

# Consensus of Multi-Agent Systems under the Special Consensus Protocols

Konghe Xie

**Abstract**—Two consensus problems are considered in this paper. One is the consensus of linear multi-agent systems with weakly connected directed communication topology. The other is the consensus of nonlinear multi-agent systems with strongly connected directed communication topology. For the first problem, a simplified consensus protocol is designed: Each child agent can only communicate with one of its neighbors. That is, the real communication topology is a directed spanning tree of the original communication topology and without any cycles. Then, the necessary and sufficient condition is put forward to the multi-agent systems can be reached consensus. It is worth noting that the given conditions do not need any eigenvalue of the corresponding Laplacian matrix of the original directed communication network. For the second problem, the feedback gain is designed in the nonlinear consensus protocol. Then, the sufficient condition is proposed such that the systems can be achieved consensus. Besides, the consensus interval is introduced and analyzed to solve the consensus problem. Finally, two numerical simulations are included to verify the theoretical analysis.

**Keywords**—Consensus, multi-agent systems, directed spanning tree, the Laplacian matrix.

## I. INTRODUCTION

OVER the last few decades, consensus has been an important research topic in the field of cooperative control of multi-agent systems. A consensus concept is that multiple agents reach a common goal by using the information of neighbors. The consensus problem plays an important role in studying the behaviors of multi-agent systems. Consensus research areas include: attitude control of satellites, sensor networks and formation control etc.

Some basic consensus issues of multi-agent systems are introduced and analyzed in [1] such as consensus in discrete-time,  $f$ -consensus problems, consensus in switching networks, iterative consensus and weighted average consensus. In [2], the consensus problem of linear multi-agent systems is considered. A new method based on observer-type protocol is introduced. The protocol is distributed and based on the relative output information of neighbors. In [3], the consensus of continuous-time and discrete-time linear multi-agent systems are studied. The consensus protocols are designed based on the observer and reduced order. In [4], the edge- and node-based protocols are designed to solve the consensus problem of multi-agent systems with linear dynamics. In both protocols, the coupling weight is variable over time. In [5] and [6], a  $r$ -consensus problem of the high-order multi-agent systems is introduced. The corresponding local control protocols are designed to solve consensus problems. In [7], consensus of multi-agent systems

with linear dynamics and high-order is studied by designing a consensus protocol. In the protocol, the state of each agent relies on other partial relevant states at the current time. In [8], the second-order consensus of nonlinear multi-agent systems is considered. The ability to achieve consensus is described by a new generalized algebraic connectivity. In [9] and [10], the pinning control of complex networks is studied by using the properties of M-matrix. A pinning plan is proposed which includes what class of nodes must be pinned and what class of nodes should be preferentially pinned. In [11], the distributed consensus algorithms are proposed to achieve the consensus of multi-agent networks. A class of smooth functions is designed in the consensus algorithms that can achieve consensus. In [12], the consensus problem of high-order linear multi-agent systems is studied by a linear transformation. The gain matrices are designed in the consensus protocol.

In this paper, some consensus problems of the general linear and nonlinear multi-agent systems are studied. The case of linear system, the corresponding communication topology has a directed spanning tree. A simplified consensus protocol is designed. In the protocol, each child agent can only obtain information from one of its neighbor agents. Under the simplified consensus protocol, the real communication topology is a minimum directed spanning tree. The complicated exchange of information in the communication topology can be reduced. By derivation, the necessary and sufficient conditions are obtained such that the multi-agent systems can be reached consensus. The case of nonlinear system, the feedback gain is designed in the consensus protocol. After some calculations, the sufficient condition without needing any global information is obtained.

The paper is roughly divided into several major parts as follows. Some correlative notations and preliminaries are reviewed in Section II. In Section III, the necessary and sufficient conditions are obtained to solve the consensus problem in linear systems. The concept of consensus interval is also introduced. The consensus problem of nonlinear systems with the consensus protocol having the feedback gain is studied in Section IV. Simulation results are presented in Section V to verify the theoretical analysis. Finally, conclusions are stated in Section VI.

## II. PRELIMINARIES AND NOTATIONS

Let  $R^{n \times n}$  denote the set of real matrices with  $n \times n$  dimensional.  $M$  is a matrix,  $M^T$  denotes its transpose.  $I_N$  represents an  $N$ -dimensional identity matrix,  $\mathbf{1}$  denotes the column vector whose elements are all ones. If the real parts

Konghe Xie is with Qingdao University of Science and Technology, Qingdao, Shandong, 266061, China (e-mail: 948860277@qq.com).

of all the eigenvalues of a matrix are negative, then the matrix is Hurwitz. For the matrices  $X$  and  $Y$ ,  $X \otimes Y$  denotes their Kronecker product.

