system is studied. Fig 5 shows the result of a system with initial value of ball angle (theta) is chosen at 0.1 (rad). From the result, the SFC can handle the system well.

However, the existence of disturbance limits the system capability, thus restrict the ability of SFC to control the system with value of theta should not exceed than 0.3 (rad). Outside of this range SFC cannot control the system anymore. These behaviors are shown in Fig. 6 and 7.

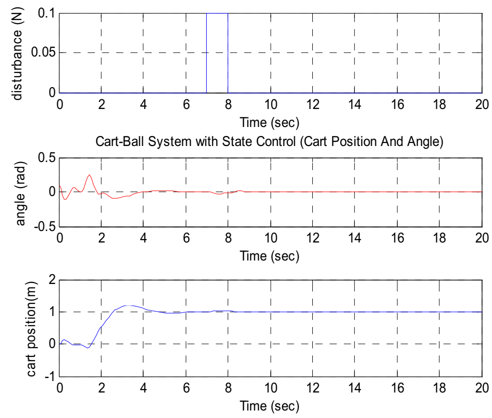

Fig. 5 Ball angle and cart position output with an angle (theta) is equal to 0.1 rad and disturbance is 0.1N.

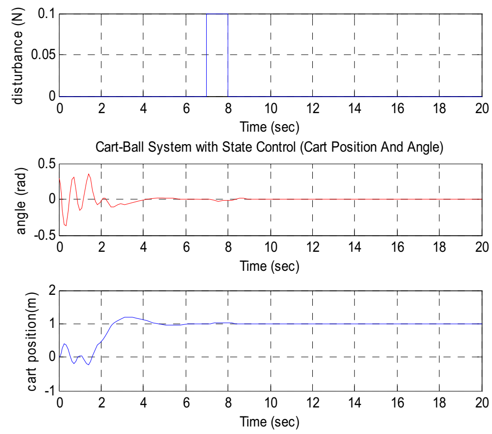

Fig. 6 Ball angle and cart position output with an angle (theta) equal to 0.3 (rad)and disturbance is 0.1(N). The maximum value of theta that can be controlled by SFC is 0.3 rad.

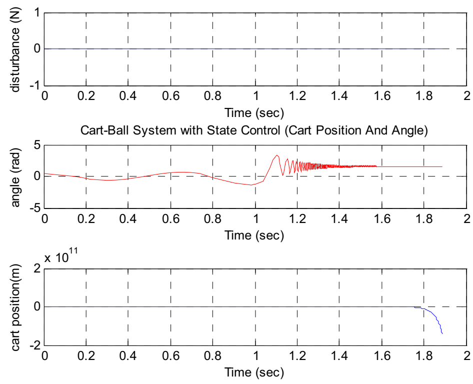

Fig. 7 Ball angle and cart position output with an angle (theta) equal to 0.4 (rad) and disturbance is 0.1(N). SFC fails to control the system.

The second part of the simulation is to determine the maximum value of the disturbance force that can be handled by SFC. In this portion, the initial value of ball angle (theta) is maintains at 0.1 (rad). Fig 8 shows the result of ball angle and cart position output with disturbance value is 0.5(N).

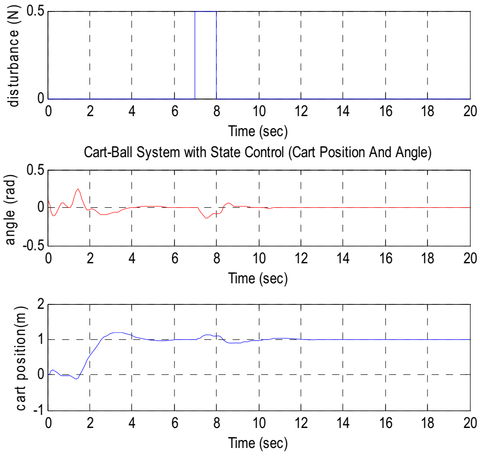

Fig. 8 Ball angle and cart position output with theta is 0.1 (rad) and disturbance force at 0.5 (N).

From Fig. 8, it shown that SFC compensates the existence of a disturbance well and the system is stable at 4 (s) after the disturbance force exist. Further increase of a disturbance force value will result higher overshoot of a system. This behavior can be depicted through Fig. 9 to 12 Based on all of these figures, the time taken to the stable location is maintain at 4 (s), whereas the overshoot of a system is increase by increasing disturbance force magnitude to the system.

