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Contact Problem for an Elastic Layered Composite Resting on Rigid Flat Supports

T. S. Ozsahin, V. Kahya, A. Birinci and A. O. Cakiroglu

***Abstract*—**In this study, the contact problem of a layered composite which consists of two materials with different elastic constants and heights resting on two rigid flat supports with sharp edges is considered. The effect of gravity is neglected. While friction between the layers is taken into account, it is assumed that there is no friction between the supports and the layered composite so that only compressive tractions can be transmitted across the interface. The layered composite is subjected to a uniform clamping pressure over a finite portion of its top surface. The problem is reduced to a singular integral equation in which the contact pressure is the unknown function. The singular integral equation is evaluated numerically and the results for various dimensionless quantities are presented in graphical forms.

***Keywords*—**Frictionless contact, Layered composite, Singular integral equation, The theory of elasticity.

I. INTRODUCTION

IN engineering mechanics, the contact problems have different applications to a variety of structures of practical interest. Foundations, roads, railways, airfield pavements, rolling mills, ball and roller bearings are some application areas of the contact problems. Although developments in the contact problems did not appear in the literature until the beginning of this century, the studies are accelerated recently because of improvements in computer technology.

In previous studies, the elastic layer resting on an elastic half space or rigid foundation is considered. In these studies, the layer is subjected to uniform or concentrated loading conditions [1, 2]. The examples for the contact problems in which the load is transmitted to the elastic layer by the rigid stamp can be found in [3, 4] . While the effect of gravity in all these studies is taken into account, it is neglected in [5]. In [6], the load is transmitted to the layer by means of an elastic stamp instead of a rigid one. The examples for the works in which the elastic layer is resting on rigid supports can be found in [7, 8].

Talat S. Ozsahin is Assistant Professor with the Civil Engineering Department, Faculty of Engineering, Karadeniz Technical University, Trabzon, Turkey (e-mail: talat@ktu.edu.tr)
Volkan Kahya is Assistant Professor with the Civil Engineering Department, Faculty of Engineering, Karadeniz Technical University, Trabzon, Turkey (corresponding author, phone: +90 462 3772631, e-mail: volkan@ktu.edu.tr)
Ahmet Birinci is Associate Professor with the Civil Engineering Department, Faculty of Engineering, Karadeniz Technical University, Trabzon, Turkey (e-mail: ahbirinci@hotmail.com)
A. Osman Cakiroglu is Professor with the Civil Engineering Department, Faculty of Engineering, Karadeniz Technical University, Trabzon, Turkey (e-mail: cakir@ktu.edu.tr)

In this study, the contact problem of the two elastic layers with different constant heights h_1 and h_2 resting on rigid flat supports is solved according to the theory of elasticity. The layered medium is subjected to a uniform clamping pressure over a portion of width $2a$ on its top surface as seen in Fig. 1. In solution, the effect of gravity is neglected. The friction between the layers is taken into account. However, it is assumed that there is no friction between the supports and the layered composite so that only compressive traction can be transmitted across the interface.

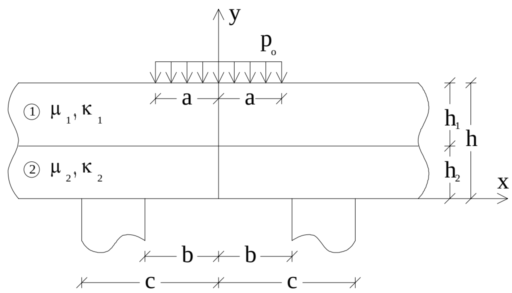

Fig. 1 Layered composite resting on rigid flat supports subjected to uniform clamping pressure

II. FORMULATION OF THE PROBLEM

The Navier equations to be used in the solution of two-dimensional contact problem may be expressed as follows.

$$\mu_i \nabla^2 u_i + (\lambda_i + \mu_i)\frac{\partial}{\partial x}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0\,, \tag{1a}$$

$$\mu_i \nabla^2 v_i + (\lambda_i + \mu_i)\frac{\partial}{\partial y}\left(\frac{\partial u_i}{\partial x} + \frac{\partial v_i}{\partial y}\right) = 0\,, \tag{1b}$$

where subscript $i=1,2$ indicates the layer number.

In these expressions, λ_i and μ_i are the Láme constant and the shear modulus, u_i and v_i are the displacement components in x and $y-$directions, respectively. The problem is symmetrical according to the $y-$axis and the following conditions must also be satisfied.

$$u_i(x,y) = -u_i(-x,y)\,,\qquad v_i(x,y) = v_i(-x,y)\,. \tag{2}$$

Due to symmetry, it is enough to consider the problem in the region $0 \le x < \infty$. Displacements of each layer may be expressed as the Fourier sine and Fourier cosine transforms of the unknown functions $\phi_i(\alpha, y)$ and $\Psi_i(\alpha, y)$ as

$$u_i(x,y) = \frac{2}{\pi}\int_0^\infty \phi_i(\alpha,y)\sin(\alpha x)d\alpha\,,\;\; i=1,2, \tag{3a}$$

$$v_i(x,y)=\frac{2}{\pi}\int_0^{\infty}\Psi_i(\alpha,y)\cos(\alpha x)d\alpha\,,\ i=1,2. \tag{3b}$$

