

# Construction Methods for Sign Patterns Allowing Nilpotence of Index $k$

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**Abstract**—In this paper, the smallest such integer  $k$  is called by the index (of nilpotence) of  $B$  such that  $B^k = 0$ . In this paper, we study sign patterns allowing nilpotence of index  $k$  and obtain four methods to construct sign patterns allowing nilpotence of index at most  $k$ , which generalizes some recent results.

**Keywords**—Sign pattern, Nilpotence, Jordan block.

## I. INTRODUCTION

THE sign of a real number  $a$ , denoted by  $\text{sgn}(a)$ , is defined to be 1,  $-1$  or 0, according to  $a > 0$ ,  $a < 0$ ,  $a = 0$ , respectively. A *sign pattern matrix* (or a *sign pattern*, for short) is a matrix whose entries are from the set  $\{1, -1, 0\}$ . The sign pattern of a real matrix  $B$ , denoted by  $\text{sgn}(B)$ , is the sign pattern matrix obtained from  $B$  by replacing each entry by its sign.

Let  $m$  be a positive integer. The integers  $a$  and  $b$  are congruent modulo  $m$  if and only if there is an integer  $t$  such that  $a = b + tm$  (for short, written as  $a \equiv b \pmod{m}$ ).

Let  $Q_n$  be the set of all sign patterns of order  $n$ . For  $A \in Q_n$ , the set of all real matrices with the same sign pattern as  $A$  is called the *qualitative class* of  $A$ , and is denoted by  $Q(A)$  ([2]).

Suppose that a real matrix has the property  $p$ . Then a sign pattern  $A$  is said to *require*  $p$  if every real matrix in  $Q(A)$  has property  $p$ , or to *allow*  $p$  if some real matrix in  $Q(A)$  has property  $p$  ([1]).

In this paper, we investigate the property  $N$  of being nilpotent. Recall that a real matrix  $B$  is said to be *nilpotent* if  $B^k = 0$  for some positive integer  $k$ . The smallest such integer  $k$  is called the *index* (of nilpotence) of  $B$ .

Let  $k$  be a positive integer. We now consider sign patterns that allow nilpotence of index at most  $k$ . These sign patterns that allow nilpotence, are also referred to as the potentially nilpotent sign patterns (see [1], [4], [5], [6]). For convenience, we denote the class of all sign patterns that allow nilpotence of index at most  $k$  by  $N_k$ . In [7], it is reported that it is an open problem to determine necessary and/or sufficient conditions for a sign pattern to allow nilpotence of index  $k \geq 4$ . Eschenbach and Li [4] studied  $N_2$  and Gao, Li and Shao [1] studied  $N_3$ . In this paper, we mainly extend these results to any  $N_k$ .

## II. PRELIMINARY

**Lemma 1** ([4]). *The set  $N_k$  is closed under the following operations:*

1) *negation;*

2) *transposition;*

3) *permutational similarity, and*

4) *signature similarity.*

As defined in [1], two sign patterns are equivalent if one can be obtained from the other by performing a sequence of operations listed in Lemma 1. This is indeed an equivalence relation.

**Lemma 2** ([1]). *A real matrix  $B$  is nilpotent if and only if its eigenvalues are equal to zero.*

Recall that a reducible (real or sign pattern) matrix is permutationally similar to a matrix in Frobenius normal form (see page 57 in [8]). Consequently, a reducible sign pattern  $A$  allows nilpotence if and only if each irreducible component (see [8]) of  $A$  allows nilpotence.

**Lemma 3.** *Let  $B$  be a nilpotent real matrix of index at most  $k$ , and  $J$  the Jordan form of  $B$ . Then each Jordan block in  $J$  is one of the following:*

$$J_1 = [0], \quad J_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{for } i = 2, 3, \dots, k.$$

Let  $A$  be a sign pattern matrix. The minimal rank of  $A$ , denoted by  $\text{mr}(A)$ , is defined as  $\text{mr}(A) = \min\{\text{rank} B : B \in Q(A)\}$  ([3]).