The communication topology of a multi-agent network can be described by a directed graph  $G = (V, E)$  where  $V$  denotes a node set,  $E \subseteq V \times V$  denotes an edge set.  $(j, i) \in E$  means the node  $j$  has a directed edge to the node  $i$ . The neighbors of agent  $i$  are denoted by  $N_i = \{j \in V : (j, i) \in E\}$ . For a digraph, a directed spanning tree contains all the nodes of the digraph and it is a directed tree. If a directed graph exists a node that has a directed path to every other node, then we can call the directed graph has a directed spanning tree.

Suppose that a digraph has  $n$  nodes.  $W = (w_{ij}) \in R^{n \times n}$  denotes the adjacency matrix of the digraph, where  $w_{ii} = 0$ ,  $w_{ij} = 1$  if  $(j, i) \in E$  and  $w_{ij} = 0$  otherwise.  $L = (l_{ij}) \in R^{n \times n}$  denotes the corresponding Laplacian matrix, where  $l_{ii} = \sum_{j \neq i} w_{ij}$  and  $l_{ij} = -w_{ij}$ ,  $j \neq i$ .

**Lemma 1.** [2] *The Laplacian matrix has a zero eigenvalue, the corresponding right eigenvector is  $\mathbf{1}$  and the other eigenvalues have positive real parts. A digraph has a directed spanning tree if and only if zero is a simple eigenvalue of the corresponding Laplacian matrix.*

**Lemma 2.** (Hurwitz criterion) *Consider a polynomial equation*

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0,$$

*the necessary and sufficient condition of all roots having negative real part is*

$$T_k = \begin{vmatrix} a_1 & 1 & 0 & \dots & 0 \\ a_3 & a_2 & a_1 & \dots & 0 \\ a_5 & a_4 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2k-1} & a_{2k-2} & a_{2k-3} & \dots & a_k \end{vmatrix} > 0,$$

$k = 1, 2, \dots, n$ ,  $a_j = 0$  if  $j > n$ .

**Lemma 3.** [8] *For the matrices  $E, F, G$  and  $H$  having appropriate dimensions, they satisfy:*

- 1)  $(\gamma E) \otimes F = E \otimes (\gamma F)$ , where  $\gamma$  is a constant.
- 2)  $(E + F) \otimes G = E \otimes G + F \otimes G$ .
- 3)  $(E \otimes F)(G \otimes H) = (EG) \otimes (FH)$ .
- 4)  $(E \otimes F)^T = E^T \otimes F^T$ .

**Lemma 4.** [8] *For a network, the adjacency matrix is irreducible if and only if the network is strongly connected.*

**Lemma 5.** [8] *If a matrix  $L_{N \times N}$  is irreducible, then there exists a positive left eigenvector  $c = (c_1, c_2, \dots, c_N)^T$  satisfies  $c^T L = 0$ .*

**Lemma 6.** [10] *For real vectors  $u = (u_1, u_2, \dots, u_n)^T$ ,  $v = (v_1, v_2, \dots, v_n)^T$  and a nonnegative matrix  $M = (m_{ij})_{n \times n}$  with  $m_{ij} \geq 0$ , having:*

$$u^T M v \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (m_{ij} u_i^2 + m_{ji} v_j^2).$$

### III. CONSENSUS UNDER THE SIMPLIFIED PROTOCOL

In this section, we consider a consensus problem of systems with  $N$  agents and linear dynamics. The dynamics of agent  $i$  as in [2] can be described by

$$\dot{x}_i = P x_i + Q u_i, i = 1, \dots, N \quad (1)$$

where  $x_i \in R^n$  and  $u_i \in R^n$  denote the state and control input of agent  $i$ , respectively.  $P$  and  $Q$  are constant  $n \times n$  dimensional matrices. Assume that  $(P, Q)$  is stabilizable and detectable.

The communication topology can be denoted by a directed graph  $G$ . Suppose that the directed graph  $G$  has a directed spanning tree. The information exchange between the agents can be restricted to the directed spanning tree. The real communication topology is the directed spanning tree. Each child node of the simplified graph can only get information from one of its neighbors except the root node. The exchange of information in the original directed graph  $G$  is simplified. Thus, every child node can obtain information from the root node easily and the complex exchange of information can be reduced. Let the root node number be 1 and the numbers of other nodes be named according to their distance from the root node.

The simplified consensus protocol is given by

$$\begin{aligned} \dot{x}_i &= P x_i + Q u_i \\ u_i &= \begin{cases} 0 & i = 1 \\ \sigma(x_i - x_j) & i = 2, \dots, N \end{cases} \end{aligned} \quad (2)$$

where the real number  $\sigma > 0$  denotes the coupling strength, agent  $j \in N_i$ , agent  $i$  can obtain information from agent  $j$  and  $j < i$ . Under the simplified consensus protocol, each child agent can only obtain information from one of its neighbors.

