

Conformal Invariance in $F(R, T)$ Gravity

Pyotr Tsyba, Olga Razina, Ertan Güdekli, Ratbay Myrzakulov

Abstract—In this paper we consider the equation of motion for the $F(R, T)$ gravity on their property of conformal invariance. It is shown that in the general case, such a theory is not conformal invariant. Studied special cases for the functions v and u in which can appear properties of the theory. Also we consider cosmological aspects $F(R, T)$ theory of gravity, having considered particular case $F(R, T) = \mu R + \nu T^2$. Showed that in this case there is a nonlinear dependence of the parameter equation of state from time to time, which affects its evolution.

Keywords—Conformally invariance, $F(R, T)$ gravity, metric FRW, equation of motion, dark energy.

I. INTRODUCTION

COSMOLOGICAL phenomenon of the current accelerated expansion of the Universe is consistent with recent observational data such as observation of supernova [1], the large-scale structure [2], [3], baryon acoustic oscillations [4], the cosmic microwave radiation [5] and weak gravitational lensing [6].

To describe the accelerated expansion of the universe, there are several models such as k -essence model [7], the modified theory of gravity [8], [9], the holographic model of dark energy [10]. There are a huge number of theories describing different effects in cosmology and gravitation. To describe some require certain symmetry conditions such as Lorentz invariant so-called general covariance and conformal invariance. This invariance is a fundamental property of space-time. For other effects, these conditions can not be satisfied. For example, the conformal symmetry can be broken at the quantum level. Earlier question is conformal invariant theory of gravitation was considered in [11]–[13]. One new feature is the theory of gravitation theories $F(R, T)$ gravity proposed in [14]. It is interesting to explore its properties.

II. EQUATION OF MOTION IN $F(R, T)$ GRAVITY

We consider, action proposed in [9], for $F(R, T)$ gravity

$$S = \int d^4x \sqrt{-\hat{g}} \left[F(\hat{R}, \hat{T}) \right], \quad (1)$$

with the metric Friedmann-Robertson-Walker

$$ds^2 = -dt^2 + \hat{a}^2(t) (dx^2 + dy^2 + dz^2). \quad (2)$$

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Introducing the undetermined Lagrange multipliers, we obtain the Lagrangian corresponding to the action (1)

$$L = (F - F_{\hat{T}}\hat{T} - F_{\hat{R}}\hat{R} + F_{\hat{T}}v + F_{\hat{R}}u)\hat{a}^3 - 6(F_{\hat{R}} - F_{\hat{T}})\hat{a}^2\dot{\hat{a}} - 6(F_{\hat{R}\hat{R}}\dot{\hat{R}} + F_{\hat{R}\hat{T}}\dot{\hat{T}})\hat{a}\dot{\hat{a}}^2 \quad (3)$$

where $\hat{R} = u + 6\left(\frac{\ddot{\hat{a}}}{\hat{a}} + \frac{\dot{\hat{a}}^2}{\hat{a}^2}\right)$ — the curvature of space-time, $\hat{T} = v + 6\frac{\dot{\hat{a}}^2}{\hat{a}^2}$ — torsion of space-time, u and v — functions depend on the scale factor and its derivatives, but clearly independent of the time. Applying the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \quad (4)$$

and the condition that the total energy of the system

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = 0 \quad (5)$$

to (4), we obtain the equations of motion for our system. Conformal transformation of the metric $\hat{g}_{\mu\nu}$ in another metric $g_{\mu\nu}$

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \sqrt{-g} = \Omega^4 \sqrt{-\hat{g}}. \quad (6)$$

In general form for an arbitrary metric conversion scalar curvature and torsion scalar have the form

$$\hat{R} = \Omega^2 (R + 6\Box \ln \Omega - 6\partial_\mu \ln \Omega \partial^\mu \ln \Omega), \quad (7)$$

$$\hat{T} = \Omega^2 T - 4\Omega^{-1} \partial^\mu \Omega T^\rho_{\rho\mu} + 6\Omega^{-2} \partial_\mu \Omega \partial^\mu \Omega. \quad (8)$$

where \Box — D'Alembertian operator in the new conformal frame in four-dimensional space-time.