**Theorem 1.** *Let  $A \in Q_n$ . If  $A \in N_k$ , then*

$$\text{mr}(A) \leq \frac{k-1}{k}n.$$

**Proof.** Let  $A \in Q_n$  and  $A \in N_k$ . Then there exists a real matrix  $B \in Q(A)$  such that  $B^k = 0$ . By Lemma 3 we can assume that the Jordan form  $J$  of  $B$  is a direct sum of  $k_i$  copies of  $J_i$  ( $i = 1, 2, \dots, k$ ), where  $\sum_{i=1}^k ik_i = n$ . Then

$$\begin{aligned} \text{rank}(B) &= \text{rank}(J) = \sum_{i=1}^k (i-1)k_i \\ &\leq \frac{k-1}{k}k_1 + \frac{k-1}{k}2k_2 + \cdots + \frac{k-1}{k}kk_k = \frac{k-1}{k}n. \end{aligned}$$

Hence  $\text{mr}(A) \leq \text{rank}(B) \leq \frac{k-1}{k}n$ .  $\square$

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**Remark 1.** Note that the sign pattern

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

satisfies  $\text{mr}(A) = 3 \leq \frac{3}{4} \times 5$ . However,  $A^4 \neq 0$ ,  $A \notin N_4$ . So the condition in Theorem 1 is not a sufficient one.

**Theorem 2.** Let  $B$  be a real matrix of order  $n$  with  $\text{rank}(B) = r$ . Then  $B^4 = 0$  if and only if there exist nonnegative integers  $l, m$  and nonzero real column vectors  $\alpha_1, \alpha_2, \dots, \alpha_r$  and  $\beta_1, \beta_2, \dots, \beta_r$  of order  $n$  with  $l \leq \frac{r}{3}$ ,  $m \leq \frac{r}{2}$ ,  $2r - 2l - m \leq n$  and

$$\beta_j^T \alpha_i = \begin{cases} 1 & j \equiv 1(\text{mod } 3), 1 \leq j \leq 3l - 1, \text{ and } i = j + 1, \\ 1 & j \equiv 2(\text{mod } 3), 1 \leq j \leq 3l - 1, \text{ and } i = j + 1, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

such that

$$B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T. \quad (2)$$

**Proof. Sufficiency.** Let  $B = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T$ . By (1), we have

$$\begin{aligned} B^2 &= (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T) \\ &\quad (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T) \\ &= \alpha_1 \beta_2^T + \alpha_2 \beta_3^T + \alpha_4 \beta_5^T + \dots + \alpha_{3l-1} \beta_{3l}^T, \\ B^3 &= (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T) \\ &\quad (\alpha_1 \beta_2^T + \alpha_2 \beta_3^T + \alpha_4 \beta_5^T + \dots + \alpha_{3l-1} \beta_{3l}^T) \\ &= \alpha_1 \beta_3^T + \alpha_4 \beta_6^T + \dots + \alpha_{3l-2} \beta_{3l}^T \end{aligned}$$

and

$$\begin{aligned} B^4 &= (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T) \\ &\quad (\alpha_1 \beta_3^T + \alpha_4 \beta_6^T + \dots + \alpha_{3l-2} \beta_{3l}^T) \\ &= 0. \end{aligned}$$