By (2), (1) can be written as

$$\dot{x} = (I_N \otimes P + \sigma L \otimes Q)x \quad (3)$$

where  $x = (x_1^T, \dots, x_N^T)^T$  and  $L \in R^{N \times N}$  is the Laplacian matrix of the simplified directed graph.  $L$  has a simple zero-eigenvalue, and the other eigenvalues are all ones.

If the states of (3) satisfy

$$x_i(t) \rightarrow x_j(t) \quad \forall i, j = 1, \dots, N, \quad \text{as } t \rightarrow \infty, \quad (4)$$

one can say (2) solves the consensus problem.

Let  $c^T = (c_1, \dots, c_N)^T$  denote a left eigenvector of  $L$  that satisfies  $c^T L = 0$  and  $c^T \mathbf{1}_N = 1$ . Then, the error variables can be given by:

$$\ell(t) = ((I_N - \mathbf{1}_N c^T) \otimes I_n)x(t) \quad (5)$$

where  $\ell = (\ell_1^T, \dots, \ell_N^T)^T$  that satisfies  $(c^T \otimes I_n)\ell = 0$ . One can obtain the following error dynamical systems:

$$\dot{\ell} = (I_N \otimes P + \sigma L \otimes Q)\ell. \quad (6)$$

The following theorem presents a necessary and sufficient condition that solves the consensus problem.

**Theorem 1.** *Suppose that the multi-agent network (3) whose communication topology  $G$  has a directed spanning tree. Consensus in the dynamic protocol (2) can be reached if and only if the matrix  $P + \sigma Q$  is Hurwitz.*

*Proof:* Obviously,  $I_N - \mathbf{1}_N c^T$  has a simple zero-eigenvalue, the corresponding right eigenvector is  $\mathbf{1}$ , and 1 is the other eigenvalue,

the multiplicity is  $N-1$ . Hence, the consensus in (3) can be reached if and only if  $\ell(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

Due to  $G$  has a directed spanning tree, by Lemma 1 and the simplified consensus protocol, zero is a simple eigenvalue of  $L$  and the other eigenvalues are all ones. Let  $A \in R^{N \times (N-1)}$ ,  $B \in R^{(N-1) \times N}$ ,  $C \in R^{N \times N}$ , and lower triangular  $D \in R^{(N-1) \times (N-1)}$

satisfy  $C = (\mathbf{1} \ A)$ ,  $C^{-1} = \begin{pmatrix} c^T \\ B \end{pmatrix}$ ,  $C^{-1}LC = J = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ , where the diagonal entries of  $D$  are all ones. Let  $\delta = (C^{-1} \otimes I_n)\ell$ , where  $\ell$  is defined in (6) and  $\delta = (\delta_1^T, \dots, \delta_N^T)^T$ . Then, (6) can be written as:

$$\dot{\delta} = (I_N \otimes P + \sigma J \otimes Q)\delta. \quad (7)$$

Due to  $\delta_1 = (c^T \otimes I_n)\ell \equiv 0$ , then the  $N-1$  subsystems along the diagonal

$$\dot{\delta}_i = (P + \sigma Q)\delta_i, i = 2, \dots, N \quad (8)$$

are asymptotically stable if and only if (7) are asymptotically stable. By  $P + \sigma Q$  is Hurwitz, the proof is completed. ■

**Remark 1.** Theorem 1 implies that the consensus problem of multi-agent systems can be seen as the consensus problem of multiple subsystems. If each subsystem of multi-agent systems can be reached consensus, then multi-agent systems can be reached too. If the communication graph has a disconnected node, obviously, the consensus cannot be reached. The given communication graph has a directed spanning tree, the condition is general and weak.

It is worth mentioning that the necessary and sufficient condition of consensus in [2] need global information. The consensus is affected by eigenvalues of the relevant Laplacian matrix. In this paper, the given the necessary and sufficient condition of consensus does not need any global information, thus the condition is better.

**Remark 2.** In Theorem 1, a simplified consensus protocol is used, which is based on child node can only obtain information from one of neighbor nodes. From the conditions of Theorem 1, we know that how to choose the coupling strength  $\sigma$ , matrices  $P$  and  $Q$  are the key points. The coupling strength is similar to the coupling strength in [2], [3] and [4]. For the stabilizable and detectable matrices  $(P, Q)$ , the only additional task is appropriately adjusting the coupling strength  $\sigma$ . In the following section A, the concept of consensus interval is introduced and analyzed for solving the problem and the Hurwitz criterion is used for solving consensus interval problems.