Substituting (3) into (1) and solving the resulting ordinary differential equation system, one may obtain the unknown functions $\phi_i(\alpha,y)$ and $\Psi_i(\alpha,y)$. Using these solutions into (3), the displacements u_i and v_i for each layer can be determined as

$$u_i(x,y)=\frac{2}{\pi}\int_0^{\infty}\left[(A_i+B_iy)e^{-\alpha y}+(C_i+D_iy)e^{\alpha y}\right]\sin(\alpha x)\,d\alpha\,, \tag{4}$$

$$v_i(x,y)=\frac{2}{\pi}\int_0^{\infty}\left\{\left[A_i+\left(\frac{\kappa_i}{\alpha}+y\right)B_i\right]e^{-\alpha y}+\left[-C_i+\left(\frac{\kappa_i}{\alpha}-y\right)D_i\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha, \tag{5}$$

where $\kappa_i, i=1,2$ is an elastic constant and $\kappa_i=(3-4\nu_i)$ for plane strain, $\kappa_i=(3-\nu_i)/(1+\nu_i)$ for plane stress. ν_i is the Poisson' s ratio. σ_{xi}, σ_{yi} and τ_{xyi} stress components may be expressed in terms of u_i and v_i as follows.

$$\sigma_{xi}=(\lambda_i+2\mu_i)\frac{\partial u_i}{\partial x}+\lambda_i\frac{\partial v_i}{\partial y}\,,\quad i=1,\ 2, \tag{6}$$

$$\sigma_{yi}=(\lambda_i+2\mu_i)\frac{\partial v_i}{\partial y}+\lambda_i\frac{\partial u_i}{\partial x}\,,\quad i=1,\ 2, \tag{7}$$

$$\tau_{xyi}=\mu_i\left(\frac{\partial u_i}{\partial y}+\frac{\partial v_i}{\partial x}\right),\quad i=1,\ 2. \tag{8}$$

Substituting (4) and (5) into equations from (6) to (8), the stress expressions for each layer may readily be obtained as follows.

$$\sigma_{xi}(x,y)=\frac{4\mu_i}{\pi}\int_0^{\infty}\left\{\left[\alpha(A_1+B_iy)-\frac{3-\kappa_i}{2}B_i\right]e^{-\alpha y}+\left[\alpha(C_i+D_iy)+\frac{3-\kappa_i}{2}D_i\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha, \tag{9}$$

$$\sigma_{y_i}(x,y)=\frac{4\mu_i}{\pi}\int_0^{\infty}\left\{-\left[\alpha(A_i+B_iy)+\frac{1+\kappa_i}{2}B_i\right]e^{-\alpha y}+\left[-\alpha(C_i+D_iy)+\frac{1+\kappa_i}{2}D_i\right]e^{\alpha y}\right\}\cos(\alpha x)d\alpha, \tag{10}$$

$$\tau_{xy_i}(x,y)=\frac{4\mu_i}{\pi}\int_0^{\infty}\left\{-\left[\alpha(A_i+B_iy)+\frac{\kappa_i-1}{2}B_i\right]e^{-\alpha y}+\left[\alpha(C_i+D_iy)-\frac{\kappa_i-1}{2}D_i\right]e^{\alpha y}\right\}\sin(\alpha x)d\alpha, \tag{11}$$

where A_i, B_i, C_i and D_i, $i=1,2$ are the unknown constants which are determined from the following boundary conditions.

$$\tau_{xy1}(x,h)=0\,,\quad -\infty<x<\infty\,, \tag{12a}$$

$$\sigma_{y1}(x,h)=-p_o\,,\quad -a<x<a\,, \tag{12b}$$

$$u_1(x,h_2)=u_2(x,h_2)\,,\quad -\infty<x<\infty\,, \tag{12c}$$

$$v_1(x,h_2)=v_2(x,h_2)\,,\quad -\infty<x<\infty\,, \tag{12d}$$

$$\sigma_{y1}(x,h_2)=\sigma_{y2}(x,h_2)\,,\quad -\infty<x<\infty\,, \tag{12e}$$

$$\tau_{xy1}(x,h_2)=\tau_{xy2}(x,h_2)\,,\quad -\infty<x<\infty\,, \tag{12f}$$

$$\tau_{xy2}(x,0)=0\,,\quad -\infty<x<\infty\,, \tag{12g}$$

$$\sigma_{y2}(x,0)=-q(x)\,,\quad b<x<c\,, \tag{12h}$$

$$\frac{\partial v_2(x,0)}{\partial x}=0\,,\quad b<x<c\,. \tag{13}$$

where $q(x)$ in (12h) is the unknown contact pressure.

Applying the boundary conditions (12) to the displacement and the stress expressions given in (4), (5) and (9) to (11), one may obtain the coefficients A_i, B_i, C_i and D_i, $i=1,2$ in terms of the unknown contact pressure $q(x)$. These coefficients are given in Appendix.

3. The Singular Integral Equation

The unknown contact pressure $q(x)$ is determined by making use of the remaining boundary condition (13). If constants A_2, B_2, C_2 and D_2 are substituted into (5), after some routine manipulations, one may obtain the following singular integral equation.

$$\int_b^c\left[\frac{1}{t-x}-\frac{1}{t+x}+\frac{2}{(1+\kappa_2)}k(x,t)\right]q(t)dt=-p_0\frac{\mu_2}{\mu_1}\frac{2}{(1+\kappa_2)}l(x)\,, \tag{14}$$

where $k(x,t)$ and $l(x)$ are given in the Appendix. The equilibrium condition for the problem may be written as

$$\int_b^c q(t)dt=ap_0\,. \tag{15}$$

The kernel, $k(x,t)$ of the singular integral equation is bounded in closed interval $b\le x\le c$.