**Necessity.** Let  $B^4 = 0$  with  $\text{rank}(B) = r$ . By Lemma 3, the Jordan form  $J$  of  $B$  is a direct sum of  $l$  copies of  $J_4$ ,  $m$  copies of  $J_3$ ,  $r - 3l - 2m$  copies of  $J_2$  and  $n - 4l - 3m - 2(r - 3l - 2m) = n - 2r + 2l + m$  copies of  $J_1$ , where  $0 \leq l \leq \frac{r}{3}$ ,  $0 \leq m \leq \frac{r}{2}$  and  $2r - 2l - m \leq n$ . It implies that there exists a nonsingular real matrix  $D$  of order  $n$  such that

$$\begin{aligned} D^{-1}BD &= J \\ &= \begin{bmatrix} J_{11} & & & \\ & J_{22} & & \\ & & \ddots & \\ & & & J_{n-r+2m+4l, n-r+2m+4l} \end{bmatrix}. \end{aligned} \quad (3)$$

where

$$J_{11} = \dots = J_{ll} = J_4, \quad J_{l+1, l+1} = \dots = J_{l+m, l+m} = J_3,$$

$$J_{l+m+1, l+m+1} = \dots = J_{r-m-2l, r-m-2l} = J_2,$$

and

$$\begin{aligned} J_{r-m-2l+1, r-m-2l+1} &= J_{r-m-2l+2, r-m-2l+2} \\ &= \dots = J_{n-r+2m+4l, n-r+2m+4l} = J_1. \end{aligned}$$

Write

$$D = (u_1, u_2, \dots, u_n) \text{ and } D^{-1} = (v_1, v_2, \dots, v_n)^T,$$

where  $u_1, u_2, \dots, u_n$  are column vectors of  $D$  and  $v_1, v_2, \dots, v_n$  are column vectors of  $D^{-1}$ . Clearly,  $v_i^T u_i = 1$ , for  $i = 1, 2, \dots, n$ , and  $v_j^T u_i = 0$ , for  $i \neq j$ . Let

$$\alpha_{3i-2} = u_{4i-3}, \quad \alpha_{3i-1} = u_{4i-2}, \quad \alpha_{3i} = u_{4i-1} \text{ for } i = 1, 2, \dots, l,$$

$$\alpha_{3l+2j-1} = u_{4l+3j-2}, \quad \alpha_{3l+2j} = u_{4l+3j-1}, \text{ for } j = 1, 2, \dots, m,$$

$$\alpha_{3l+2m+s} = u_{4l+3m+2s-1}, \text{ for } s = 1, 2, \dots, r - 3l - 2m,$$

$$\beta_{3i-2} = v_{4i-3}, \quad \beta_{3i-1} = v_{4i-2}, \quad \beta_{3i} = v_{4i-1} \text{ for } i = 1, 2, \dots, l,$$

$$\beta_{3l+2j-1} = v_{4l+3j-2}, \quad \beta_{3l+2j} = v_{4l+3j-1}, \text{ for } j = 1, 2, \dots, m,$$

$$\beta_{3l+2m+s} = v_{4l+3m+2s-1}, \text{ for } s = 1, 2, \dots, r - 3l - 2m.$$

It is easy to see that  $\alpha_i$  and  $\beta_i$  satisfy the condition (1). By (3), we have

$$B = DJD^{-1} = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T.$$

The conclusion follows.  $\square$

Next, we generalize the above result to any  $B^k = 0$ , that is,  $N_k$ .

**Theorem 3.** Let  $B$  be a real matrix of order  $n$  with  $\text{rank}(B) = r$ . Then  $B^k = 0$  if and only if there exist nonnegative integers  $l_1, l_2, \dots, l_k$  and nonzero real column vectors

$\alpha_1, \alpha_2, \dots, \alpha_r$  and  $\beta_1, \beta_2, \dots, \beta_r$  of order  $n$  with  $\sum_{i=1}^k il_i = n$ ,  $\sum_{i=1}^k (i-1)l_i = r$ , and

$$\beta_j^T \alpha_i = \begin{cases} 1, & j \equiv s(\text{mod } k-1), s = 1, 2, \dots, k-2, \\ & 1 \leq j \leq (k-1)l_k - 1, i = j + 1, \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