**Theorem 2.** For the systems (3), if conditions in Theorem 1 are satisfied, then

$$x_i(t) \rightarrow \phi(t) \triangleq (c^T \otimes e^{Pt}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix}, \quad i = 1, 2, \dots, N, \quad \text{as } t \rightarrow \infty \quad (9)$$

where  $c \in R^N$  satisfies  $c^T L = 0$  and  $c^T \mathbf{1} = 1$ .

*Proof:* The solution of (3) can be obtained as

$$\begin{aligned} x(t) &= e^{(I_N \otimes P + \sigma L \otimes Q)t} x(0) \\ &= (C \otimes I_n) e^{(I_N \otimes P + \sigma J \otimes Q)t} (C^{-1} \otimes I_n) x(0) \\ &= (C \otimes I_n) \begin{bmatrix} e^{Pt} & 0 \\ 0 & e^{(I_{N-1} \otimes P + \sigma D \otimes Q)t} \end{bmatrix} (C^{-1} \otimes I_n) x(0) \end{aligned} \quad (10)$$

where matrices  $C, J$  and  $D$  are defined in the proof of Theorem 1, and from the conditions of Theorem 1, one knows that  $I_{N-1} \otimes P + \sigma D \otimes Q$  is Hurwitz. Thus, one can obtain the following from (10):

$$\begin{aligned} e^{(I_N \otimes P + \sigma L \otimes Q)t} &\rightarrow (C \otimes I_n) \begin{bmatrix} e^{Pt} & 0 \\ 0 & 0 \end{bmatrix} (C^{-1} \otimes I_n) \\ &= (\mathbf{1} \otimes I_n) e^{Pt} (c^T \otimes I_n) \\ &= (\mathbf{1} c^T) \otimes e^{Pt}, \quad \text{as } t \rightarrow \infty \end{aligned} \quad (11)$$

implying that

$$x_i \rightarrow (c^T \otimes e^{Pt}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix}, \quad \text{as } t \rightarrow \infty,$$

where  $i = 1, 2, \dots, N$ . ■

**Remark 3.** In Theorem 2, the consensus value problem of multi-agent systems has been solved. If  $P$  is Hurwitz, then the consensus value is zero. If one of the eigenvalues of matrix  $P$  have positive real parts, the agents can reach consensus at a nonzero value under the protocol (2).

#### A. Consensus Interval

Clearly, the stability of systems (8) depends on the coupling strength  $\sigma$ . For an interval  $S$ , when the coupling strength  $\sigma \in S$ , (8) is asymptotically stable, then the interval  $S$  can be called the consensus interval. Consensus in (3) can be reached if and only if  $\sigma \in S$ .  $S$  can be an interval or several intervals. The stability of the matrix  $P + \sigma Q$  can be determined by the consensus interval  $S$ . The consensus interval problem can be solved by using Hurwitz criterion.

### IV. CONSENSUS OF NONLINEAR MULTI-AGENT SYSTEMS

In this section, the consensus problem of nonlinear multi-agent systems is considered and the corresponding communication graph is strongly connected. The multi-agent systems with nonlinear dynamics can be described by

$$\dot{x}_i(t) = f(t, x_i(t)) + u_i(t), i = 1, 2, \dots, N, \quad (12)$$

where  $x_i = (x_{i1}, x_{i2}, \dots, x_{in})^T$  is the state of the  $i$ th agent,  $f(t, x_i) = (f_1(t, x_i), f_2(t, x_i), \dots, f_n(t, x_i))^T \in R^n$  is a continuous vector-valued function, the function describes inherent dynamics of the  $i$ th agent, and  $u_i$  is the control input of the  $i$ th agent which needs to be designed.

Consider the following control algorithm to reach consensus of multi-agent systems (12):

$$\begin{aligned} \dot{x}_i(t) &= f(t, x_i(t)) + u_i(t) \\ u_i(t) &= -\gamma x_i(t) + \sum_{j=1}^N w_{ij}(x_j(t) - x_i(t)), \\ i &= 1, 2, \dots, N. \end{aligned} \quad (13)$$

where  $\gamma$  is the state feedback gain of the  $i$ th agent,  $w_{ij}$  is a entry of adjacency matrix  $W$ . The consensus protocol is similar to the protocol of [12]. If (4) is satisfied, we can say that the multi-agent systems (12) achieve consensus.

**Lemma 7.** For the function  $f$  in (12), there is a nonnegative matrix  $H = (h_{ij})_{n \times n}$  satisfies

$$|f_j(t, x) - f_j(t, y)| \leq \sum_{k=1}^n h_{jk} |x_k - y_k|, \quad j = 1, 2, \dots, n, \quad (14)$$

where  $x = (x_1, x_2, \dots, x_n)^T$  and  $y = (y_1, y_2, \dots, y_n)^T$ .