To simplify the numerical analysis of the integral equation, the following dimensionless quantities can be introduced.

$$\alpha=z/h\,,\quad x=\frac{c-b}{2}r+\frac{c+b}{2}\,,\quad t=\frac{c-b}{2}s+\frac{c+b}{2}\,,$$

$$L(r)=\frac{l(\frac{c-b}{2}r+\frac{c+b}{2})}{p_0}\,,\quad g(s)=\frac{q(\frac{c-b}{2}s+\frac{c+b}{2})}{p_0}\,,$$

$$k(r,s)=k\left(\frac{c-b}{2}r+\frac{c+b}{2},\frac{c-b}{2}s+\frac{c+b}{2}\right). \tag{16}$$

Substituting the dimensionless quantities given in (16) into (14) and (15), these equations may be written as follows.

$$\int_{-1}^{1}\left[\frac{1}{s-r}-\frac{1}{(s+r)+2(c+b)/(c-b)}+\frac{(c-b)}{h}\frac{1}{(1+\kappa_2)}k(r,s)\right]g(s)ds=-\frac{\mu_2}{\mu_1}\frac{2}{(1+\kappa_2)}L(r),$$

$$-1<r<1\,, \tag{17}$$

$$\int_{-1}^{1} g(s)ds = \frac{2a}{c-b}. \quad (18)$$

The function $g(s)$ has singularities at $s = \pm 1$ and thus the index of the integral equation is +1 [10]. Assuming the solution of integral equation as

$$g(s) = G(s)/\sqrt{1-s^2}, \quad (-1 < s < 1), \quad (19)$$

and using the appropriate Gauss-Chebyshev integration formula [11], (17) and (18) may then be replaced by

$$\sum_{i=1}^{n} W_i \left[\frac{1}{s_i - r_j} - \frac{1}{(s_i + r_j) + 2(c+b)/(c-b)} + \frac{(c-b)}{h} \frac{1}{(1+\kappa_2)} k(r_j, s_i) \right] G(s_i) = -\frac{\mu_2}{\mu_1} \frac{2}{(1+\kappa_2)} L(r_j),$$

$$j=1,\ldots,n-1, \quad (20)$$

$$\sum_{i=1}^{n} W_i G(s_i) = \frac{2a}{c-b}, \quad (21)$$

where

$$W_1 = W_n = \frac{\pi}{2n-2}, \quad W_i = \frac{\pi}{n-1}, \quad i=2,\ldots,n-1,$$

$$s_i = \cos\left(\frac{i-1}{n-1}\pi\right), \quad i=1,\ldots,n,$$

$$r_j = \cos\left(\frac{2j-1}{2n-2}\pi\right), \quad j=1,\ldots,n-1. \quad (22)$$

Equations (20) and (21) constitute n linear algebraic equations for n unknowns, $G(s_i)$, $i=1,\ldots,n$. Solution of these algebraic equations and use of (19) yield the unknown contact pressure, $q(x)$. Once the contact stress is obtained, the stress components at any point in the medium may be found easily by making use of (9) to (11).

4. Results

Results for normalised contact pressure $q(x)/p_0$ are shown in Figs. 2 to 5.

Fig. 2 shows the normalised contact pressure distribution with variation of load width. It should be mentioned that the calculated contact pressure distribution was found to depend essentially on a/h only, that is, the normalized contact pressure $q(x)/p_0$ independent of the magnitude of the applied load. As expected, the contact pressure has singularities at the corners of the supports. The normalised contact pressure decreases with decreasing of a/h. As a/h increases, the normalised contact pressure has more great values at where close to outer edge with respect to ones at where close to inner edge of the rigid stamp.

In Fig. 3, the normalised contact pressure distribution with variation of support width is given. As support width increases, it is observed that the normalised contact pressure decreases at where close to the outer edge of the rigid stamp. Also, as c/h increases, the normalised contact pressure gets smaller. Therefore, it shows the possibility of separation between the rigid stamp and layered composite.

As seen in Fig. 4, the normalised contact pressure decreases at where close to inner edge whereas increases at where close to outer edge of the rigid stamp with decreasing of h_2/h.

As μ_2/μ_1 increases, the normalised contact pressure increases interior region of the rigid stamp while decreases in the region close to corners as seen in Fig. 5.

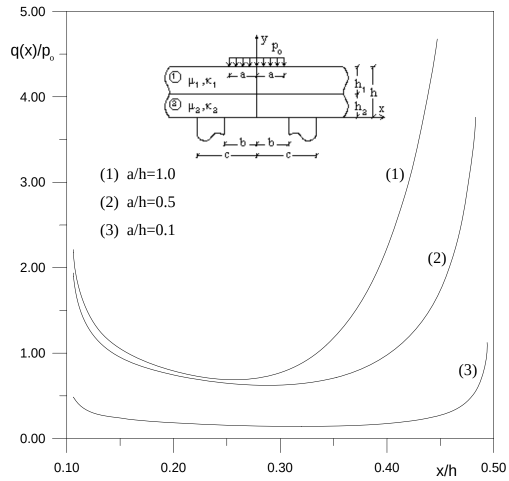

Fig. 2 The normalised contact pressure distribution with variation of load width ($h_2/h=0.2$, $\mu_2/\mu_1=6.48$, $b/h=0.1$, $c/h=0.5$)

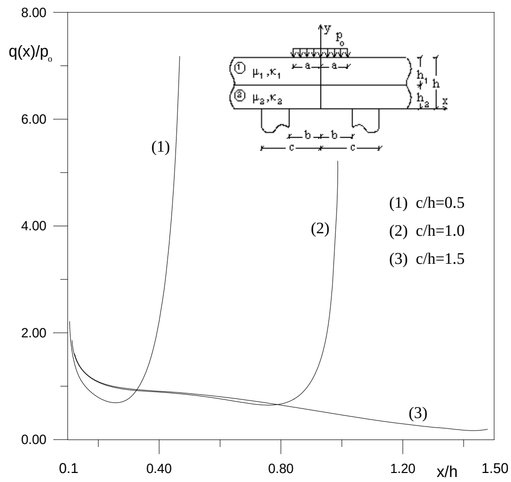

Fig. 3 The normalised contact pressure distribution with variation of support width ($h_2/h=0.2$, $\mu_2/\mu_1=6.48$, $b/h=0.1$, $a/h=1$)

Results for $\sigma_x(0,y)/p_0$ dimensionless stress are shown in Figs. 6 to 8.