such that

$$B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T. \quad (5)$$

**Proof. Sufficiency.** Let  $B = \alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T$ . By (4), we have

$$\begin{aligned} B^2 &= (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T) \\ &\quad (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T) \\ &= (\alpha_1 \beta_2^T + \alpha_2 \beta_3^T + \dots + \alpha_{k-2} \beta_{k-1}^T) \\ &\quad + (\alpha_k \beta_{k+1}^T + \dots + \alpha_{2k-3} \beta_{2k-2}^T) + \dots + \\ &\quad (\alpha_{(k-1)(l_k-1)+1} \beta_{(k-1)(l_k-1)+2}^T + \dots + \\ &\quad \alpha_{(k-1)l_k-1} \beta_{(k-1)l_k}^T), \\ B^{k-1} &= (\alpha_1 \beta_1^T + \alpha_2 \beta_2^T + \dots + \alpha_r \beta_r^T) [(\alpha_1 \beta_{k-2}^T \\ &\quad + \alpha_2 \beta_{k-1}^T) + (\alpha_k \beta_{2k-3}^T + \alpha_{k+1} \beta_{2k-2}^T) + \dots + \\ &\quad (\alpha_{(k-2)l_k+1} \beta_{(k-1)l_k-1}^T + \alpha_{(k-2)l_k+2} \beta_{(k-1)l_k}^T)] \\ &= \alpha_1 \beta_{k-1}^T + \alpha_k \beta_{k+1}^T + \dots + \alpha_{(k-2)l_k+1} \beta_{(k-1)l_k}^T \end{aligned}$$

and

$$\begin{aligned} B^k &= BB^{k-1} \\ &= (\alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T)(\alpha_1\beta_{k-1}^T \\ &\quad + \alpha_k\beta_{k+1}^T + \cdots + \alpha_{(k-2)l_k+1}\beta_{(k-1)l_k}^T) \\ &= 0. \end{aligned}$$

*Necessity.* Let  $B^k = 0$  with  $\text{rank}(B) = r$ . By Lemma 3, the Jordan form  $J$  of  $B$  is a direct sum of  $l_i$  copies of  $J_i$ , where  $\sum_{i=1}^k il_i = n$ ,  $\sum_{i=1}^k (i-1)l_i = r$  and it implies that there exists a nonsingular real matrix  $D$  of order  $n$  such that

$$D^{-1}BD = J = \begin{bmatrix} J_{11} & & & & \\ & J_{22} & & & \\ & & \ddots & & \\ & & & J_{n-r-1, n-r-1} & \\ & & & & J_{n-r, n-r} \end{bmatrix}, \quad (6)$$

where

$$J_{11} = \cdots = J_{l_k l_k} = J_k,$$

$$J_{l_k+1, l_k+1} = \cdots = J_{l_k+l_{k-1}, l_k+l_{k-1}} = J_{k-1},$$

$\cdots,$

$$J_{1+\sum_{i=2}^k l_i, 1+\sum_{i=2}^k l_i} = \cdots = J_{n-r, n-r} = J_1.$$

Write

$$D = (u_1, u_2, \cdots, u_n) \text{ and } D^{-1} = (v_1, v_2, \cdots, v_n)^T,$$

where  $u_1, u_2, \cdots, u_n$  are column vectors of  $D$  and  $v_1, v_2, \cdots, v_n$  are column vectors of  $D^{-1}$ ,  $v_i^T u_i = 1$ , for  $i = 1, 2, \cdots, n$ , and  $v_j^T u_i = 0$ , for  $i \neq j$ . Let

$$\alpha_{(s-1)i+m+\sum_{j=s+1}^k jl_j} = u_{si+m+\sum_{j=s+1}^k jl_j},$$

$$\alpha_{(s-1)i+m+\sum_{j=s+1}^k jl_j} = u_{si+m+\sum_{j=s+1}^k jl_j}$$

for  $s = 1, 2, \cdots, k$ ,  $i = 1, 2, \cdots, l_s - 1$ , and  $m = 1, 2, \cdots, s - 1$ . It is easy to see that  $\alpha_i$  and  $\beta_i$  satisfy the condition (4). By (6), we have that

$$B = DJD^{-1} = \alpha_1\beta_1^T + \alpha_2\beta_2^T + \cdots + \alpha_r\beta_r^T.$$

The proof is completed.  $\square$

### III. MAIN RESULTS

Based on the above analysis, one can obtain the following construction methods to find a sign pattern in  $N_k$ .