By Lemma 7 and the nonnegative adjacency matrix  $W$ , let

$$\rho = \frac{1}{2} \max_{1 \leq j \leq n} \sum_{k=1}^n (w_{jk} + w_{kj}), \quad (15)$$

where  $\rho$  is nonnegative. The systems (13) can be written as:

$$\dot{x} = f(t, x) - ((L + \gamma I_N) \otimes I_n) x \quad (16)$$

where  $f(t, x) = (f^T(t, x_1), f^T(t, x_2), \dots, f^T(t, x_N))^T$  and  $x = (x_1^T, x_2^T, \dots, x_N^T)^T$ .

Let a positive vector  $c^T = (c_1, \dots, c_N)^T$  satisfy  $c^T L = \mathbf{0}$  and  $c^T \mathbf{1}_N = 1$ . Then, one can obtain the following error variable:

$$\ell(t) = ((I_N - \mathbf{1}_N c^T) \otimes I_n) x(t) \quad (17)$$

where  $\ell(t) = (\ell_1^T(t), \ell_2^T(t), \dots, \ell_N^T(t))^T$  and  $\ell_i = (\ell_{i1}, \ell_{i2}, \dots, \ell_{in})^T$ ,  $i = 1, 2, \dots, N$ . Let  $\bar{x} = (\mathbf{1}_N c^T) x$ . From (16) and (17), one can get:

$$\begin{aligned} \dot{\ell} &= ((I_N - \mathbf{1}_N c^T) \otimes I_n) (f(t, x) - ((L + \gamma I_N) \otimes I_n) x) \\ &= ((I_N - \mathbf{1}_N c^T) \otimes I_n) f(t, x) - ((L + \gamma I_N) \otimes I_n) \ell \end{aligned} \quad (18)$$

**Theorem 3.** For the systems (12), the communication graph is strongly connected and (14) satisfies. Then, the consensus problem of system (13) can be solved if  $\rho < \gamma$ , where  $\rho, \gamma$  are defined in (15) and (13), respectively.

*Proof:* Consider the Lyapunov function:

$$V(t) = \frac{1}{2} \ell^T(t) (\Xi \otimes I_n) \ell(t) \quad (19)$$

where  $\Xi = \text{diag}(c_1, c_2, \dots, c_N) > 0$ . Obviously,  $V(t) \geq 0$ , and  $V(t) = 0$  if and only if  $\ell = \mathbf{0}$ . The time derivative of  $V(t)$  is:

$$\begin{aligned} \dot{V}(t) &= \ell^T (\Xi \otimes I_n) \left( ((I_N - \mathbf{1}_N c^T) \otimes I_n) f(t, x) \right. \\ &\quad \left. - ((L + \gamma I_N) \otimes I_n) \ell \right) \\ &= \ell^T (\Xi \otimes I_n) ((I_N - \mathbf{1}_N c^T) \otimes I_n) f(t, x) \\ &\quad - \ell^T (\Xi (L + \gamma I_N) \otimes I_n) \ell \\ &= \ell^T (\Xi \otimes I_n) (f(t, x) - \mathbf{1} \otimes f(t, \bar{x})) \\ &\quad - \ell^T (\Xi \otimes I_n) ((\mathbf{1} c^T) \otimes I_n) f(t, x) \\ &\quad + \ell^T (\Xi \otimes I_n) (\mathbf{1} \otimes f(t, \bar{x})) \\ &\quad - \ell^T (\Xi (L + \gamma I_N) \otimes I_n) \ell \end{aligned} \quad (20)$$

Since  $\ell(t) = ((I_N - \mathbf{1}_N c^T) \otimes I_n) x(t)$  and  $c^T \mathbf{1}_N = 1$ , one has

$$\begin{aligned} \ell^T (\Xi \otimes I_n) ((\mathbf{1} c^T) \otimes I_n) f(t, x) &= f^T(t, x) ((c \mathbf{1}^T) \otimes I_n) (\Xi \otimes I_n) \ell \\ &= f^T(t, x) ((c \mathbf{1}^T) \otimes I_n) (\Xi \otimes I_n) \\ &\quad \times ((I_N - \mathbf{1}_N c^T) \otimes I_n) x(t) \\ &= f^T(t, x) ((c c^T) \otimes I_n) ((I_N - \mathbf{1}_N c^T) \otimes I_n) x(t) \\ &= f^T(t, x) ((c c^T) (I_N - \mathbf{1}_N c^T) \otimes I_n) x(t) = 0 \end{aligned} \quad (21)$$

$$\begin{aligned} \ell^T (\Xi \otimes I_n) (\mathbf{1} \otimes f(t, \bar{x})) &= (\mathbf{1}^T \otimes f^T(t, \bar{x})) (\Xi \otimes I_n) \ell \\ &= (\mathbf{1}^T \otimes f^T(t, \bar{x})) (\Xi \otimes I_n) \\ &\quad \times ((I_N - \mathbf{1}_N c^T) \otimes I_n) x(t) \\ &= (r^T (I_N - \mathbf{1}_N c^T) \otimes f^T(t, \bar{x})) x(t) = 0 \end{aligned} \quad (22)$$