In Fig. 6, it can be seen that the layer 2 has tensile stress distribution along the y-axis, although very small compressive stresses appear at the upper surface for small values of a/h. In the layer 1, compressive stress distribution is examined. On the contrary, stresses are tensile in the layer 1 and compressive in the layer 2 for larger values of a/h, although small compressive stresses appear close to the lower surface. For small values of a/h, the layered composite behaves like a simply supported beam whereas it behaves like an overhanging beam for larger values of a/h.

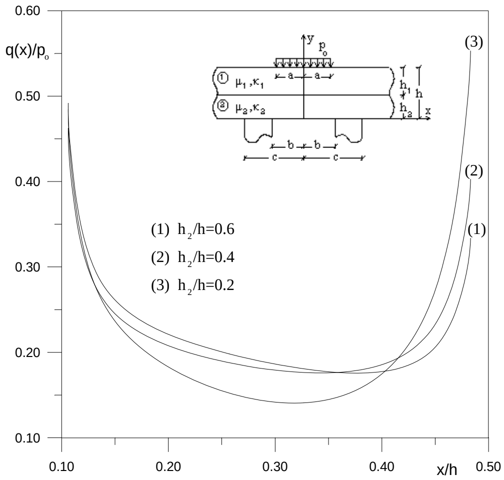

Fig. 4 The normalised contact pressure distribution with variation of h_2/h (*μ₂/μ₁=6.48, b/h=0.1, c/h=0.5, a/h=0.1*)

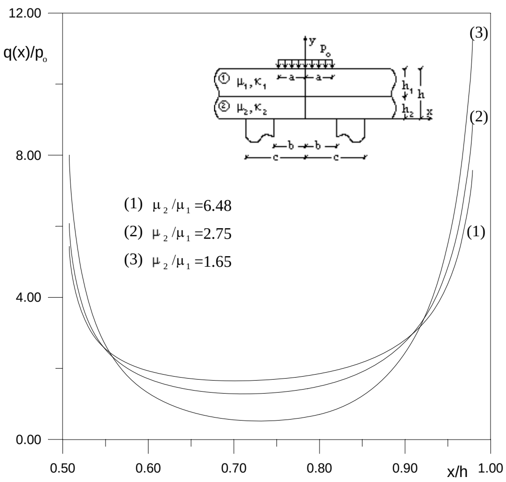

Fig. 5 The normalised contact pressure distribution with variation of elastic constants (*h₂/h=0.2, b/h=0.5, c/h=1, a/h=1.5*)

Variation of $\sigma_x(0,y)/p_0$ dimensionless stress distribution with elastic constants is shown in Fig. 7. In the layer 2, for every μ_2/μ_1 ratio, tensile stress distribution is observed. As μ_2/μ_1 increases, the values of the tensile stresses increase at the lower surface of the layer 2. In the layer 1, both tensile and compressive stresses are determined. As μ_2/μ_1 ratio decreases, the region where the tensile stresses appear becomes closer to the middle of the layer 1.

As the support width increases, the values of dimensionless stress $\sigma_x(0,y)/p_0$ increase as seen in Fig. 8. Although only compressive stresses are determined in the layer 1, both tensile and compressive stresses are observed in the layer 2.

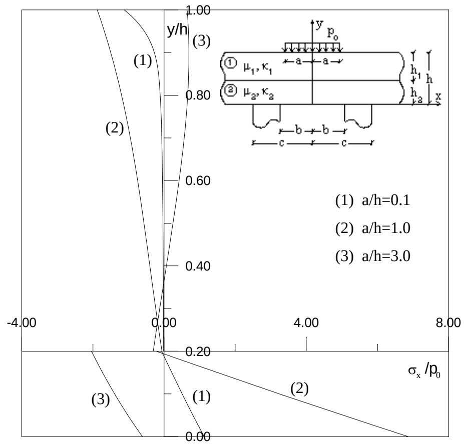

Fig. 6 The axial stress distribution $\sigma_x(0,y)/p_0$ with variation of load width (*h₂/h=0.2, μ₂/μ₁=6.48, b/h=1.0, c/h=1.5*)

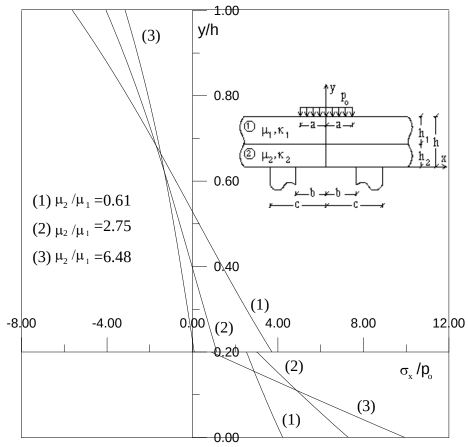

Fig. 7 The axial stress distribution $\sigma_x(0,y)/p_0$ with variation of elastic constants (*h₂/h=0.2, b/h=1.5, c/h=2, a/h=1*)