#### A. Construction Method 1—Jordan Method

By Lemma 3, we may obtain the Jordan form method to construct a sign patterns in  $N_k$ . For example, let

$$J = \begin{bmatrix} J_4 & \\ & J_2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 & 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D^{-1} = \begin{bmatrix} -1 & -1 & -2 & 2 & 2 & 3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 & -2 & -2 \\ -1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

Note that

$$B = DJD^{-1} = \begin{bmatrix} 0 & 2 & 2 & 0 & -2 & -1 \\ 1 & 0 & 1 & -1 & -1 & -2 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -2 & 1 & 0 & 2 & 1 & 3 \\ 1 & 0 & 1 & -1 & -1 & -2 \\ 1 & 1 & 1 & -1 & -1 & -2 \end{bmatrix}, \quad B^4 = 0,$$

Then

$$A = \text{sgn}(B) = \begin{bmatrix} 0 & 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \end{bmatrix}.$$

#### B. Construction Method 2—vectors spanning method

Let  $l_1, l_2, \cdots, l_k$  be nonnegative integers with  $\sum_{i=1}^k il_i = n$ ,  $\sum_{i=1}^k (i-1)l_i = r$ . Let real column vectors  $\alpha_1, \alpha_2, \cdots, \alpha_r$  and  $\beta_1, \beta_2, \cdots, \beta_r$  of order  $n$  satisfy the condition

$$\beta_j^T \alpha_i = \begin{cases} 1 & j \equiv s(\text{mod } k-1), s = 1, 2, \cdots, k-2, \\ 1 \leq j \leq (k-1)l_k - 1, i = j+1, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

By Theorem 3, the real matrix

$$B = \sum_{1 \leq i \leq r} \alpha_i \beta_i^T \quad (8)$$

is nilpotent of index at most  $k$ , and its sign pattern is in  $N_k$ . For example, let  $n = 8$ ,  $r = 6$ ,  $l = m = 1$ ,

$$\alpha_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$\alpha_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \alpha_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \alpha_6 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

$$\beta_1 = (1, 1, 0, -1, 0, 0, -1, -1), \beta_2 = (-1, 1, 2, -1, 1, -1, -1, 1),$$

$$\beta_3 = (-1, 0, 0, 0, 0, 0, 1, 0), \beta_4 = (2, 0, 0, -1, 1, 0, -1, -1),$$

$$\beta_5 = (1, -1, -1, 1, -1, 1, -1, 1), \beta_6 = (-1, 0, 0, 0, 0, 0, 0, 1),$$

$$B = \sum_{1 \leq i \leq 6} \alpha_i \beta_i = \begin{bmatrix} 1 & 2 & 1 & -3 & 1 & 0 & -4 & 1 \\ 2 & 5 & 5 & -7 & 4 & -2 & -8 & 1 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 2 & 2 & 2 & -3 & 2 & -1 & -3 & -1 \\ 3 & 1 & 1 & -3 & 1 & 0 & -4 & 0 \\ -1 & 2 & 3 & -3 & 2 & -1 & -4 & 3 \\ 1 & 2 & 1 & -3 & 1 & 0 & -4 & 1 \\ 1 & 2 & 1 & -3 & 1 & 0 & -4 & 1 \end{bmatrix},$$

$$D^4 = 0.$$

Then

$$A = \text{sgn}(B) \in N_4.$$

### C. Construction Method 3—block method

**Theorem 4.** Suppose  $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \in N_k$ , where  $A_1$  and  $A_4$  are square, then for any positive integer  $m$ , we have

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 & \cdots & A_2 \\ A_3 & A_4 & \cdots & A_4 \\ \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & \cdots & A_4 \end{bmatrix} \in N_k,$$

where  $\tilde{A}$  has  $(m+1)^2$  blocks.