Note that  $(1/2)(\Xi L + L^T \Xi) \mathbf{1}_N = 0$  and  $(1/2)(\Xi L + L^T \Xi)$  is a matrix whose off-diagonal elements are all nonpositive and diagonal elements are all positive, so the matrix is a Laplacian matrix and all the eigenvalues of the matrix are nonnegative i.e.

$$\lambda_i((1/2)(\Xi L + L^T \Xi)) \geq 0, i = 1, 2, \dots, N. \quad (23)$$

By Lemma 7, one can obtain

$$\begin{aligned} \ell_i^T (f(t, x_i) - f(t, \bar{x}_i)) &= (\ell_{i1}, \dots, \ell_{in}) \begin{pmatrix} f_1(t, x_i) - f_1(t, \bar{x}_i) \\ \vdots \\ f_n(t, x_i) - f_n(t, \bar{x}_i) \end{pmatrix} \\ &= \ell_{i1}(f_1(t, x_i) - f_1(t, \bar{x}_i)) + \dots + \ell_{in}(f_n(t, x_i) - f_n(t, \bar{x}_i)) \\ &\leq |\ell_{i1}| |f_1(t, x_i) - f_1(t, \bar{x}_i)| + \dots + |\ell_{in}| |f_n(t, x_i) - f_n(t, \bar{x}_i)| \\ &\leq |\ell_{i1}| \sum_{k=1}^n m_{1k} |x_{ik} - \bar{x}_{ik}| + \dots + |\ell_{in}| \sum_{k=1}^n m_{nk} |x_{ik} - \bar{x}_{ik}| \\ &= |\ell_{i1}| \sum_{k=1}^n m_{1k} |\ell_{ik}| + \dots + |\ell_{in}| \sum_{k=1}^n m_{nk} |\ell_{ik}| \\ &= \sum_{j=1}^n |\ell_{ij}| \sum_{k=1}^n m_{jk} |\ell_{ik}| \\ &= \hat{\ell}_i^T M \hat{\ell}_i \end{aligned} \quad (24)$$

where  $\hat{\ell}_i = (|\ell_{i1}|, \dots, |\ell_{in}|)^T$ .

By (15), (20)–(24) and Lemma 6, one has

$$\begin{aligned} \dot{V}(t) &= \ell^T (\Xi \otimes I_n) (f(t, x) - \mathbf{1} \otimes f(t, \bar{x})) \\ &\quad - \ell^T (\Xi (L + \gamma I_N) \otimes I_n) \ell \\ &= \sum_{i=1}^N c_i \ell_i^T (f(t, x_i) - f(t, \bar{x}_i)) \\ &\quad - \ell^T (\Xi (L + \gamma I_N) \otimes I_n) \ell \\ &\leq \sum_{i=1}^N c_i \hat{\ell}_i^T M \hat{\ell}_i - \ell^T (\Xi (L + \gamma I_N) \otimes I_n) \ell \\ &\leq \frac{1}{2} \sum_{i=1}^N c_i \left( \sum_{j=1}^n \sum_{k=1}^n (m_{jk} + m_{kj}) \hat{\ell}_{ij}^2 \right) \\ &\quad - \ell^T (\Xi (L + \gamma I_N) \otimes I_n) \ell \\ &= \rho \sum_{i=1}^N c_i \sum_{j=1}^n \hat{\ell}_{ij}^2 - \ell^T (\Xi (L + \gamma I_N) \otimes I_n) \ell \\ &= \rho \sum_{i=1}^N c_i \sum_{j=1}^n \hat{\ell}_{ij}^2 - \ell^T (\gamma \Xi \otimes I_n) \ell \\ &\quad - \frac{1}{2} \ell^T ((\Xi L + L^T \Xi) \otimes I_n) \ell \\ &\leq (\rho - \gamma) \sum_{i=1}^N c_i \sum_{j=1}^n \hat{\ell}_{ij}^2 \end{aligned} \quad (25)$$

By the conditions of Theorem 3, one knows that  $\dot{V}(t) \leq 0$  and  $V(t) \equiv 0$  holds if and only if  $\ell(t) = \mathbf{0}$ . Hence, by LaSalle's invariance principle [13], the consensus problem of systems (13) is solved. ■

**Remark 4.** From Theorem 3, one knows that if the feedback gain  $\gamma$  is bigger than the positive scale  $\rho$ , the nonlinear multi-agent systems (13) can reach consensus. A favorable feature of the consensus protocol (13) is that its feedback gain  $\gamma$  can be independently designed. It is worth mentioning that the sufficient conditions do not need any global information.