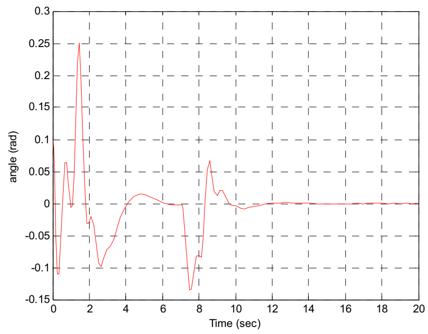

Fig. 9 Ball angle output of theta is 0.1 (rad) and disturbance force is 0.5 (N).

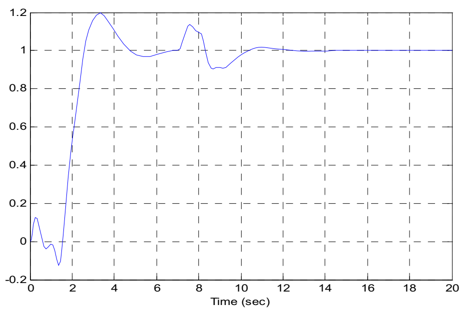

Fig. 10 Cart position output of theta is 0.1 (rad) and disturbance force is 0.5 (N).

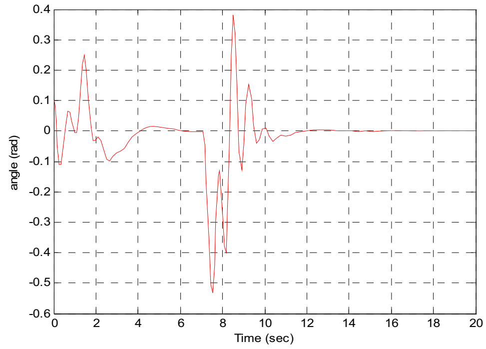

Fig. 11 Ball angle output of theta is 0.1 (rad) and disturbance force is 2 (N)

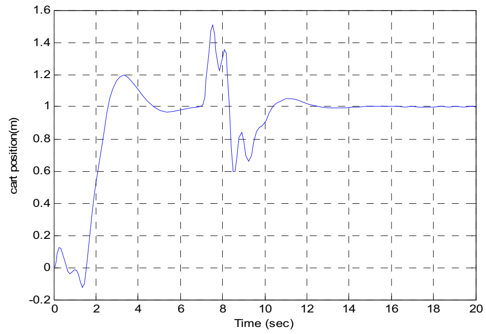

Fig. 12 Cart position output of theta is 0.1 rad and disturbance force is 2N. %OS at the disturbance location is greater than 20%.

However, increasing the disturbance force to 3 (N) results the system becomes unstable. So, SFC manages to control the disturbance force only up to 2N in this system. It can be shown in Fig. 14 where 3 (N) disturbance force is applied to the system.

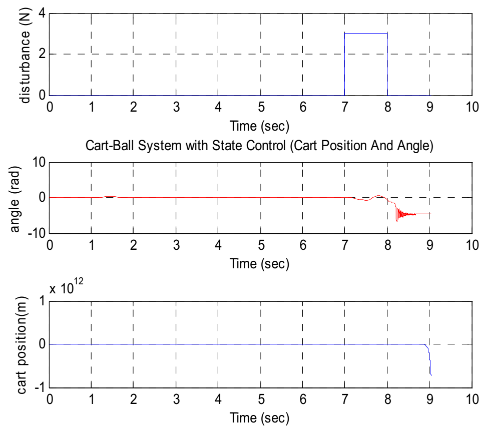

Fig.14: Ball angle and cart position output with theta is equal to 0.1 rad and disturbance force is 3 N. SFC fails to control the system.

VI. Conclusion

The modeling and control of a cart-ball system are presented in this paper. The formulation produced an unstable nonlinear state equation model with disturbance. Performance of a State-Feedback Controller using the pole placement method has also been highlighted. Through the result it can be concluded that the SFC needs to be complemented with the Integral Control to eliminate the steady state error. In terms of the disturbance rejection, the SFC can only control with the small rolling angle of the ball which is at 0.3(rad) and also can control just up to 2 N disturbance forces as shown in the result in *Part V*. Therefore, it can be said that the performance of SFC become poor with the existence of a disturbance to a system. However, the performance of SFC in this system is good in condition where there are no disturbances force applied to the system. Further investigation is still needed before it can be adapted in this system in a real implementation..

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