**Proof.** Note that, if  $A \in N_k$ , then there is a real matrix

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \in Q(A),$$

where  $B_i \in Q(A_i) (i = 1, 2, 3, 4)$ , such that  $B^k = 0$ .

Let

$$B^j = \begin{bmatrix} f_{1j}(B_1, B_2, B_3, B_4) & f_{2j}(B_1, B_2, B_3, B_4) \\ f_{3j}(B_1, B_2, B_3, B_4) & f_{4j}(B_1, B_2, B_3, B_4) \end{bmatrix}$$

for  $j = 1, 2, \dots, k-1$ .

For short, we denote

$$B^j = \begin{bmatrix} f_{1j} & f_{2j} \\ f_{3j} & f_{4j} \end{bmatrix}$$

for  $j = 1, 2, \dots, k-1$ . And

$$\tilde{B} = \begin{bmatrix} B_1 & \frac{1}{m}B_2 & \cdots & \frac{1}{m}B_2 \\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \\ \vdots & \vdots & \ddots & \vdots \\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \end{bmatrix}.$$

When  $k = 2$ , it follows that

$$B^2 = \begin{bmatrix} B_1^2 + B_2B_3 & B_1B_2 + B_2B_4 \\ B_3B_1 + B_4B_3 & B_3B_2 + B_4^2 \end{bmatrix},$$

$$\tilde{B}^2 = \begin{bmatrix} B_1^2 + B_2B_3 & \frac{1}{m}(B_1B_2 + B_2B_4) \cdots \frac{1}{m}(B_1B_2 + B_2B_4) \\ B_3B_1 + B_4B_3 & \frac{1}{m}(B_1B_2 + B_2B_4) \cdots \frac{1}{m}(B_1B_2 + B_2B_4) \\ \vdots & \vdots \\ B_3B_1 + B_4B_3 & \frac{1}{m}(B_1B_2 + B_2B_4) \cdots \frac{1}{m}(B_1B_2 + B_2B_4) \end{bmatrix}$$

$$= \begin{bmatrix} f_{12} & \frac{1}{m}f_{22} & \cdots & \frac{1}{m}f_{22} \\ f_{32} & \frac{1}{m}f_{42} & \cdots & \frac{1}{m}f_{42} \\ \vdots & \vdots & \ddots & \vdots \\ f_{32} & \frac{1}{m}f_{42} & \cdots & \frac{1}{m}f_{42} \end{bmatrix}.$$

So  $\tilde{B}^2 = 0$ . Thus  $\tilde{A} \in N_k$ .

Suppose that we have

$$\tilde{B}^s = \begin{bmatrix} f_{1s} & \frac{1}{m}f_{2s} & \cdots & \frac{1}{m}f_{2s} \\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s} \end{bmatrix} \text{ for } 2 \leq s < k,$$

then

$$B^{s+1} = B^s B = \begin{bmatrix} f_{1s} & f_{2s} \\ f_{3s} & f_{4s} \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$$

$$= \begin{bmatrix} f_{1s}B_1 + f_{2s}B_3 & f_{1s}B_2 + f_{2s}B_4 \\ f_{3s}B_1 + f_{4s}B_3 & f_{3s}B_2 + f_{4s}B_4 \end{bmatrix}$$

and

$$\tilde{B}^{s+1} = \begin{bmatrix} f_{1s} & \frac{1}{m}f_{2s} & \cdots & \frac{1}{m}f_{2s} \\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s} \\ \vdots & \vdots & \ddots & \vdots \\ f_{3s} & \frac{1}{m}f_{4s} & \cdots & \frac{1}{m}f_{4s} \end{bmatrix} \begin{bmatrix} B_1 & \frac{1}{m}B_2 & \cdots & \frac{1}{m}B_2 \\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \\ \vdots & \vdots & \ddots & \vdots \\ B_3 & \frac{1}{m}B_4 & \cdots & \frac{1}{m}B_4 \end{bmatrix}$$