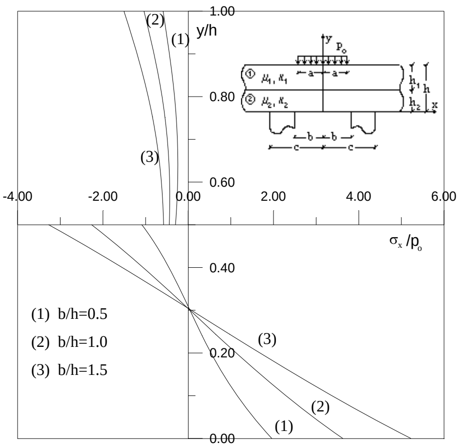

Fig. 8 The axial stress distribution $\sigma_x(0,y)/p_0$ with variation of support width (*$h_2/h=0.5$, $\mu_2/\mu_1=6.48$, $a/h=0.5$*)

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Appendix

Here, we gave the coefficients A_i, B_i, C_i and D_i, *i=1,2* determined from the boundary conditions and the functions $k(x,t)$ and $l(x)$ seen in (14).

$$\begin{aligned}\Delta=\,&x_1(e_1z_9(3-4\alpha^2z_8))+(e_{17}+e_2)(z_1x_2x_3+x_4x_5)\\&+e_1(2z_{13}(1+\kappa_2x_4-mx_2)+\kappa_2^2z_9+m(4\alpha^2h((-h+4h_2)\\&+m(h-2h_2)))+2\kappa_1((1+z_{10})+\kappa_1m(1+.5z_9))\\&+2\kappa_2(.5z_{15}+\kappa_1(-1+z_2z_{12}))))\\&+(e_9+e_{10})((1-4\alpha^2z_8)x_4x_2-x_3x_5)(1+e_3)x_3(-x_4)\\&+(e_4+e_{11})x_2x_5\end{aligned}\qquad\text{(A1)}$$

$$\begin{aligned}\alpha A1=\,&(-(.5(P(e_7(x_2x_3(z_4+z_9z_4-z_1\kappa_1)-x_4x_5(z_4+\kappa_1))\\&+e_5(x_1z_9(z_5\kappa_1+2(1+\alpha h_2z_5))+z_9(\kappa_2^2+\kappa_1m(-2\kappa_2+\kappa_1m))\\&+z_2m(z_7(1-\kappa_2)-\kappa_1^2x_6)+(1-z_2z_7)(1+\kappa_2^2)\\&+z_5(\kappa_1(1+\kappa_2^2)+2\kappa_1m(-1+\kappa_2+m))\\&+m(2\kappa_1(1-\kappa_2+\kappa_1m)+2z_2(z_7x_2+\alpha h\kappa_1x_6)))\\&+(e_{12}+e_{13})(x_2x_4z_5(z_2+\kappa_1)-x_3x_5)\\&+(z_4+\kappa_1)(e_6x_3x_4-e_{14}x_2x_5))+T(x_6(x_2(z_2(-\kappa_1m)(e_2-z_{11}e_{10})\\&+m(-e_2z_2z_3+(z_6+4\alpha^2h(h-h_2+z_2(h-h_2)))e_{10}))\\&+m(\kappa_1-\kappa_2)(e_2+e_{10}2\alpha h))+(m(z_2(x_6m(1-\kappa_1)+(-1+\kappa_2^2))\\&+x_6(\kappa_1\kappa_2-1)))+e_1m(z_3\kappa_1(1+\kappa_1mx_6)+x_6((1+z_{15})\kappa_2\\&+(1+2\alpha hz_{12})m+2\alpha(h_2z_4\kappa_1m-h(1-\kappa_1\kappa_2)))+(1-\kappa^2{}_2)z_{10}\\&-\kappa_1\kappa_2(1+\kappa_2z_2))+e_{10}(x_6\kappa_1mx_5))))))/\Delta\end{aligned}\qquad\text{(A2)}$$

$$\begin{aligned}B1=\,&(-(P(e_5(m(z_2((2-m)+\kappa_1x_6)+z_7(-1+\kappa_2+m))\\&+(1+z_{12})((1+\kappa_2x_4-mx_2)+z_9x_1))-e_6x_4x_3\\&+x_2(e_{14}x_5+(e_{12}+e_{13})(1+z_{12})x_4)+e_7(z_1x_2x_3+x_4x_5))\\&+T(x_2mx_6(e_{10}2\alpha(-h-h_2z_{11})+e_2z_3)+e_1m(-z_3(1+\kappa_1mx_6)\\&-z_{12}(m+\kappa_2x_4))-m(x_6(e_{10}x_5+x_4)+e_1z_4\kappa_2)))))/\Delta\end{aligned}\qquad\text{(A3)}$$

$$\begin{aligned}\alpha C1=\,&((.5(P(e_6(x_1(2z_9(-1+\alpha h_2z_{11})+(-z_{11}-z_9z_{11})\kappa_1)\\&+(-1+z_2z_4)(1-\kappa_2x_5)-z_{11}\kappa_1(\kappa_2x_4+m^2)\\&-z_2z_4m(2x_2+(1-\kappa_2))-x_5(2(1+.5z_9)\kappa_1m-z_9\kappa_2)\\&+2\kappa_1m(-\alpha h_2\kappa_1x_6-(1+.5z_{10})-\alpha h\kappa_2))\\&+e_{12}(x_3x_2(z_7+z_9z_7+z_1\kappa_1)-x_5x_4(z_7-\kappa_1))\\&+(e_{15}+e_7)(x_2x_4(z_2z_{11}-z_{11}\kappa_1)+x_3x_5)+e_{16}(-x_2x_5(z_7-\kappa_1))\\&+e_5(x_4x_3(z_7-\kappa_1))))+T(mx_6(x_2((e_9(-1-z_{15}+z_2-z_{10}z_5)\\&+e_{17}z_2z_6)+(\kappa_1z_2(e_9z_5-e_{17})))+(\kappa_1-\kappa_2)(2\alpha he_9-e_{17}))\\&+x_6m(-\kappa_1x_5e_9+(1-\kappa_1\kappa_2)(-2\alpha he_1+e_3))\\&+e_1m((-1-2\alpha hz_{12})(m+\kappa_2x_4)-\kappa_2(1+z_{15})\\&+z_2(-z_7\kappa_1mx_6-2\alpha h)+\kappa_1(x_3x_6+z_2(x_3+\kappa_2x_5)))\\&+e_3(z_2m((1-\kappa_1)mx_6+(-1+\kappa_2^2))))))))/\Delta\end{aligned}\qquad\text{(A4)}$$