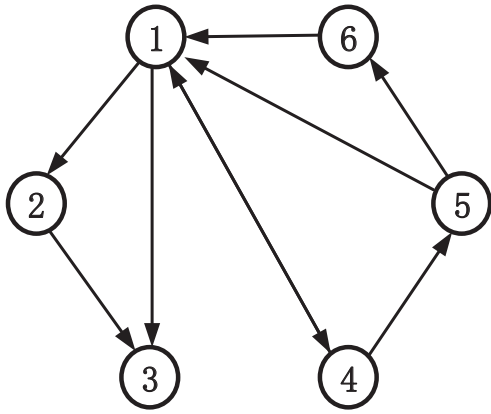


Fig. 1 The communication topology of linear multi-agent systems

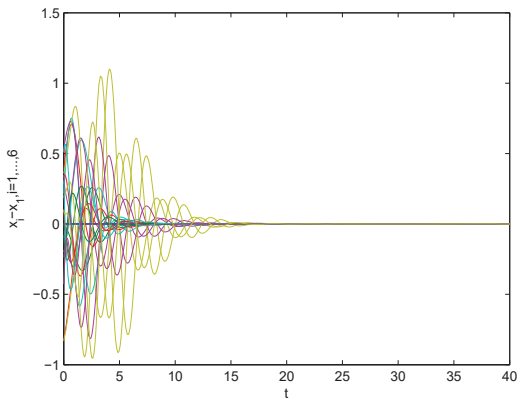


Fig. 2 The consensus errors  $x_i - x_1$  of six-agent network (3) when  $c = 0.5$

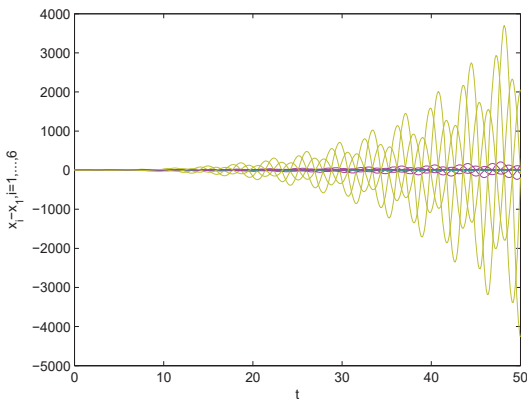


Fig. 3 The consensus errors  $x_i - x_1$  of six-agent network (3) when  $c = 0.4$

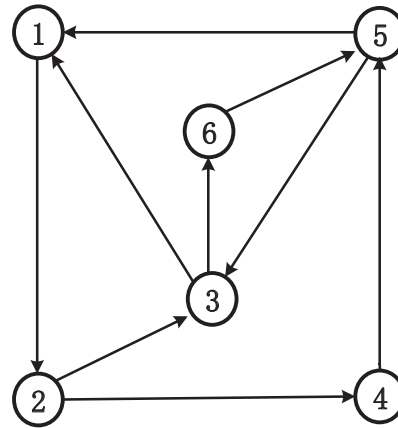


Fig. 4 The communication topology of nonlinear multi-agent systems

## V. SIMULATION

In this section, two examples are given to illustrate the effectiveness of the above theoretical analyses.

**Example 1.** Consider linear multi-agent systems with six agents as following:

$$\dot{x}_i = Px_i + Qu_i, i = 1, \dots, 6$$

and consensus protocol is given by

$$\begin{aligned} \dot{x}_i &= Px_i + Qu_i \\ u_i &= \begin{cases} 0 & i = 1 \\ \sigma(x_i - x_j) & i = 2, \dots, 6 \end{cases} \end{aligned}$$

where  $x_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})^T$ ,

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & -3 & -4 \end{pmatrix}.$$

The characteristic equation of  $P + \sigma Q$  is

$$\det(\lambda I - (P + \sigma Q)) = \lambda^4 + 4\sigma\lambda^3 + 9\sigma\lambda^2 + 12\sigma\lambda + 6\sigma.$$

By Hurwitz criterion,  $P + \sigma Q$  is stable if and only if

$$T_1 = 4\sigma > 0, T_2 = \begin{pmatrix} 4\sigma & 1 \\ 12\sigma & 9\sigma \end{pmatrix} > 0,$$

$$T_3 = \begin{pmatrix} 4\sigma & 1 & 0 \\ 12\sigma & 9\sigma & 4\sigma \\ 0 & 6\sigma & 12\sigma \end{pmatrix} > 0, T_4 = 6\sigma T_3 > 0.$$

By computing, we have  $\sigma > 0.4286$ . So the consensus interval is  $\sigma > 0.4286$ . Let the corresponding communication graph be  $G$  (See Fig1). The adjacency matrix is

$$W = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

By using the protocol (2), the consensus errors  $x_i - x_1$  of the six-agent network are depicted with different  $\sigma$  (See Fig 2 and Fig 3). The consensus interval issue is similar to the consensus region issue in [2]. The consensus interval can be easily obtained by using Hurwitz criterion even if the systems are high dimensional.