$$= \begin{bmatrix} f_{1s}B_1 + f_{2s}B_3 & \frac{1}{m}(f_{1s}B_2 + f_{2s}B_4) & \cdots & \frac{1}{m}(f_{1s}B_2 + f_{2s}B_4) \\ f_{3s}B_1 + f_{4s}B_3 & \frac{1}{m}(f_{3s}B_2 + f_{4s}B_4) & \cdots & \frac{1}{m}(f_{3s}B_2 + f_{4s}B_4) \\ \vdots & \vdots & \ddots & \vdots \\ f_{3s}B_1 + f_{4s}B_3 & \frac{1}{m}(f_{3s}B_2 + f_{4s}B_4) & \cdots & \frac{1}{m}(f_{3s}B_2 + f_{4s}B_4) \end{bmatrix}.$$

So

$$B^k = \begin{bmatrix} f_{1(k-1)}B_1 + f_{2(k-1)}B_3 & f_{1(k-1)}B_2 + f_{2(k-1)}B_4 \\ f_{3(k-1)}B_1 + f_{4(k-1)}B_3 & f_{3(k-1)}B_2 + f_{4(k-1)}B_4 \end{bmatrix} \in N_k,$$

$$\tilde{B}^k = 0.$$

By the principle of mathematical induction, we have  $\tilde{A} \in N_k$ .  $\square$

### D. Construction Method 4—null space method

**Theorem 5.** Let  $B$  and  $C$  be nilpotent real matrices of indices of  $k$  with order  $n_1$  and  $n_2$ , respectively. Let  $p$  be a positive integer. The kernel of a matrix  $B$ , denoted by  $\text{Ker}(B)$ , also called the null space, is the kernel of the linear map defined by the matrix  $B$ . Suppose that the following conditions hold:

$$u_1, u_2, \dots, u_p \in \text{Ker}(B^i), v_1, v_2, \dots, v_p \in \text{Ker}((C^j)^T), \quad (9)$$

where  $1 \leq i < k$ ,  $1 \leq j < k$ , and  $i + j \leq k$ . Then the following partitioned block real matrix of order  $n_1 + n_2$

$$D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$$

is nilpotent of index at most  $k$  and  $A = \text{sgn}(D) \in N_k$ , where  $X = u_1 v_1^T + u_2 v_2^T + \cdots + u_p v_p^T$ .

**Proof.** In fact, let  $D = \begin{bmatrix} B & X \\ 0 & C \end{bmatrix}$ , where  $B$  and  $C$  are square. Then

$$D^k = \begin{bmatrix} B^k & B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} \\ 0 & C^k \end{bmatrix}.$$

Thus  $D^k = 0$  if and only if  $B^k = 0$ ,  $C^k = 0$  and

$$B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} = 0.$$

It is obvious that  $B^k = 0$  and  $C^k = 0$ . In addition, we observe that

$$\begin{aligned} & B^{k-1}X + B^{k-2}XC + \cdots + XC^{k-1} \\ &= B^{k-2}(Bu_1 v_1^T + Bu_2 v_2^T + \cdots + Bu_p v_p^T) \\ & \quad + B^{k-3}(u_1, \cdots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C \\ & \quad + \cdots + (u_1, \cdots, u_p) \begin{pmatrix} v_1^T \\ \vdots \\ v_p^T \end{pmatrix} C^{k-1}. \end{aligned}$$

Therefore, we get the desired result with the above condition (9).  $\square$

#### IV. CONCLUSION

In this paper, sign patterns allowing nilpotence of index at most  $k$  are researched and four methods to construct sign patterns under the condition that allows nilpotence of index at most  $k$  are obtained, which generalizes some recent results in [1], [4] and has a certain theoretical and practical value.

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