$$\begin{aligned}D1=\,&(-(P(z_{11}(e_6(1+\kappa_2x_4-mx_2)+(e_{15}+e_7)x_2x_4)\\&+e_{12}(-z_1x_2x_3-x_4x_5)+e_6(z_9z_{11}x_1+m(-z_4(x_2-\kappa_2)+z_2((2-m)\\&+\kappa_1x_6)))-e_{16}x_2x_5+e_5x_3x_4)+T(x_2mx_6(e_{17}z_6\\&+e_92\alpha(-h-h_2z_5))+e_1(m(-z_6(1+\kappa_1mx_6)\\&-z_7\kappa_2-z_{12}(\kappa_2x_4+m)))+mx_6(e_3x_4+e_9x_5)))))/\Delta\end{aligned}\qquad\text{(A5)}$$

$$\begin{aligned}\alpha A2=\,&(-(.5(P(e_6(z_2(1+\kappa_1x_3)+z_{12}(\kappa_2x_7+m)\\&+m(x_7(\kappa_1\kappa_2-1)+2\alpha h\kappa_1))+e_7(x_7(x_2(z_{11}-z_2z_{11}\kappa_2)-\kappa_2x_5))\\&+e_{12}(x_7(z_2z_5x_2-z_{12}\kappa_2-m(z_5\kappa_2+\kappa_1)))+e_5(\kappa_2(z_5(\kappa_2x_7+m)\\&-z_2(1+\kappa_1x_3)-z_7\kappa_1m)+x_7x_3))+T(e_1(x_1(8\alpha^2h_2(h_2\\&+\alpha(-h^2-h_2^2+2h_2h+z_2z_{16}))+(-1+z_2-z_{15}-z_9+2z_{10}\\&+8\alpha^3h_2z_{16})\kappa_2)+z_3(\kappa_2+\kappa_1m(2\kappa_2+\kappa_1m(-1+\kappa_2)))\\&+(1+z_{15})(\kappa_2^2+m(2\kappa_2+m))+(m(z_{12}(-1+\kappa_2^2x_7)\\&-2\alpha(h+h_2)\kappa_1)+(2z_{10}(-\kappa_2^2+m(x_7-m+\kappa_2(\kappa_1-1)))\\&+z_9(\kappa_2^2+\kappa_1m(-2\kappa_2+\kappa_1m)))+z_2(-2+m(2-m)))\\&+\kappa_2(\kappa_2-2\kappa_1m))+e_{10}(1-4\alpha^2z_8)((1-\kappa_2)(\kappa_2-m^2)\\&+m(1+\kappa_2^2)-2\kappa_2m)+((1-\kappa_2)(\kappa_2-\kappa_1^2m^2)\\&+\kappa_1m(-1+2\kappa_2-\kappa_2^2)))+(e_2+e_{17})(-z_3x_2(\kappa_2x_3+z_2x_3)\\&+x_4x_5)+(1-\kappa_2)(e_{11}x_2x_5-x_4x_3))))))/\Delta\end{aligned}\qquad\text{(A6)}$$

$$\begin{aligned}B2=\,&(P(z_5(e_{12}x_2x_7+e_5(\kappa_2x_7+m))+x_7(e_7(-z_2z_{11}x_2-x_5)+e_6x_3)\\&+e_5(-z_2(1+\kappa_1x_3)-z_7\kappa_1m))+T(x_2(-z_3x_3(e_2+e_{17})\\&+e_{10}(-1-4\alpha^2z_{16})\,x_4-e_{11}x_5)+x_3(e_{10}x_5+x_4)\\&+e_1(x_1(-z_9+2z_{10}-8\alpha^3h_2z_8+(-1-z_{15}+z_2))+z_3(1+\kappa_1mx_3)\\&+z_{12}(m(1+\kappa_2x_7))+z_4\kappa_1m)))/\Delta\end{aligned}\qquad\text{(A7)}$$

$$\begin{aligned}\alpha C2=\,&(.5(P(x_7(x_2(-e_{12}z_5(1+\kappa_2z_2)+e_7(z_2z_{11}-\kappa_2z_{12}))+e_{12}\kappa_2x_5\\&+e_7m(\kappa_1-\kappa_2))+e_5(z_2(x_7+m(-1+\kappa_1^2))\end{aligned}$$