Then, taking into account the (2) conformal transformation of the metric (6) takes the form

$$a = \Omega \hat{a}, a^3 = \Omega^4 \hat{a}^3. \quad (9)$$

Accordingly, expressions for the scalar curvature, torsion and the Lagrangian for the case of metrics FRW transformed to the form

$$\hat{R} = u + 6\left(\dot{H} + 2H^2\right) + 6\left(\frac{\dot{\Omega}^2}{\Omega^2} - \frac{\ddot{\Omega}}{\Omega} - 4\frac{\dot{\Omega}}{\Omega}H\right), \quad (10)$$

$$\hat{T} = v + 6H^2 + 6\left(2\frac{\dot{\Omega}}{\Omega}H - \frac{\dot{\Omega}^2}{\Omega^2}\right), \quad (11)$$

$$L = \Omega^{-4}(F - F_T T - F_R R + F_T v + F_R u)a^3 - 6\Omega^{-4}(F_R - F_T)\dot{a}^2 a - 6\Omega^{-4}(F_{RR}\dot{R} + F_{RT}\dot{T})\dot{a}a^2 + (6\Omega^{-6}\dot{\Omega}^2 F_T - 24\Omega^{-6}\dot{\Omega}^2 F_R + 6\Omega^{-5}\dot{\Omega}(F_{RR}\dot{R} + F_{RT}\dot{T}))a^3 - 6\Omega^{-5}\dot{\Omega}(3F_R + 2F_T)\dot{a}a^2. \quad (12)$$

Obtain the equation of motion for our system after making a substitute in (4) and (5) according to (9).

A. Case $u = u(a)$ and $v = v(a)$

We set $u = u(a)$ and $v = v(a)$, then performing elementary calculations we obtain

$$\begin{aligned} -2p = & -F + A_1 F_R + A_2 F_T + \\ & + B_1 F_{RR} + B_2 F_{TT} + B_3 F_{RT} + B_4 F_{TR} + \\ & + C_1 F_{RRR} + C_2 F_{RRT} + C_3 F_{RTT} + C_4 F_{RTT} \end{aligned} \quad (13)$$

$$F_R : A_1 = 16 \frac{\dot{\Omega}}{\Omega} \frac{\dot{a}}{a} + 54 \frac{\dot{\Omega}^2}{\Omega^2} - 6 \frac{\ddot{\Omega}}{\Omega} + 2 \frac{\ddot{a}}{a} + 4 \frac{\dot{a}^2}{a^2} - u_a \frac{a}{3}, \quad (14)$$

$$F_T : A_2 = -16 \frac{\dot{\Omega}}{\Omega} \frac{\dot{a}}{a} + 14 \frac{\dot{\Omega}^2}{\Omega^2} - 4 \frac{\ddot{\Omega}}{\Omega} + 4 \frac{\ddot{a}}{a} + 4 \frac{\dot{a}^2}{a^2} - v_a \frac{a}{3}, \quad (15)$$

$$F_{RR} : B_1 = -4 \frac{\dot{a}}{a} \dot{R} - 4 \frac{\dot{\Omega}}{\Omega} \dot{R} - 2 \ddot{R}, \quad (16)$$

$$F_{TT} : B_2 = 4 \frac{\dot{a}}{a} \dot{T} - 4 \frac{\dot{\Omega}}{\Omega} \dot{T}, \quad (17)$$

$$F_{RT} : B_3 = -4 \frac{\dot{a}}{a} \dot{T} - 4 \frac{\dot{\Omega}}{\Omega} \dot{T} - 2 \ddot{T}, \quad (18)$$

$$F_{TR} : B_4 = 4 \frac{\dot{a}}{a} \dot{R} - 4 \frac{\dot{\Omega}}{\Omega} \dot{R}, \quad (19)$$

$$F_{RRR} : C_1 = -2 \dot{R}^2, \quad (20)$$

$$F_{RRT} : C_2 = -2 \dot{R} \dot{T}, \quad (21)$$

$$F_{RTT} : C_3 = -2 \dot{R} \dot{T}, \quad (22)$$

$$F_{RTT} : C_4 = -2 \dot{T}^2. \quad (23)$$

$$-2\rho = -F + D_1 F_R + D_2 F_T + E_1 F_{RR} + E_2 F_{RT} \quad (24)$$

$$F_R : D_1 = 6 \frac{\ddot{a}}{a} - 24 \frac{\dot{\Omega}^2}{\Omega^2}, \quad (25)$$