**Example 2.** Consider the nonlinear multi-agent systems composed



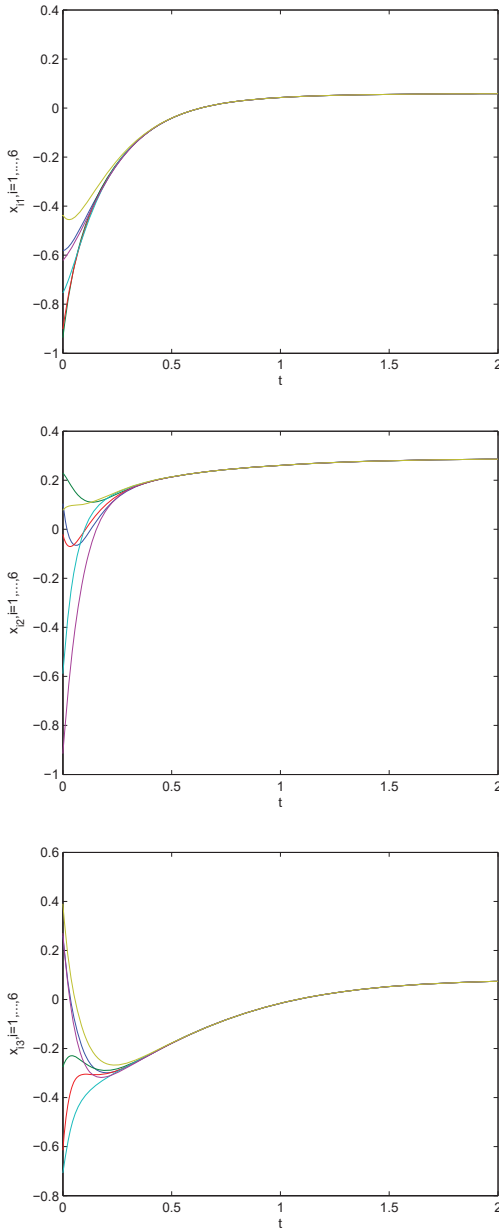


Fig. 5 The states of six agents with nonlinear dynamics

of six agents as (13):

$$\begin{aligned}\dot{x}_i(t) &= f(t, x_i(t)) + u_i(t) \\ u_i(t) &= -\gamma x_i(t) + \sum_{j=1}^N w_{ij} (x_j(t) - x_i(t)) \\ , i &= 1, 2, \dots, N\end{aligned}$$

where  $x_i = (x_{i1}, x_{i2}, x_{i3})^T$  and the nonlinear function

$$f(t, x_i) = (f_1(t, x_i), f_2(t, x_i), f_3(t, x_i))^T, \quad (26)$$

in which

$$\begin{aligned}f_1(t, x_i) &= |0.5x_{i1} + 1| - |2.5x_{i1} - 1| + \sin(x_{i2}) \\ f_2(t, x_i) &= |1.2x_{i2} + 1| + |0.8x_{i2} - 1| + \sin(1.5x_{i3}) \\ f_3(t, x_i) &= |1.7x_{i1} + 1| - |2.3x_{i1} - 1| + \sin(5x_{i3})\end{aligned}$$

After some calculations, one can obtain the following matrix satisfies Lemma 7 with the nonlinear function (26):

$$M = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1.5 \\ 4 & 0 & 5 \end{pmatrix}.$$

By (15), one can get  $\rho = 7.75$ . The communication topology of nonlinear multi-agent systems (13) is shown in Fig 4. One can choose  $\gamma = 7.76 > \rho = 7.75$ . With the consensus protocol, the states of six agents can be reached consensus that is shown in Fig 5.

## VI. CONCLUSION

In this paper, the linear multi-agent systems and the nonlinear multi-agent systems are studied, and their corresponding communication topologies are directed. For the linear systems, under a special consensus protocol, each child agent can only obtain information from one of its neighbor agents. Therefore, the information transmission paths of the communication graph are optimized. The concept of consensus interval is introduced, and the consensus interval is analyzed by using the Hurwitz criterion. For nonlinear systems, the feedback gain is designed in the consensus protocol. The conditions we obtained from both systems do not require any global information. The cases of time delay, switching topologies, and high order multi-agent systems are interesting topics in future work.

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