$$
\begin{aligned}
&+ x_7(z_{12}\kappa_2 + m(z_7 - \kappa_1\kappa_2))) + e_6(-z_{11}\kappa_2(\kappa_2 x_7 + m) \\
&- z_2\kappa_2(1 + \kappa_1 x_3) - x_7 x_3 - z_4\kappa_1\kappa_2 m)) + T(e_1(x_1(8\alpha^2 h_2(-h_2 \\
&+ \alpha(-h^2 - h_2^2 + 2h_2(h + \alpha z_8))) + \kappa_2(1 + z_2 + z_{15} + z_9 \\
&+ 8\alpha^2 h_2(-h - \alpha z_8))) + z_6(\kappa_2 + m^2(-1 + \kappa_1^2\kappa_2)) \\
&+ \kappa_1 m((-1 - z_2 z_6)(\kappa_1 m - 2\kappa_2) + 2\alpha(-h - h_2)) \\
&+ 2(\kappa_2((-1 + z_{10})(\kappa_2 + m(1 - \kappa_1)) + 2\alpha^2\kappa_2(-h^2 - h_2^2)) \\
&+ z_2(-x_2 + 2\alpha hm(-1 - \kappa_1 + m)) + m(.5z_{12}(\kappa_2^2 x_7 - 1) \\
&+ .5z_{15}(-2\kappa_2 - m)))) + e_9((1 - 4\alpha^2 z_8)((\kappa_2 - 1)(\kappa_2 - m^2) \\
&+ m(-1 + \kappa_2(2 - \kappa_2))) + (x_3(\kappa_2^2 + \kappa_1 m) - \kappa_2(\kappa_1 m(\kappa_1 m + 2) \\
&+ 1))) + (e_2 + e_{17})(x_2 x_3 z_6(z_2 - \kappa_2) - x_5 x_4) \\
&+ (1 - \kappa_2)(-e_4 x_2 x_5 + e_3 x_4 x_3))))))/\Delta \qquad \text{(A8)}
\end{aligned}
$$

$$
\begin{aligned}
D2=&(-(P(x_7(x_2(e_7 z_{11} + e_{12}z_2 z_5) - e_{12}x_5 + e_5 x_3) \\
&+ e_6(z_4\kappa_1 m + z_{11}(m + \kappa_2 x_7) + z_2(1 + \kappa_1 x_3))) \\
&+ T(x_2(z_6(e_{17} + e_2)x_3 - e_4 x_5) + e_3 x_4 x_3 + e_1(-z_6(2x_2 + \kappa_1 m x_3 \\
&+ m^2) - z_{12}(m(1 + \kappa_2 x_7)) - z_7\kappa_1 m) + e_9((-1 + 4\alpha^2 z_8)x_4 x_2 \\
&+ x_3 x_5) + e_1((-z_{15} - z_9 + 2z_{10} + 8\alpha^3 h_2 z_8)x_1)))))/\Delta \qquad \text{(A9)}
\end{aligned}
$$

where

$$
\begin{aligned}
&e_1 = e^{(-2\alpha h)},\ e_2 = e^{(-2\alpha h_2)},\ e_3 = e^{(-4\alpha h)},\ e_4 = e^{(-4\alpha h_2)},\ e_5 = e^{(-\alpha h)}, \\
&e_6 = e^{(-3\alpha h)},\ e_7 = e^{(-3\alpha h+2\alpha h_2)},\ e_8 = e^{(-4\alpha h+2\alpha h_2)},\ e_9 = e^{(-2\alpha h-2\alpha h_2)}, \\
&e_{10} = e^{(-2\alpha h+2\alpha h_2)},\ e_{11} = e^{(-4\alpha h+4\alpha h_2)},\ e_{12} = e^{(-\alpha h-2\alpha h_2)}, \\
&e_{13} = e^{(-\alpha h+2\alpha h_2)},\ e_{14} = e^{(-3\alpha h+4\alpha h_2)},\ e_{15} = e^{(-3\alpha h-2\alpha h_2)}, \\
&e_{16} = e^{(-\alpha h-4\alpha h_2)},\ e_{17} = e^{(-4\alpha h+2\alpha h_2)} \qquad \text{(A10)}
\end{aligned}
$$

$$
\begin{aligned}
&z_1=(-1-4\alpha^2 h_2^2),\quad z_2=2\alpha h_2,\quad z_3=(-1+2\alpha h_2),\quad z_4=(-1+2\alpha h), \\
&z_5=(-1+2\alpha(-h+h_2)),\quad z_6=(1+2\alpha h_2),\quad z_7=(1+2\alpha h), \\
&z_8=(-h^2+h_2(2h-h_2)),\quad z_9=4\alpha^2 h_2^2,\quad z_{10}=4\alpha^2 hh_2, \\
&z_{11}=(-1+2\alpha(h-h_2)),\quad z_{12}=2\alpha(h-h_2),\quad z_{13}=(1+2\alpha^2 h(h-2h_2)), \\
&z_{14}=2\alpha^2 h,\quad z_{15}=4\alpha^2 h^2,\quad z_{16}=(h_2^2+h(h-2h_2)) \qquad \text{(A11)}
\end{aligned}
$$

$$
\begin{aligned}
&x_1=(1+m(-2+m)),\quad x_2=(1-m),\quad x_3=(1+\kappa_1 m),\quad x_4=(\kappa_2+m), \\
&x_5=(\kappa_1 m-\kappa_2),\quad x_6=(1+\kappa_2),\quad x_7=(1+\kappa_1) \qquad \text{(A12)}
\end{aligned}
$$

$$
\begin{aligned}
&m=\mu_2/\mu_1,\quad T=-(1/2\mu_2)\int_b^c q(t)\cos(\alpha t)dt, \\
&P=-p_0\sin(\alpha a)/2\mu_1\alpha \qquad \text{(A13)}
\end{aligned}
$$