$$F_T : D_2 = 12 \frac{\dot{a}^2}{a^2} + 6 \frac{\dot{\Omega}^2}{\Omega^2}, \quad (26)$$

$$F_{RR} : E_1 = 6 \frac{\dot{a}}{a} \dot{R} + 6 \frac{\dot{\Omega}}{\Omega} \dot{R}, \quad (27)$$

$$F_{RT} : E_2 = 6 \frac{\dot{a}}{a} \dot{T} + 6 \frac{\dot{\Omega}}{\Omega} \dot{T}. \quad (28)$$

B. Case $u = u(\dot{a})$ and $v = v(\dot{a})$

We set $u = u(\dot{a})$ and $v = v(\dot{a})$, then performing elementary calculations we obtain

$$\begin{aligned} -2p = & -F + A_1 F_R + A_2 F_T + \\ & + B_1 F_{RR} + B_2 F_{TT} + B_3 F_{RT} + B_4 F_{TR} + \\ & + C_1 F_{RRR} + C_2 F_{RRT} + C_3 F_{RTT} + C_4 F_{RTT} + \\ & + C_5 F_{RRR} \end{aligned} \quad (29)$$

$$F_R : A_1 = R - u - 54 \frac{\dot{\Omega}^2}{\Omega^2} - 6 \frac{\ddot{\Omega}}{\Omega} + \dot{a} u_{\dot{a}} - \quad (30)$$

$$\begin{aligned} & -4 \frac{\ddot{a}}{a} + 4 \frac{\dot{a}}{a} \frac{\dot{\Omega}}{\Omega} + \frac{4}{3} \frac{\dot{\Omega}}{\Omega} a u_{\dot{a}} + \frac{1}{3} a u_{\dot{a}\dot{a}} - \\ & - \frac{a}{3} u_a + \frac{1}{3} \frac{\dot{a}}{a} u_{a\dot{a}}, \end{aligned}$$

$$F_T : A_2 = T - v + 14 \frac{\dot{\Omega}^2}{\Omega^2} - 4 \frac{\ddot{\Omega}}{\Omega} + \dot{a} v_{\dot{a}} + \quad (31)$$

$$\begin{aligned} & + 2 \frac{\dot{a}^2}{a^2} + 4 \frac{\ddot{a}}{a} - 4 \frac{\dot{a}}{a} \frac{\dot{\Omega}}{\Omega} - \frac{4}{3} \frac{\dot{\Omega}}{\Omega} a v_{\dot{a}} + \frac{4}{3} \frac{\dot{\Omega}}{\Omega} a u_{\dot{a}} - \\ & - \frac{4}{3} \frac{\dot{\Omega}}{\Omega} a v_{\dot{a}} + \frac{1}{3} a v_{\dot{a}\dot{a}} - \frac{a}{3} v_a + \frac{1}{3} \frac{\dot{a}}{a} v_{a\dot{a}}, \end{aligned}$$

$$F_{RR} : B_1 = -4 \dot{R} \frac{\dot{\Omega}}{\Omega} - 2 \ddot{R} - 4 \frac{\dot{a}}{a} + \frac{1}{3} \dot{R} a u_{\dot{a}}, \quad (32)$$

$$F_{TT} : B_2 = -4 \dot{T} \frac{\dot{\Omega}}{\Omega} + 4 \frac{\dot{a}}{a} T + \frac{1}{3} a \dot{T} u_{\dot{a}}, \quad (33)$$

$$F_{RT} : B_3 = -4 \dot{R} \frac{\dot{\Omega}}{\Omega} - 4 \dot{T} \frac{\dot{\Omega}}{\Omega} - 2 \ddot{T} - 4 \frac{\dot{a}}{a} \dot{T} + \quad (34)$$

$$+ 4 \frac{\dot{a}}{a} \dot{R} + \frac{1}{3} a u_{\dot{a}} \dot{T} + \frac{1}{3} a v_{\dot{a}} \dot{R},$$

$$F_{RRR} : C_1 = -2 \dot{R}^2, \quad (35)$$

$$F_{RRT} : C_2 = -4 \dot{R} \dot{T}, \quad (36)$$

$$F_{RTT} : C_4 = -2 \dot{T}^2 \quad (37)$$

$$-2p = -F + D_1 F_R + D_2 F_T + E_1 F_{RR} + E_2 F_{RT} \quad (38)$$

$$F_R : D_1 = 24 \frac{\dot{\Omega}^2}{\Omega^2} + \dot{a} u_{\dot{a}} + 6 \frac{\ddot{a}}{a}, \quad (39)$$

$$F_T : D_2 = -6 \frac{\dot{\Omega}^2}{\Omega^2} + \dot{a} v_{\dot{a}} + 12 \frac{\dot{a}}{a}, \quad (40)$$

$$F_{RR} : E_1 = -6 \frac{\dot{\Omega}}{\Omega} \dot{T}, \quad (41)$$

$$F_{RT} : E_2 = -6 \frac{\dot{\Omega}}{\Omega} \dot{R}. \quad (42)$$

III. COSMOLOGICAL SOLUTIONS

In this section we consider $F(R, T)$ gravity and show, what obtained models which can be based on $F(R, T)$ theory of gravity and able to describe evolution of universe. For this we consider a special case

$$F(R, T) = \mu R + \nu T^2. \quad (43)$$

Rewrite (4) as follow

$$\begin{aligned} L = & (F - F_T T - F_R R + F_T v + F_R u) \dot{a}^3 - \\ & - 6(\epsilon_1 F_R - \epsilon_2 F_T) \dot{a}^2 \dot{a} - 6\epsilon_1 (F_{RR} \dot{R} + F_{RT} \dot{T}) \dot{a} \dot{a}^2 \end{aligned} \quad (44)$$

where ϵ_1, ϵ_2 — constants, depending on the signature of the metric. They may take the following values $\epsilon_1 = -1$ and $\epsilon_2 = -1$, or $\epsilon_1 = 1$ and $\epsilon_2 = 1$. After substituting (44) and (43) in (4) we obtain the equations of motion for the scale factor a

$$\begin{aligned} A\mu + 2B\nu + 2C\nu T - D(\nu R + \nu T^2) &= 6\dot{a}^2 p, \\ E\mu + 2F\nu T - G(\nu R + \mu T^2) &= 2\dot{a}^3 \rho, \\ \dot{\rho} + 3H(p + \rho) &= 0, \end{aligned} \quad (45)$$

where

$$A = 12\epsilon_1 \dot{a}^2 + 6\epsilon_1 \dot{a} \ddot{a} + 3\dot{a}^2 \dot{u}_{\dot{a}} + \dot{a}^3 \dot{u}_{\dot{a}} - \dot{a}^3 u_{\dot{a}}, \quad (46)$$

$$B = 12\epsilon_2 \dot{T} \dot{a} \dot{a} + \dot{a}^3 \dot{T} v_{\dot{a}}, \quad (47)$$

$$C = 24\dot{a}^2 + 12\dot{a} \ddot{a} + 3\dot{a}^2 \dot{v}_{\dot{a}} + \dot{a}^3 \dot{v}_{\dot{a}} - \dot{a}^3 v_{\dot{a}}, \quad (48)$$

$$D = -3\dot{a}^3, \quad (49)$$

$$E = 6\epsilon_1 \dot{a} \ddot{a} + \dot{a}^3 u_{\dot{a}} \dot{a}, \quad (50)$$

$$F = 12\epsilon_2 \dot{a} \dot{a}^2 + \dot{a}^3 v_{\dot{a}} \dot{a}, \quad (51)$$

$$G = -\dot{a}^3. \quad (52)$$

After simple transformations of equations (45)-(46) we obtain the following expression for the energy density and pressure, respectively

$$\begin{aligned} 3xH^2 - \frac{\dot{a}}{2}(y_{\dot{a}} - y) &= \rho, \\ -x(3H^2 + 3\dot{H}) + 6\epsilon_2 \dot{H} T \nu + 3\dot{a}(y_{\dot{a}} - y) + \\ &+ \dot{a}(\dot{y}_{\dot{a}} - y_{\dot{a}}) = p, \\ \dot{\rho} + 3H(\rho + p) &= 0 \end{aligned} \quad (53)$$

where $x = u\mu + 3vT\nu$ $y = \epsilon_1\mu + 3\epsilon_2T\nu$.