$$
\begin{aligned}
k(x,t)=\int_0^\infty &((e_{10}(\kappa_2(x_2((-1 - 4\alpha^2 z_{16})x_4) + x_3 x_5) - .5(\kappa_1(-1 + 2\kappa_2 \\
&- \kappa_2^2)m + (1 - \kappa_2)(\kappa_2 - \kappa_1^2 m^2) + (1 - 4\alpha^2 z_8)(-2\kappa_2 m \\
&+ (1 + \kappa_2^2)m + (1 - \kappa_2)(\kappa_2 - m^2)))) \\
&+ (-\kappa_2 - .5(1 - \kappa_2))(-x_2 x_5(-e_{11} + e_4) + x_3(e_3 x_4 \\
&+ (e_2 + e_8)2z_2 x_2)) + e_9(-\kappa_2((-1 + 4\alpha^2 z_8)x_2 x_4 + x_3 x_5) \\
&- .5(x_3(\kappa_2^2 + \kappa_1 m) - \kappa_2(1 + \kappa_1 m(2 + \kappa_1 m)) \\
&+ (1 - 4\alpha^2 z_8)((-1 + (2 - \kappa_2)\kappa_2)m + (-1 + \kappa_2)(\kappa_2 - m^2)))) \\
&+ e_1(-x_1(-8\alpha^3 h_2 z_{16} + \kappa_2(z_2(1 + 4\alpha^2 z_{16}) - 4\alpha^2(h^2 + 2z_2 z_{16}))) \\
&+ z_3(\kappa_2(1 + \kappa_1 m x_3) - .5(\kappa_2 + \kappa_1 m(+2\kappa_2 + \kappa_1(-1 + \kappa_2)m))) \\
&+ z_6(\kappa_2(2x_2 + m(m + \kappa_1 x_3)) - .5(\kappa_2 + (-1 + \kappa_1^2\kappa_2)m^2)) \\
&+ ((2\alpha h + z_2)\kappa_1 m - z_{12}(-1 + x_7\kappa_2^2)m + 2\kappa_2(z_{12}(1 + x_7\kappa_2)m \\
&+ 2\alpha h\kappa_1 m)) + \kappa_2((-1 - z_{15} + z_2)x_1) - .5((\kappa_2(\kappa_2 - 2\kappa_1 m) \\
&+ (1 + z_{15})(\kappa_2^2 + m(2\kappa_2 + m)) + 2\alpha(h_2(-2 + (2 - m)m) \\
&+ z_2(2h(-\kappa_2^2 + (x_7 + (-1 + \kappa_1)\kappa_2 - m)m) + h_2(\kappa_2^2 \\
&+ \kappa_1 m(-2\kappa_2 + \kappa_1 m))))) + (\kappa_1 m((-1 - z_2 z_6)(-2\kappa_2 + \kappa_1 m)) \\
&+ ((z_{15}(-2\kappa_2 - m))m + 2z_2(-1 + m + 2\alpha hm(-x_7 + m)) \\
&+ \kappa_2(-(z_{15} + z_9)\kappa_2 + 2(-1 + z_{10})(\kappa_2 + (1 - \kappa_1)m)))))) \\
&+ .5x_3 x_4 x_6)/delta + .5x_6)\ [\sin\alpha(t + x) - \sin\alpha(t - x)]\,d\alpha \qquad \text{(A14)}
\end{aligned}
$$

$$
\begin{aligned}
l(x)=\int_0^\infty &1/\alpha((x_7(e_7(x_2(z_{11}(-\kappa_2 - .5z_6) + \kappa_2\alpha((1 - z_{12})h_2 \\
&- (-h + h_2))) + .5(-x_5\kappa_2 - (\kappa_1 - \kappa_2)m)) + e_{12}(x_2 z_5(-\kappa_2 z_3 - \alpha h_2) \\
&+.5\ (z_5(\kappa_2 m + (1 + z_2\kappa_2)x_2) + (\kappa_1 m + \kappa_2(x_5 + z_{12})))) \\
&+ x_3(.5 + \kappa_2)(e_6 - e_5)) + .5(e_5(-(z_2(x_7 + (-1 + \kappa_1^2)m) \\
&+ x_7(z_{12}\kappa_2 + (z_7 - \kappa_1\kappa_2)m)) + \kappa_2(z_5(x_7\kappa_2 + m) \\
&- z_2(1 + \kappa_1 x_3) - z_7\kappa_1 m)) + e_6(-z_2(1 + \kappa_1 x_3)x_6 \\
&- (m(2\alpha h\kappa_1 + x_7(-1 + \kappa_1\kappa_2))) + (-z_4(\kappa_1\kappa_2 m)) \\
&+ (x_7\kappa_2 + m)(\kappa_2 - z_{12}x_6))))/delta) \\
&.\ [\cos\alpha(a - x) - \cos\alpha(a + x)]\,d\alpha \qquad \text{(A15)}
\end{aligned}
$$

Talat S. Ozsahin was born in Hopa, Turkey, in 1971. He graduated as a Civil Engineer from Civil Engineering Department at Karadeniz Technical University, Trabzon, Turkey in 1992. He received M.Sc. and Ph.D. in the same university in 1995 and 2000, respectively. Presently, he is Assistant Professor in the Civil Engineering Department at Karadeniz Technical University. His research interests are in the area of crack and contact problems in mechanics.

Volkan Kahya was born in Istanbul, Turkey, in 1973. He graduated from Civil Engineering Department at Karadeniz Technical University, Trabzon, Turkey in 1994. He received M.Sc. and Ph.D. in the same university in 1997 and 2003, respectively. He has still working as an Assistant Professor in the Civil Engineering Department at Karadeniz Technical University. His research interests are related to crack and contact problems in the theory of elasticity and moving load problems.

Ahmet Birinci was born in Caykara, Turkey, in 1968. He graduated as a Civil Engineer from Civil Engineering Department at Karadeniz Technical University, Trabzon, Turkey in 1991. He received M.Sc. and Ph.D. in the same university in 1994 and 1998, respectively. Presently he is Associate Professor in the Civil Engineering Department at Karadeniz Technical University. His research interests are in the area of crack and contact problems in mechanics.

A. Osman Cakiroglu was born in Salpazari, Turkey, in 1946. He graduated from Civil Engineering Department at Karadeniz Technical University, Turkey, in 1967. He received M.Sc. and Ph.D. in the same department in 1968 and 1974, respectively. Presently, he is a Professor in the same department. His research interests include contact mechanics and fracture mechanics.