Then the parameter equation of state takes the form

$$\omega = \frac{p}{\rho} = -1 - \frac{2x\dot{H} - 6\epsilon_2 \dot{H} T \nu - \frac{\dot{a}}{6}(\dot{x}_{\dot{a}} - x_{\dot{a}})}{3xH^2 - \frac{\dot{a}}{2}(x_{\dot{a}} - x)}. \quad (54)$$

Suggesting in (56) $x = 0$, we have

$$\omega = \frac{p}{\rho} = -1 + \frac{6\epsilon_2 \dot{H} T \nu - \frac{\dot{a}}{6}(\dot{x}_{\dot{a}} - x_{\dot{a}})}{\frac{\dot{a}}{2}(x_{\dot{a}} - x)}. \quad (55)$$

A. Example 1

We suggest that $x = \alpha \dot{a}^l$ and $\epsilon_2 \nu = 1$ then the parameter equation of state takes the form

$$\omega = \frac{p}{\rho} = -1 - \frac{36\dot{H}T + \alpha l \dot{a}^l}{3\dot{a}x}. \quad (56)$$

It follows that if

$$36\dot{H}T + \alpha l \dot{a}^l = 0, \quad (57)$$

then $\omega = -1$. This value corresponds to the state Λ CD model. The numerical solution of the equation (57) for $\alpha > 0$, $l < 0$ is shown in Fig. 1 with the initial conditions $\dot{a}(0) = 10^{-5}$, and $\dot{a}(0) = 10^{-10}$. The Fig. 2 shows the evolution of the deceleration parameter $q = -\ddot{a}/\dot{a}$ over time.

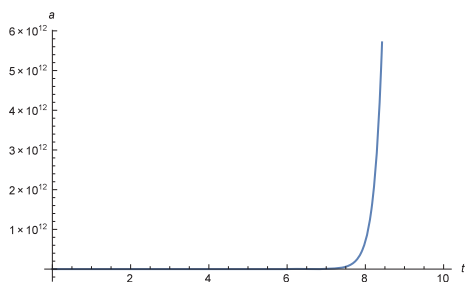


Fig. 1 The time dependence scale factor a .

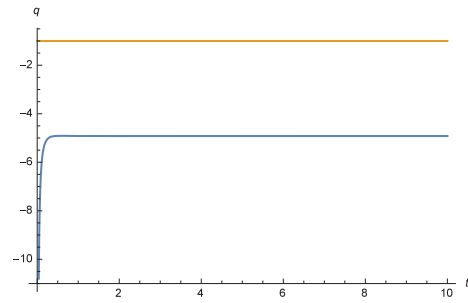


Fig. 2 The time dependence deceleration parameter q .

B. Example 2

Now we discuss the case where the scale factor varies according to a power law $\dot{a} = \dot{a}_0 t^n$. Then the system (53) takes the form

$$\begin{aligned} 3xn^2t^{-2} + 2xnt^{-2} - 36\epsilon_2^2 n^3 t^{-4} - \frac{\dot{a}_0^l k l}{6} t^{n(l+1)-1} &= p, \\ 3xn^2t^{-2} - \frac{\dot{a}_0^l k}{2} t^{nl} &= \rho, \\ \dot{\rho} + 3H(\rho + p) &= 0. \end{aligned} \quad (58)$$

Parameter equation of state

$$\omega = -1 + \frac{2xnt^{-2} - 36\epsilon_2^2 n^3 t^{-4} - \frac{\dot{a}_0^l k l}{6} t^{n(l+1)-1}}{3xn^2t^{-2} - \frac{\dot{a}_0^l k}{2} t^{nl}} \quad (59)$$

provided that $t \rightarrow \infty$

$$\omega = -1 + \frac{l}{3} t^{n-1}. \quad (60)$$

IV. CONCLUSION

Thus we have shown that there are additional terms in the equations of motion in the case of metric Friedmann-Robertson-Walker which modify it. That is, $F(R, T)$ gravity is not conformal invariant, but picking function are essential u and v we can find a combination in which these additional terms disappear. We also show that $F(R, T)$ gravity model able to describe the accelerated expansion of the universe. It will focus on further research.

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